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THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions* take this opportunity to acquaint the public that it fully appears, as well from the Council-books and Journals of the Society as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the public that their usual meetings were then continued, for the improvement of knowledge and benefit of mankind: the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions;* which was accordingly done upon the 26th of March, 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body, upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report, and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

1899.

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" " AB " " " " Series A and B, and Proceedings.
" " p " " Proceedings only.

America (Central).
Mexico.
- *p.* Sociedad Científica "Antonio Alzate."

America (North). (See United States and Canada.)

America (South).
Buenos Ayres.
- AB. Museo Nacional.

Caracas.
- B. University Library.

Cordova.
- AB. Academia Nacional de Ciencias.

Demerara.
- *p.* Royal Agricultural and Commercial Society, British Guiana.

La Plata.
- B. Museo de La Plata.

Rio de Janeiro.
- *p.* Observatorio.

Australia.
Adelaide.
- *p.* Royal Society of South Australia.

Brisbane.
- *p.* Royal Society of Queensland.

Melbourne.
- *p.* Observatory.
- *p.* Royal Society of Victoria.
- AB. University Library.

Sydney.
- *p.* Australian Museum.
- *p.* Geological Survey.
- *p.* Linnean Society of New South Wale
- AB. Royal Society of New South Wales.
- AB. University Library.

Austria.
Agram.
- *p.* Jugoslavenska Akademija Znanosti i Umjetnosti.
- *p.* Societas Historico-Naturalis Croatica.

Austria (continued).
Brünn.
- AB. Naturforschender Verein.

Gratz.
- AB. Naturwissenschaftlicher Verein für Steiermark.

Innsbruck.
- AB. Das Ferdinandeum.
- *p.* Naturwissenschaftlich - Medicinischer Verein.

Prague.
- AB. Königliche Böhmische Gesellschaft der Wissenschaften.

Trieste.
- B. Museo di Storia Naturale.
- *p.* Società Adriatica di Scienze Naturali.

Vienna.
- *p.* Anthropologische Gesellschaft.
- AB. Kaiserliche Akademie der Wissenschaften.
- *p.* K.K. Geographische Gesellschaft.
- AB. K.K. Geologische Reichsanstalt.
- B. K.K. Naturhistorisches Hof-Museum.
- B. K.K. Zoologisch-Botanische Gesellschaft.
- *p.* Oesterreichische Gesellschaft für Meteorologie.
- A. Von Kuffner'sche Sternwarte.

Belgium.
Brussels.
- B. Académie Royale de Médecine.
- AB. Académie Royale des Sciences.
- B. Musée Royal d'Histoire Naturelle de Belgique.
- *p.* Observatoire Royal.
- *p.* Société Belge de Géologie, de Paléontologie, et d'Hydrologie.
- *p.* Société Malacologique de Belgique.

Ghent.
- AB. Université.

Belgium (continued).
Liége.
AB. Société des Sciences.
p. Société Géologique de Belgique.
Louvain.
B. Laboratoire de Microscopie et de Biologie Cellulaire.
AB. Université.

Canada.
Fredericton, N.B.
p. University of New Brunswick.
Halifax, N.S.
p. Nova Scotian Institute of Science.
Hamilton.
p. Hamilton Association.
Montreal.
AB. McGill University.
p. Natural History Society.
Ottawa.
AB. Geological Survey of Canada.
AB. Royal Society of Canada.
St. John, N.B.
p. Natural History Society.
Toronto.
p. Astronomical and Physical Society.
p. Canadian Institute.
AB. University.
Windsor, N.S.
p. King's College Library.

Cape of Good Hope.
A. Observatory.
AB. South African Library.

Ceylon.
Colombo.
a. Museum.

China.
Shanghai.
p. China Branch of the Royal Asiatic Society.

Denmark.
Copenhagen.
AB. Kongelige Danske Videnskabernes Selskab.

Egypt.
Alexandria.
AB. Bibliothèque Municipale.

England and Wales.
Aberystwith.
AB. University College.
Bangor.
AB. University College of North Wales.
Birmingham.
AB. Free Central Library.
AB. Mason College.
p. Philosophical Society.

England and Wales (continued).
Bolton.
p. Public Library.
Bristol.
p. Merchant Venturers' School.
AB. University College.
Cambridge.
AB. Philosophical Society.
p. Union Society.
Cooper's Hill.
AB. Royal Indian Engineering College.
Dudley.
p. Dudley and Midland Geological and Scientific Society.
Essex.
p. Essex Field Club.
Falmouth.
p. Royal Cornwall Polytechnic Society.
Greenwich.
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Kew.
B. Royal Gardens.
Leeds.
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AB. Yorkshire College.
Liverpool.
AB. Free Public Library.
p. Literary and Philosophical Society.
A. Observatory.
AB. University College.
London.
AB. Admiralty.
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AB. British Museum (Nat. Hist.).
AB. Chemical Society.
A. City and Guilds of London Institute.
p. "Electrician," Editor of the.
B. Entomological Society.
AB. Geological Society.
AB. Geological Survey of Great Britain.
p. Geologists' Association.
AB. Guildhall Library.
A. Institution of Civil Engineers.
p. Institution of Electrical Engineers.
A. Institution of Mechanical Engineers.
A. Institution of Naval Architects.
p. Iron and Steel Institute.
AB. King's College.
B. Linnean Society.
AB. London Institution.
p. London Library.
A. Mathematical Society.
p. Meteorological Office.
p. Odontological Society.

England and Wales (continued).

London (continued).

- *p.* Pharmaceutical Society.
- *p.* Physical Society.
- *p.* Quekett Microscopical Club.
- *p.* Royal Agricultural Society.
- A. Royal Astronomical Society.
- B. Royal College of Physicians.
- B. Royal College of Surgeons.
- *p.* Royal Engineers (for Libraries abroad, six copies).
- AB. Royal Engineers. Head Quarters Library.
- *p.* Royal Geographical Society.
- *p.* Royal Horticultural Society.
- *p.* Royal Institute of British Architects.
- AB. Royal Institution of Great Britain.
- B. Royal Medical and Chirurgical Society.
- *p.* Royal Meteorological Society.
- *p.* Royal Microscopical Society.
- *p.* Royal Statistical Society.
- AB. Royal United Service Institution.
- AB. Society of Arts.
- *p.* Society of Biblical Archæology.
- *p.* Society of Chemical Industry (London Section).
- *p.* Standard Weights and Measures Department.
- AB. The Queen's Library.
- AB. The War Office.
- AB. University College.
- *p.* Victoria Institute.
- B. Zoological Society.

Manchester.

- AB. Free Library.
- AB. Literary and Philosophical Society.
- *p.* Geological Society.
- AB. Owens College.

Netley.

- *p.* Royal Victoria Hospital.

Newcastle.

- AB. Free Library.
- *p.* North of England Institute of Mining and Mechanical Engineers.
- *p.* Society of Chemical Industry (Newcastle Section).

Norwich.

- *p.* Norfolk and Norwich Literary Institution.

Nottingham.

- AB. Free Public Library.

Oxford.

- *p.* Ashmolean Society.
- AB. Radcliffe Library.
- A. Radcliffe Observatory.

England and Wales (continued).

Penzance.

- *p.* Geological Society of Cornwall.

Plymouth.

- B. Marine Biological Association.
- *p.* Plymouth Institution.

Richmond.

- A. "Kew" Observatory.

Salford.

- *p.* Royal Museum and Library.

Stonyhurst.

- *p.* The College.

Swansea.

- AB. Royal Institution.

Woolwich.

- AB. Royal Artillery Library.

Finland.

Helsingfors.

- *p.* Societas pro Fauna et Flora Fennica.
- AB. Société des Sciences.

France.

Bordeaux.

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- *p.* Faculté des Sciences.
- *p.* Société de Médecine et de Chirurgie.
- *p.* Société des Sciences Physiques et Naturelles.

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Cherbourg.

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- *p.* Académie des Sciences.

Lille.

- *p.* Faculté des Sciences.

Lyons.

- AB. Académie des Sciences, Belles-Lettres et Arts.
- AB. Université.

Marseilles.

- AB. Faculté des Sciences.

Montpellier.

- AB. Académie des Sciences et Lettres.
- B. Faculté de Médecine.

Nantes.

- *p.* Société des Sciences Naturelles de l'Ouest de la France.

Paris.

- AB. Académie des Sciences de l'Institut.
- *p.* Association Française pour l'Avancement des Sciences.
- *p.* Bureau des Longitudes.
- A. Bureau International des Poids et Mesures.
- *p.* Commission des Annales des Ponts et Chaussées.
- *p.* Conservatoire des Arts et Métiers.

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  - Paris (continued).
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    - AB. École des Mines.
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    - B. Société Entomologique.
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    - *p.* Société Météorologique de France.
  - Toulouse.
    - AB. Académie des Sciences.
    - A. Faculté des Sciences.
- **Germany.**
  - Berlin.
    - A. Deutsche Chemische Gesellschaft.
    - A. Die Sternwarte.
    - *p.* Gesellschaft für Erdkunde.
    - AB. Königliche Preussische Akademie der Wissenschaften.
    - A. Physikalische Gesellschaft.
  - Bonn.
    - AB. Universität.
  - Bremen.
    - *p.* Naturwissenschaftlicher Verein.
  - Breslau.
    - *p.* Schlesische Gesellschaft für Vaterländische Kultur.
  - Brunswick.
    - *p.* Verein für Naturwissenschaft.
  - Carlsruhe. See Karlsruhe.
  - Charlottenburg.
    - A. Physikalisch-Technische Reichsanstalt.
  - Danzig.
    - AB. Naturforschende Gesellschaft.
  - Dresden.
    - *p.* Verein für Erdkunde.
  - Emden.
    - *p.* Naturforschende Gesellschaft.
  - Erlangen.
    - AB. Physikalisch-Medicinische Societät.
- **Germany** (continued).
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    - AB. Senckenbergische Naturforschende Gesellschaft.
    - *p.* Zoologische Gesellschaft.
  - Frankfurt-am-Oder.
    - *p.* Naturwissenschaftlicher Verein.
  - Freiburg-im-Breisgau.
    - AB. Universität.
  - Giessen.
    - AB. Grossherzogliche Universität.
  - Görlitz.
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  - Göttingen.
    - AB. Königliche Gesellschaft der Wissenschaften.
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  - Hamburg.
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    - AB. Naturwissenschaftlicher Verein.
  - Heidelberg.
    - *p.* Naturhistorisch-Medizinischer Verein.
    - AB. Universität.
  - Jena.
    - AB. Medicinisch-Naturwissenschaftliche Gesellschaft.
  - Karlsruhe.
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    - *p.* Technische Hochschule.
  - Kiel.
    - *p.* Naturwissenschaftlicher Verein für Schleswig-Holstein.
    - A. Sternwarte.
    - AB. Universität.
  - Königsberg.
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  - Leipsic.
    - *p.* Annalen der Physik und Chemie.
    - AB. Königliche Sächsische Gesellschaft der Wissenschaften.
  - Magdeburg.
    - *p.* Naturwissenschaftlicher Verein.
  - Marburg.
    - AB. Universität.
  - Munich.
    - AB. Königliche Bayerische Akademie der Wissenschaften.
    - *p.* Zeitschrift für Biologie.

Germany (continued).

- Münster.
  - AB. Königliche Theologische und Philosophische Akademie.
- Potsdam.
  - A. Astrophysikalisches Observatorium.
- Rostock.
  - AB. Universität.
- Strasburg.
  - AB. Universität.
- Tübingen.
  - AB. Universität.
- Würzburg.
  - AB. Physikalisch-Medicinische Gesellschaft.

Greece.

- Athens.
  - A. National Observatory.

Holland. (See Netherlands.)

Hungary.

- Buda-pest.
  - *p*. Königl. Ungarische Geologische Anstalt.
  - AB. Á Magyar Tudós Társaság. Die Ungarische Akademie der Wissenschaften.
- Hermannstadt.
  - *p*. Siebenbürgischer Verein für die Naturwissenschaften.
- Klausenburg.
  - AB. Az Erdélyi Muzeum. Das Siebenbürgische Museum.
- Schemnitz.
  - *p*. K. Ungarische Berg- und Forst-Akademie.

India.

- Bombay.
  - AB. Elphinstone College.
  - *p*. Royal Asiatic Society (Bombay Branch).
- Calcutta.
  - AB. Asiatic Society of Bengal.
  - AB. Geological Museum.
  - *p*. Great Trigonometrical Survey of India.
  - AB. Indian Museum.
  - *p*. The Meteorological Reporter to the Government of India.
- Madras.
  - B. Central Museum.
  - A. Observatory.
- Roorkee.
  - *p*. Roorkee College.

Ireland.

- Armagh.
  - A. Observatory.
- Belfast.
  - AB. Queen's College.

Ireland (continued).

- Cork.
  - *p*. Philosophical Society.
  - AB. Queen's College.
- Dublin.
  - A. Observatory.
  - AB. National Library of Ireland.
  - B. Royal College of Surgeons in Ireland.
  - AB. Royal Dublin Society.
  - AB. Royal Irish Academy.
- Galway.
  - AB. Queen's College.

Italy.

- Acireale.
  - *p*. Accademia di Scienze, Lettere ed Arti.
- Bologna.
  - AB. Accademia delle Scienze dell' Istituto.
- Catania.
  - AB. Accademia Gioenia di Scienze Naturali.
- Florence.
  - *p*. Biblioteca Nazionale Centrale.
  - AB. Museo Botanico.
  - *p*. Reale Istituto di Studi Superiori.
- Genoa.
  - *p*. Società Ligustica di Scienze Naturali e Geografiche.
- Milan.
  - AB. Reale Istituto Lombardo di Scienze, Lettere ed Arti.
  - AB. Società Italiana di Scienze Naturali.
- Modena.
  - *p*. Le Stazioni Sperimentali Agrarie Italiane.
- Naples.
  - *p*. Società di Naturalisti.
  - AB. Società Reale, Accademia delle Scienze.
  - B. Stazione Zoologica (Dr. Dohrn).
- Padua.
  - *p*. University.
- Palermo.
  - A. Circolo Matematico.
  - AB. Consiglio di Perfezionamento (Società di Scienze Naturali ed Economiche).
  - A. Reale Osservatorio.
- Pisa.
  - *p*. Il Nuovo Cimento.
  - *p*. Società Toscana di Scienze Naturali.
- Rome.
  - *p*. Accademia Pontificia de' Nuovi Lincei.
  - *p*. Rassegna delle Scienze Geologiche in Italia.
  - A. Reale Ufficio Centrale di Meteorologia e di Geodinamica, Collegio Romano.
  - AB. Reale Accademia dei Lincei.
  - *p*. R. Comitato Geologico d' Italia.
  - A. Specola Vaticana.

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Rome (continued).
AB. Società Italiana delle Scienze.
Siena.
p. Reale Accademia dei Fisiocritici.
Turin.
p. Laboratorio di Fisiologia.
AB. Reale Accademia delle Scienze.
Venice.
p. Ateneo Veneto.
AB. Reale Istituto Veneto di Scienze, Lettere ed Arti.

Japan.
Tokiô.
AB. Imperial University.
p. Asiatic Society of Japan.

Java.
Buitenzorg.
p. Jardin Botanique.

Luxembourg.
Luxembourg.
p. Société des Sciences Naturelles.

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Amsterdam.
AB. Koninklijke Akademie van Wetenschappen.
p. K. Zoologisch Genootschap 'Natura Artis Magistra.'
Delft.
p. École Polytechnique.
Haarlem.
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p. Musée Teyler.
Leyden.
AB. University.
Rotterdam.
AB. Bataafsch Genootschap der Proefondervindelijke Wijsbegeerte.
Utrecht.
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Trondhjem.
AB. Kongelige Norske Videnskabers Selskab.

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AB. Universidade.
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AB. Academia Real das Sciencias.
p. Secção dos Trabalhos Geologicos de Portugal.
Oporto.
p. Annaes de Sciencias Naturaes.

Russia.
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AB. Université.
Irkutsk.
p. Société Impériale Russe de Géographie (Section de la Sibérie Orientale).
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AB. Imperatorsky Kazansky Universitet.
p. Société Physico-Mathématique.
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p. Section Médicale de la Société des Sciences Expérimentales, Université de Kharkow.
Kieff.
p. Société des Naturalistes.
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AB. Le Musée Public.
B. Société Impériale des Naturalistes.
Odessa.
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A. Nikolai Haupt-Sternwarte.
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AB. Académie Impériale des Sciences.
B. Archives des Sciences Biologiques.
AB. Comité Géologique.
AB. Ministry of Marine.
A. Observatoire Physique Central.

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Aberdeen.
AB. University.
Edinburgh.
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p. Royal College of Physicians (Research Laboratory).
p. Royal Medical Society.
A. Royal Observatory.
p. Royal Physical Society.
p. Royal Scottish Society of Arts.
AB Royal Society.

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AB. Real Academia de Ciencias.
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Lund.
AB. Universitet.
Stockholm.
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AB. Kongliga Svenska Vetenskaps-Akademie.
AB. Sveriges Geologiska Undersökning
Upsala.
AB. Universitet.
Switzerland.
Basel.
p. Naturforschende Gesellschaft.
Bern.
AB. Allg. Schweizerische Gesellschaft.
p. Naturforschende Gesellschaft.
Geneva.
AB. Société de Physique et d'Histoire Naturelle.
AB. Institut National Genevois.
Lausanne.
p. Société Vaudoise des Sciences Naturelles.
Neuchâtel.
p. Société des Sciences Naturelles.
Zürich.
AB. Das Schweizerische Polytechnikum.
p. Naturforschende Gesellschaft.
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p. Texas Academy of Sciences.
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AB. Johns Hopkins University.
Berkeley.
p. University of California.
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A. Technological Institute.
Brooklyn.
AB. Brooklyn Library.
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AB. Harvard University.
B. Museum of Comparative Zoology.
Chapel Hill (N.C.).
p. Elisha Mitchell Scientific Society.
Charleston.
p. Elliott Society of Science and Art of South Carolina.
Chicago.
AB. Academy of Sciences.
p. Astro-physical Journal.
p. Field Columbian Museum.
p. Journal of Comparative Neurology.
Davenport (Iowa).
p. Academy of Natural Sciences.
Ithaca (N.Y.).
A. Journal of Physical Chemistry.
p. Physical Review (Cornell University).
Madison.
p. Wisconsin Academy of Sciences.
Mount Hamilton (California).
A. Lick Observatory.
New Haven (Conn.).
AB. American Journal of Science.
AB. Connecticut Academy of Arts and Sciences.
New York.
p. American Geographical Society.
A. American Mathematical Society.
p. American Museum of Natural History.
p. New York Academy of Sciences.
p. New York Medical Journal.
p. School of Mines, Columbia College.
Philadelphia.
AB. Academy of Natural Sciences.
AB. American Philosophical Society.
p. Franklin Institute.
p. Wagner Free Institute of Science

United States (continued).
- Rochester (N.Y.).
  - *p*. Academy of Science.
- St. Louis.
  - *p*. Academy of Science.
- Salem (Mass.).
  - *p*. American Association for the Advancement of Science.
  - AB. Essex Institute.
- San Francisco.
  - AB. California Academy of Sciences.
- Washington.
  - AB. Patent Office.

United States (continued).
- Washington (continued).
  - AB. Smithsonian Institution.
  - AB. United States Coast Survey.
  - B. United States Commission of Fish and Fisheries.
  - AB. United States Geological Survey.
  - AB. United States Naval Observatory.
  - *p*. United States Department of Agriculture.
  - A. United States Department of Agriculture (Weather Bureau).
- West Point (N.Y.)
  - AB. United States Military Academy.

ADJUDICATION of the MEDALS of the ROYAL SOCIETY for the year 1899, by the PRESIDENT and COUNCIL.

The **COPLEY MEDAL** to the Right Hon. Lord RAYLEIGH, F.R.S., in recognition of his contributions to Physical Science.

A **ROYAL MEDAL** to Professor G. F. FITZGERALD, F.R.S., for his contributions to Physical Science, especially in the domains of Optics and Electricity.

A **ROYAL MEDAL** to Professor W. C. MCINTOSH, F.R.S., for his important monographs on British Marine Zoology and on the Fishery Industries.

The **DAVY MEDAL** to Dr. EDWARD SCHUNCK, F.R.S., for his researches on Madder, Indigo, and Chlorophyll.

The Bakerian Lecture for 1899, "The Crystalline Structure of Metals," was delivered by Professor J. A. EWING, F.R.S., and Mr. W. ROSENHAIN, on May 18, 1899.

The Croonian Lecture for 1899, "On the Relation of Motion in Animals and Plants to the Electrical Phenomena which are associated with it," was delivered by Professor BURDON SANDERSON, F.R.S., on March 16, 1899.

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PHILOSOPHICAL TRANSACTIONS.

I. *On the Recovery of Iron from Overstrain.*

By JAMES MUIR, *B.Sc., Trinity College, Cambridge* (1851 *Exhibition Science Research Scholar, Glasgow University*).

Communicated by Professor EWING, *F.R.S.*

Received January 25,—Read February 9, 1899.

IT has long been known that iron which has been overstrained in tension—that is to say strained beyond the yield-point so that it suffers a permanent stretch—possesses very different elastic properties from the same iron in its primitive condition. The material is said to be "hardened" by stretching,* since the ultimate effect of such treatment is to raise the elastic limit and reduce the ductility of the material.

More recently, attention has been called to the fact that, primarily, the result of tensile overstrain is to make iron assume a semi-plastic state, so that the elastic limit, instead of being raised by stretching, is first of all lowered, it may be to zero.† This plasticity may be shown by applying a comparatively small load to a bar of iron or steel which has just been overstrained by the application and removal of a large stretching load. When the small load is put on, the bar will be found to elongate further than it would had the material been in its primitive state; and a slight continued elongation—a "creeping"—may occur after the small load has been applied. If this load be withdrawn, a quite appreciable permanent, or semi-permanent, set will be found to have been produced; a set which diminishes slightly, and, if small, may vanish, provided time be allowed for backward creeping to take effect. It may also be shown that, if the re-applied load be increased, the elongation produced will increase in a greater proportion. Thus, if a stress-strain curve be obtained from a recently overstrained bar of iron or steel, it will show, even for small loads, a marked falling away from the straight line which would indicate obedience to HOOKE's law.

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* EWING, "On Certain Effects of Stress," 'Proc. Roy. Soc.,' No. 205, 1880. The raising of the elastic limit due to stretching seems to have been first noted in 1865 by THALÉN. See a translation of his paper in the 'Phil. Mag.' for September of that year.

† BAUSCHINGER, "Ueber die Veränderung der Elasticitätsgrenze," 'Civilingenieur,' 1881, or "Mittheilungen aus dem mechanisch-technischen Laboratorium der K. Polytechnischen Schule in München." An account of BAUSCHINGER'S work is given in UNWIN'S book on "Testing of Materials of Construction."

EWING, "On Measurements of Small Strains in the Testing of Materials and Structures," 'Proc. Roy. Soc.,' vol. 58, April, 1895.

It is the recovery from this semi-plastic state induced by overstrain to a condition of perfect or nearly perfect elasticity with raised elastic limit, that is referred to in the title of this paper. Such recovery is known to be effected by mere lapse of time,* and the object of the experiments about to be described is to show the effect of moderate temperature, of mechanical vibration, and of magnetic agitation, on this slow return to the elastic state, and further to illustrate this recovery by means of compression tests. One section of the paper deals with the phenomenon of hysteresis in the relation of extension to stress, which is exhibited in a marked degree by iron in the overstrained state. Incidentally, attention will be called to subsidiary points of interest.

The experiments were carried out in the Engineering Laboratory of Cambridge University, and were the outcome of suggestions by Professor EWING. It was on his suggestion that the effect of moderate temperature on recovery from overstrain was tried, and the result of that trial led to much of the work incorporated in this paper.

Before going into details of the experiments it may be of interest to give, drawn to a small scale, an ordinary complete stress-strain diagram, such as is obtained in the testing of iron or steel. The period in the history of iron subjected to tensile stress which is about to be investigated, may thus be more clearly indicated. The curve given in Diagram No. I. was sketched by hand, roughly, from data obtained from the experiments which will be described later. It applies to steel not previously submitted to overstrain.

For the portion *ab* of this curve HOOKE's law is obeyed. At *b* the yield-point occurs, and as soon as this point is passed the material becomes overstrained. During the large yielding which takes place at the yield-point the load may be reduced without causing the extension to stop. After stretching by a large amount as compared with elastic extension, the material will be found to have hardened; so that to produce further yielding the load must be increased. The stress-strain diagram may now be represented by some such curve as *cd*. If at *d* the load be removed, and at once gradually replaced, then the stress-strain curve may follow a path such as *de*, *ef*. These curves *de* and *ef*, when obtained in such a manner that the extensions can be plotted to a much larger scale, show the imperfect elasticity of recently overstrained iron which has been referred to above; that is, they show that the material is semi-plastic. If time be allowed to elapse between unloading and reloading, the recovery from the effect of overstrain may be shown in a diagram like the present, by some such curve as *ef'*. When no interval of time is allowed to elapse between the removal and the replacement of the load, then the stress-strain curve is continued in the manner shown by *fg*, until a point *g* is reached, at which local extension sets in. When this happens the stress may be diminished, and fracture may take place at a load lower than that at which local extension occurred. The stress per square inch of fractured area is, however, found to be much greater than the stress per square inch of the actual area when local elongation began.

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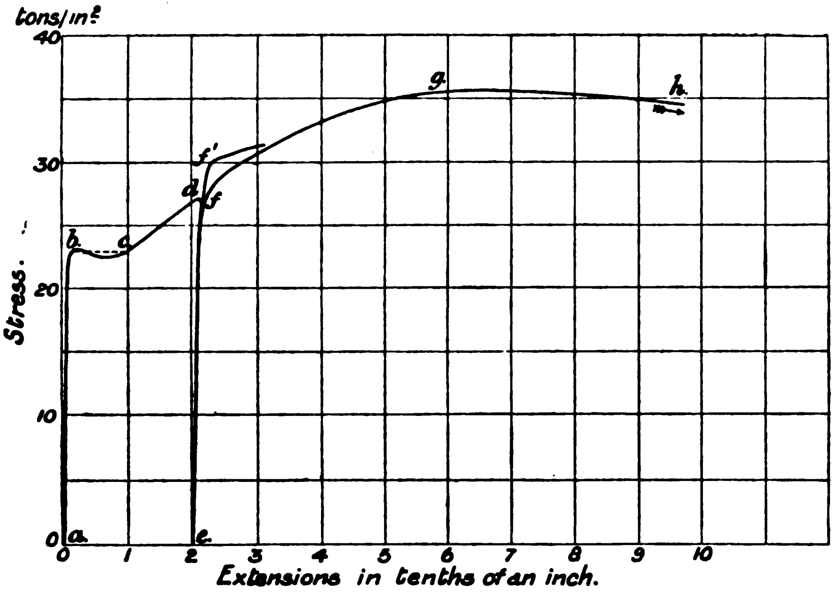
* BAUSCHINGER, 'DINGLER'S Journal,' vol. 224, p. 5, and papers already cited.

EWING, both papers cited above.

Reference might also be made to Lord KELVIN'S discovery of the effect of a Sunday's rest on wires which had been subjected to torsional vibrations throughout the preceding week.—See article, "Elasticity," 'Encycl. Brit.'

Diagram No. I.



The Apparatus and the Material.

The straining and testing in the following experiments was done by means of the 50-ton Wicksteed single-lever hydraulic testing machine of the Cambridge Engineering Laboratory. With this machine the magnitude of the load applied could be read in tons to a second decimal place by means of a vernier, and to a third decimal place roughly by estimation. Thus a load could be applied accurately if necessary to, it may be said, $\frac{1}{500}$th of a ton.

The small strains of extension were measured by Professor EWING's extensometer.*

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* For a full description of this instrument see the paper already cited "On the Measurement of Small Strains, &c.," 'Proc. Roy. Soc.,' vol. 58, April, 1895.

This instrument gave the extension occurring in the middle 8 inches of the length of the specimen under test. It enabled elongations to be measured to $\frac{1}{50000}$th of an inch, and to be measured to that degree of precision with ease and confidence. The instrument was found especially convenient on account of the facility with which it could (by the aid of a distance-piece) be immediately re-applied to a specimen which had just been strained beyond its yield-point; and also on account of the readiness with which the correct adjustment of the instrument itself could be tested.

The specimens employed were, with one or two exceptions, 18 to 20-inch lengths of steel-rod, 1 inch in diameter, of a quality which may be described as semi-mild. The details of the particular rods employed in the various experiments will be given when these come to be described. Here, as illustrating the general character of the material, the chemical analyses and elastic characteristics of two of the bars made use of will be given. The first is that from which diagram No. IX. has been obtained. A specimen from it showed a well-defined yield-point at a stress of 23 tons to the square inch, and gave an ultimate strength of 36½ tons per square inch of original area, with an elongation of 22½ per cent. on an 8-inch length. The second bar is that from which diagrams Nos. IV. and VII. have been obtained; it was characterised by a small flaw running up the centre through the whole length of the bar. A well-defined yield-point was not obtained with specimens from this bar; there was a distinct departure from obedience to **Hooke's** law, at a stress of about 22 tons per square inch, but the yield-point should, perhaps, be placed as much as 6 or 7 tons higher than this stress, at which elastic behaviour broke down.* The ultimate strength of the material as obtained from a short specimen was 39 tons per square inch, the elongation being only about 20½ per cent. on a 3-inch length. The chemical analyses of these two bars were kindly supplied by Messrs. **Edgar Allen** and Co., Sheffield, from whom the material was obtained; they are as follows:—

	Bar of diagram No. IX.	Bar of diagrams Nos. IV. and VII.
Carbon	0·430	0·450
Silicon	0·112	0·093
Sulphur	0·010	0·012
Phosphorus	0·016	0·021
Manganese	0·450	0·410
Iron (by difference)	98·982	99·014
	100·000	100·000

---

* See the first column of the table on p. 15.

The Method of Experimenting.

The general procedure adopted in experimenting will now be described. First, the diameter of the specimen was determined from the mean of ten micrometer readings, taken at five equidistant places along the 8-inch length to which the extensometer was to be applied. For example, the following readings were obtained for a certain unturned specimen:—

$$\text{Diameter} = \left\{\begin{array}{rrrrr} 1''\cdot0069 & 71 & 65 & 64 & 61 \\ 70 & 71 & 71 & 59 & 54 \end{array}\right\} = 1''\cdot0066.$$

Not only was this done for virgin specimens, but whenever a yield-point had been passed, the diameter was re-determined by means of fresh readings. For example, the specimen already instanced was subjected to a pull gradually increasing to 35 tons to the square inch of original section, the yield-point occurring at 23 tons per square inch. After the removal of this large stretching load the new diameter was determined from the following readings:—

$$\text{Diameter} = \left\{\begin{array}{rrrrr} 0''\cdot9918 & 21 & 18 & 20 & 25 \\ 14 & 22 & 24 & 15 & 24 \end{array}\right\} = 0\cdot9920 \text{ of an inch.}$$

After the determination of the diameter of a specimen at each stage of an experiment, a table was formulated, from which the total load applied could be translated into tons per square inch of section, the stress being in every case measured with reference to the section at the beginning of each separate test.

These preliminaries having been completed, the specimen was put into the testing machine, the extensometer was attached, and the load was gradually applied. Extensometer readings were taken sometimes only after the addition of every four tons of stress, sometimes after each ton, sometimes after each half or quarter ton, according to judgment.

The following two series of readings are given for a typical experiment; they will serve to explain the usual procedure. The first series shows the elastic properties of a certain virgin specimen; the second shows the plastic nature of the same specimen immediately after the overstrain produced by the primary loading.

Stress in tons per square inch.	Total load in tons.	Extensometer readings. Unit = $\frac{1}{50000}$ of an inch.	Differences.
0	0	0	
1	0·79	30	30
2	1·58	60	30
4	3·16	120	60
6	&c.	180	60
8	...	240	60
10	...	300	60
12	...	360	60
14	...	421	61
16	...	481	60
18	...	540	59
20	...	600	60
22	...	660	60
24	...	720	60
25	...	750	30
26	...	780	30
27	...	810	30
		still 810 after 4 minutes,	
		but 830 „ 6 „	
		&c. &c. (see Curve No. 1, Diagram II.)	
		1595 after 20 minutes	
27½	...	1700 and then quickly out of range	
30 put on gradually and kept on for 3 minutes, the beam of the testing machine remaining steady			

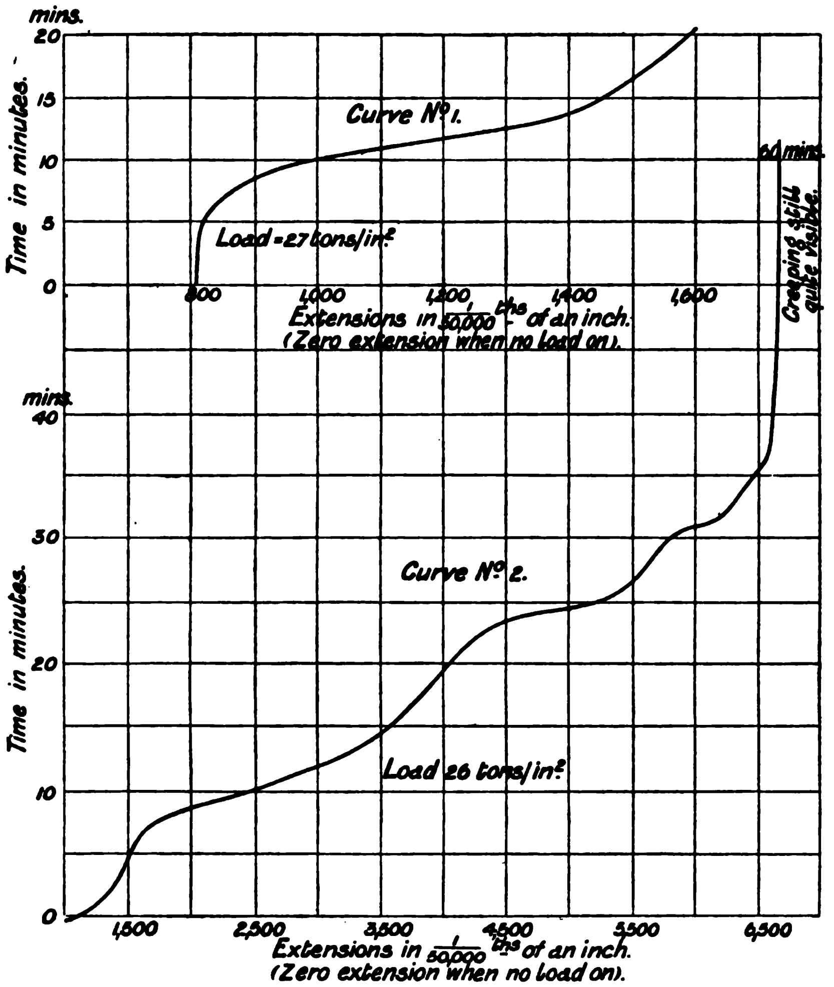
These figures show that this specimen has accurately obeyed **Hooke's** law, until a stress of 27 tons per square inch was attained. At this load the yield-point was expected to occur, and although the extensometer reading obtained gave no evidence of the proximity of such a point, by simply allowing the load to remain on for a short time (4 minutes) the creeping recorded above set in, or perhaps spread from without to within the 8-inch length under test.* The yield-point, with this specimen, has, therefore, coincided with the elastic limit as accurately as the extensometer can measure. Usually, in testing, imperfection of elasticity is shown before the yield-point is reached, and if a load less than that at the yield-point be allowed to act for some time, then a slight creeping probably supervenes. When, however, a bar, like that referred to above, has shown very perfect elasticity up to the yield-point, it is probable that a load very little under that at which the yield-point occurs could be sustained for an indefinite time without creeping taking place. Even although slight imperfection of elasticity be shown before the yield-point, experiments showed that no creeping need necessarily occur for a pause in the loading of at least a night's duration.

---

* The fact that yielding takes some time to start has already been recorded by Professor **Ewing** in his paper cited above, "On Measurements of Small Strains, &c." (see pp. 135 and 136). He has noticed that yielding may begin in a part of the bar lying outside the 8-inch length to which the extensometer is applied, and may gradually spread along the bar.

The manner in which yielding under a constant load proceeds after the yield-point has just been passed is often very irregular. The curves given on Diagram II. illustrate this yielding with time for two entirely different specimens. The first shows the creeping referred to above as having started 4 minutes after the application of the load which was its cause. The second curve shows a larger yielding of much longer duration. It occurred under a load of 26 tons per square inch, but before this load was attained—when 25 and 25½ tons per square inch were acting—considerable creeping had already taken place. The readings from which these curves have been obtained were taken at intervals of one minute.

Diagram No. II.—(Manner in which yielding occurs at the yield-point.)



To return to the table of figures given above, the maximum load (of 30 tons to the square inch of original section) was found, after its removal, to have produced a permanent set of 0·22 of an inch on the 8-inch length; this corresponds to an extensometer reading of 11,000; such a reading is, of course, far beyond the range of the extensometer. Immediately after the maximum load had been removed, the diameter of the specimen was re-measured and the reduced section determined. The extensometer was then re-applied, the specimen was re-loaded, and the following readings observed :—

Tons per sq. in. (of reduced section).	Extensometer readings.	Differences.
0	0	
1 (= 0·77 ton of total load)	30	30
2	61	31
4	125	64
6	190	65
8	260	70
10	329	69
12	399	70
14	469	70
16	539	70
18	613	74
20	687	74
22	764	77
24	845	81
26	930	85
28	1028	98
30	1150	122

The load was now removed, and during its removal the following three readings were taken :—

Tons/inch².	Extensometer.
20	830
10	477
0	60 but diminishing slightly with lapse of time

The series of increasing differences shown in this second table plainly indicates a change in the elastic state of the material. HOOKE's law is no longer obeyed.

This augmentation of the differences is, to some extent, associated with creeping, and to a greater extent, the higher is the applied load. Thus it is essential, if consistent results are to be obtained, that the interval of time which elapses between successive readings should always be kept the same. If a pause had been allowed to occur after the addition of any of the higher loads in this second table, then, owing to prolonged creeping, a larger difference would have been obtained than is recorded above. On proceeding with the loading, however, the immediately succeeding difference, or differences, would have been smaller than according to the table. For had there been no interruption, part of the creeping which occurred during the pause would have been recorded on the addition of the subsequent loads.

In experiments on a virgin piece of which the first table given above is typical, the time element does not enter, for there is no perceptible creeping until the yield-point is all but reached.

Slow Recovery of Elasticity with lapse of Time.

Before proceeding to describe the effect of special treatments on recovery from tensile overstrain, I give two instances of the slow recovery from overstrain with lapse of time, similar to the examples already given by Professor EWING.*

The curves in Diagram No. IIIA. illustrate this slow recovery for a specimen of 1 inch round steel rod, which has been strained to or very little beyond its yield-point. The material had an ultimate strength of 37 tons per square inch of original area, the total elongation being almost 23 per cent. on an 8-inch length. The yield-point was well defined and occurred at a stress slightly under 27 tons to the square inch. The readings from which the various curves have been plotted are given in the table on p. 11. The curves were obtained in the usual manner, the stresses being plotted as ordinates, and the corresponding extensions as abscissæ.

Curve No. 1 of Diagram IIIA. is a record of the primary test of the specimen. It shows that HOOKE'S law has been obeyed up to a load of 26 tons to the square inch; and that before 27 tons there was a well-marked yield-point. This load of 27 tons per square inch was kept on for two minutes, by which time rapid stretching had ceased, as was shown by the beam of the testing machine remaining stationary. There was still a slow creeping, which probably would have continued for hours or days, becoming however slower and slower. Curve No. 2 represents a test performed as shortly after the removal of this load as the remeasurement of the diameter and the calculation of the reduced area would permit: it illustrates the semi-plastic condition of the material immediately after overstrain. In the plotting of this, and all subsequent curves in the diagram, it will be noticed that the origin for the measurement of extensions has been displaced; this was merely to keep the curves distinct and to facilitate comparison.

Curves Nos. 3, 4, 5, and 6, obtained at succeeding intervals, illustrate the gradual recovery of the elasticity lost by the overstrain. This recovery will be noticed to be quickest at first, and latterly to be very slow. In Curves 3, 4, and 5 the load was not allowed to exceed 27 tons per square inch. Curve No. 6 shows the recovery to have been nearly, though not quite, perfect after the material had been allowed to rest for 6 days 3 hours. In this test the load was gradually increased beyond the 27 tons, and a new yield-point was not obtained till rather over 30 tons to the square inch was reached.

Curve No. 7 shows the plastic nature of the material immediately after this second overstrain, and No. 8 the condition after 4 days' rest. Thus after 4 days the recovery

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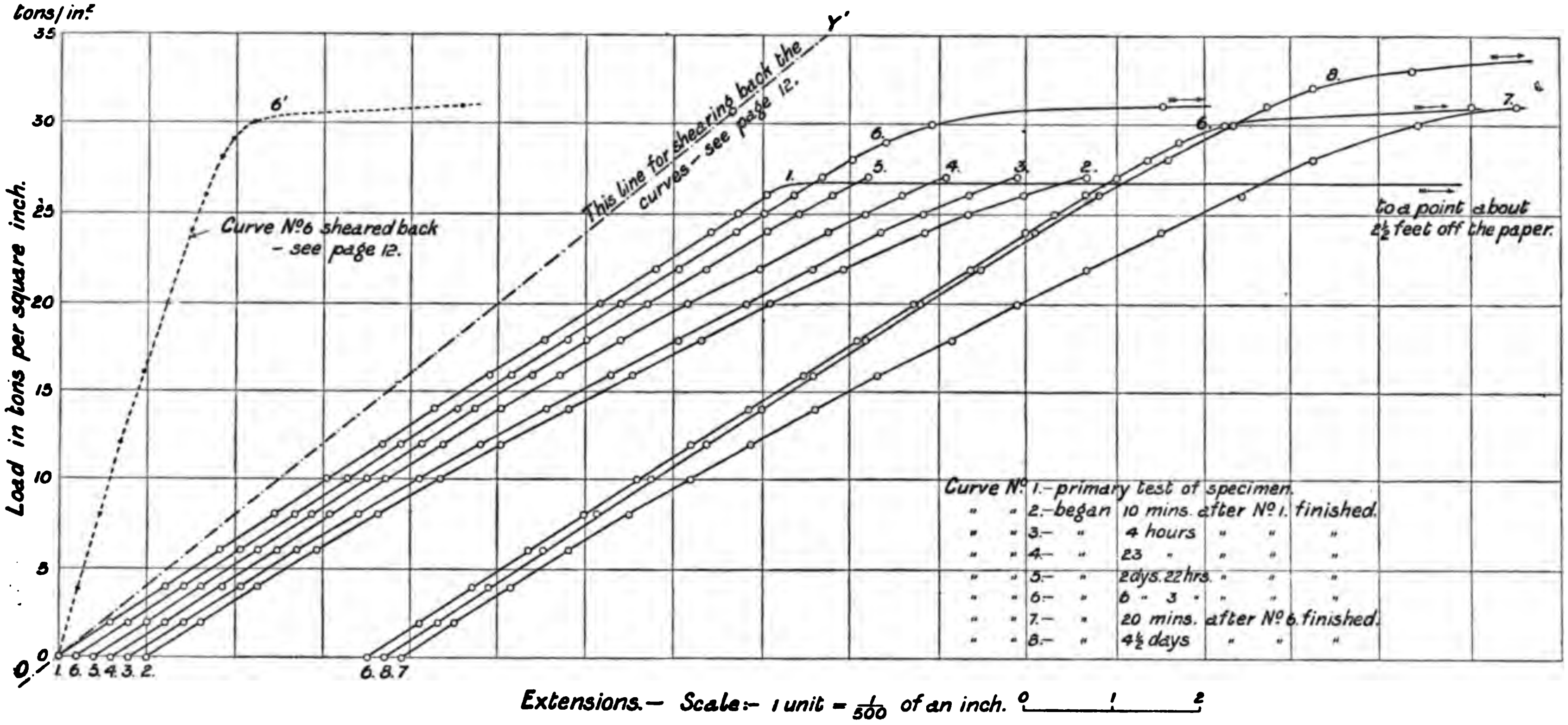
* See page 139, &c., of his paper "On Measurements of Small Strains, &c."

Diagram No. IIIA.—(Slow recovery with lapse of time.)



was by no means perfect, and on increasing the load beyond the 30 tons an almost immediate falling away was observed, and (as is shown in the curve) a third, rather indefinite, yield-point was indicated. Further yield-points might have been obtained had this treatment been proceeded with.

TABLE giving readings for Diagram No. III. (Slow recovery with time).

Load in tons/in^2.	Extensometer readings for various curves of Diagram III.							
	No. 1. Zero time.	No. 2. 10 mins.	No. 3. 4 hours.	No. 4. 23 hours.	No. 5. 2 days 22 hours.	No. 6. 6 days 3 hours.	No. 7. 20 mins. (No. 6, zero.)	No. 8. 4 days (No. 6, zero.)
0	0	0	0	0	0	0	0	0
2	60	62	62	61	61	60	61	60
4	121	125	128	125	122	122	127	120
6	182	192	191	188	186	186	191	181
8	243	262	260	249	248	249	260	243
10	304	333	330	311	310	309	330	305
12	368	405	401	378	372	370	397	368
14	428	481	477	444	434	433	471	429
16	491	553	550	511	500	498	542	489
18	553	631	625	581	562	560	626	549
20	618	709	701	658	630	621	700	610
22	680	793	779	738	698	687	778	678
24	742	883	855	817	765	750	860	740
25	772	933	902	859	801	781		
26	803	999	953	900	841	816	942	810
27	Out of range	1068 to 1117 in 3 mins.	1010 to 1035 in 3 mins.	949 to 978 in 3 mins.	881 to 897 in 3 mins.	850		
28	...	...	...	...	...	884	1035	889
29	...	...	...	...	...	921		
30	...	...	...	...	...	972	1152	964
31	...	...	...	...	...	1251 and then very large yielding	1262 to 1382 in 2 mins.	1004
32	...	...	...	...	...	...	...	1055
33	...	...	...	...	...	...	...	1166
34	...	...	...	...	...	...	...	Very large yielding
Load removed. 0	...	128 to 113 in 4 mins.	51 to 39 in 4 mins.	36 to 29 in 15 mins.	23 to 17	...	241 to 228	

The "Shearing Back" of the Curves.

It will have been noticed that, owing to the extensions being plotted to an unusually large scale, the curves occupy an inconvenient amount of space. This may be avoided by adopting a geometric artifice, suggested by Professor EWING, of "shearing back" the curves; that is, retaining the same scale of measurement, an amount is deducted from each extension proportional to the load producing it. For example, if extensions of 120, 240, 360 were produced by loads of 4, 8, and 12 tons per square inch respectively, the extensions might be diminished by, say, 100 units per 4 tons of load, and plotted as 20, 40, and 60. Another way of expressing this is to say that in the diagram the axis from which extensions are measured may simply be considered as tilted back in the manner shown below by fig. 2. Fig. 1 shows a curve drawn with ordinary rectangular axes, and fig. 2 the same curve "sheared back."

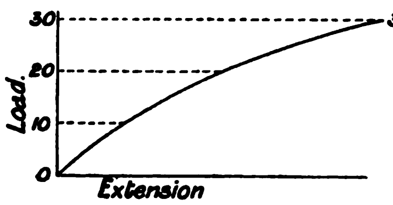


Fig. 1.

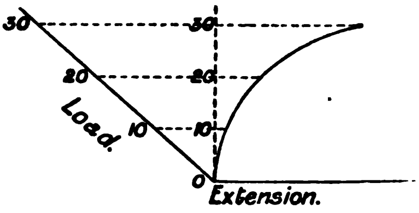


Fig. 2

Referring again to Diagram No. IIIA., the process of "shearing back" is performed graphically on the single Curve No. 6 by adopting an oblique base line, OY′, the distance from which of each point of the curve is measured horizontally and re-plotted from the vertical base line, Curve No. 6′ being obtained. Besides the convenience of space gained, this foreshortening of the curves by diminishing the rates of extension renders a more obvious comparison of similar, but slightly different, curves, and emphasises any irregularities there may be in the extensometer readings. It should thus be remembered that in a stress-strain curve which has been "sheared back" inequalities in extension are exaggerated, since the relations they bear to the total extensions are not shown.

In Diagram No. IIIB. all the curves of Diagram No. IIIA.* are shown sheared back by the method just explained, and in all similar diagrams to be shown in this paper the curves will be subjected to the same treatment. In all cases the amount of shearing of the curves is the same—the extensions are always diminished by $\frac{100}{50000}$ths of an inch for every 4 tons of stress. The scale for the measurement of extensions will be kept the same for all analogous diagrams, but that for the measurement of load will be varied, in order to get the different diagrams suitably spaced.

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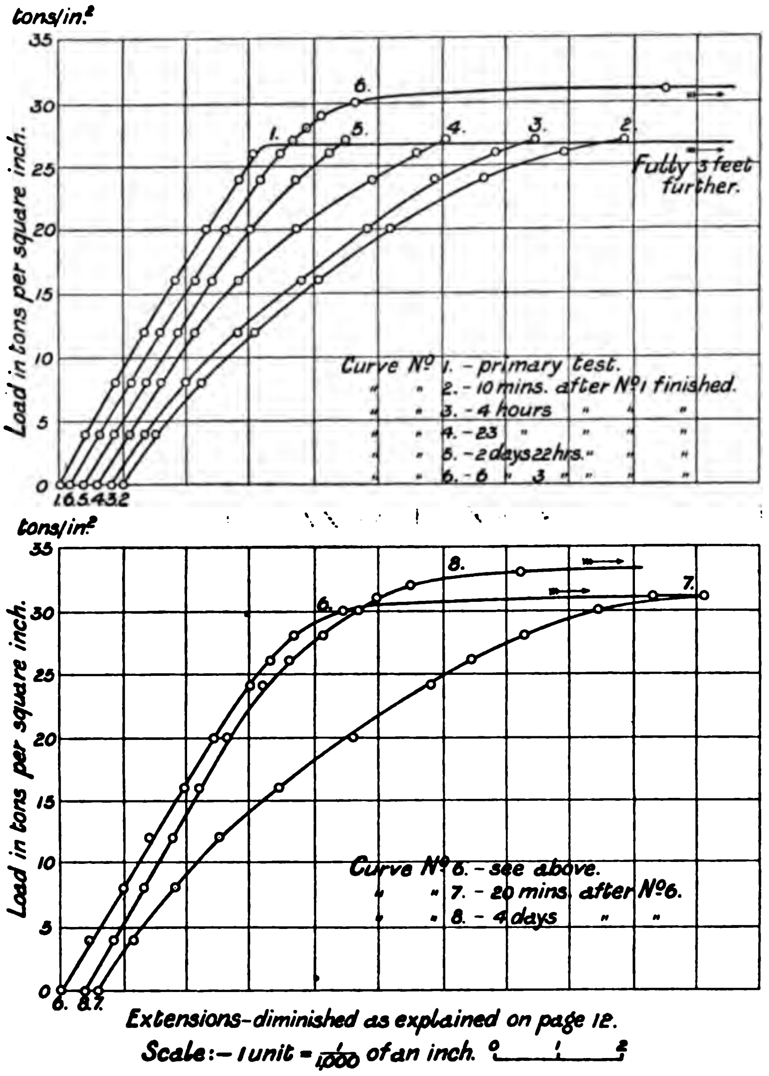
* Since Diagram IIIA. has been reproduced one-half full size, while Diagram IIIB. (and all other diagrams in the paper) have been reproduced to a two-thirds scale, the two Diagrams IIIA. and IIIB. are not absolutely comparable. Besides this difference due to reproduction, the scale for the measurement of load in IIIB. was originally only half that in IIIA.

Diagram No. IIIB.—(Slow recovery with time.)

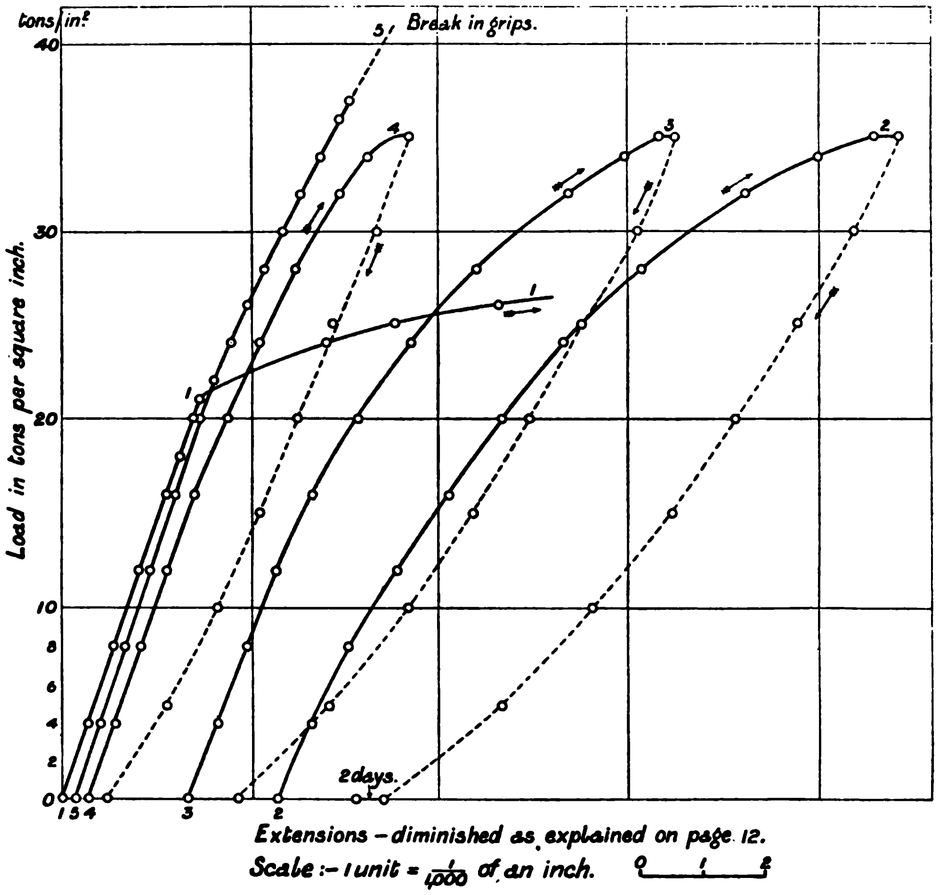


Second Example of Slow Recovery with Lapse of Time.

In Diagram No. IV. there is shown the slow recovery from overstrain of a slightly different quality of steel under somewhat different conditions from those of the last example, while in the accompanying table the figures are given from which the various curves of that diagram have been plotted. The material is that described second on p. 4.

An examination of the differences given in the first column of the table on p. 15 will show that, although during the primary loading of the specimen, considerable yielding set in at a stress of 22 tons per square inch, it was not until almost 29 tons that such yielding as is usually associated with a yield-point occurred. Before this stress of 29 tons per square inch was reached, the stretching was not sufficient to cause the skin of oxide to spring off in the manner characteristic of a yield-point, and the removal and gradual re-application of a load of 28 tons was found to show comparatively little semi-plasticity of the material. Thus the specimen cannot be said to have been thoroughly overstrained until after 29 tons per square inch had been applied.* The primary loading was continued beyond this amount until a stress of 35 tons per square inch of original area was attained.

Diagram No. IV.—(Slow recovery with time).



Curve No. 1—primary test of specimen.
" " 2—after load removed from No. 1.
" " 3—about 2 days after No. 1.
" " 4— " 7 " 1.
" " 5— " 17 " 1.

Curve No. 1, Diagram IV, illustrates, so far, the primary loading of this specimen.

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* A cause for the gradual manner in which this specimen yielded was perhaps, directly or indirectly, indicated by a small flaw which ran up the centre of the bar (see p. 4).

Curve No. 2 illustrates the semi-plastic condition of the material immediately after the removal of the overstraining load; while Curves Nos. 3, 4, and 5 show the progress made towards recovery, 2, 7, and 17 days respectively, after the material had been overstrained. The manner in which contraction takes place during the removal of the load is shown by dotted lines in Curves Nos. 2, 3, and 4, and it will be noticed that comparatively great retraction takes places as the lowest loads are removed. The test illustrated by Curve No. 5 shows that after 17 days' rest recovery was practically

TABLE of Readings for Diagram IV. (Slow recovery with time.)

Load in tons/in^2.	Extensometer readings for the various curves.					
	No. 1. Zero time.	Differences.	No. 2. 15 mins.	No. 3. 2 days.	No. 4. 7 days.	No. 5. 17 days.
0	0		0	0	0	0
2	60		62	61	61	60
4	121	61	129	122	122	120
6	181	60	190	188	182	179
8	241	60	259	247	242	239
10	301	60	329	309	302	299
12	361	60	398	370	363	359
14	422	61	468	433	424	419
16	482	60	539	499	485	479
18	543	61	609	568	545	539
20	604	61	681	637	612	600
21	634	30	719	669	643	630
22	684	50	755	706	674	661
23	750	66	791	740	705	692
24	810	60	830	779	737	725
25	889	79	869	815	770	757
26	998	109	909	851	802	788
27	1140	142	949	890	834	819
28	1440	300	991	930	865	850
29	—— Out of		1035	970	903	882
30	range, say 6500		1079	1011	934	915
31	...		1124	1058	967	948
32	...		1171	1104	1002	979
33	Load slowly increased to 35 tons, and then removed.		1225	1149	1036	1012
34			1279	1199	1074	1048
35 1 minute			1349 1368	1250 1262	1133	1079
						tons/in^2.
						35½ 1096
30	...		1207	1108	980	36 1111
25	...		1037	940	820	36½ 1129
20	...		863	772	666	37 1146
15	...		689	601	510	Extensometer removed.
10	...		501	425	352	
5	...		305	237	187	Break in grips at 41 tons/in^2.
0	0·31 of an inch.		85	39	15	
Time	...		6 mins. 78 2 days 63	6 mins. 30		

perfect, the material approximately obeying HOOKE'S law up to the maximum stress of 35 tons per square inch. The load in this test was therefore increased beyond its previous maximum amount. The extensometer, however, was shortly removed for fear of a sudden break, so that the top part of Curve No. 5 (shown dotted in the diagram) was not obtained from extensometer readings. At a stress of about 41 tons per square inch the specimen suddenly broke, unfortunately in the machine grips; before this stress was reached no yield-point had been passed, or it would have been detected by a rapid falling of the horizontal beam of the testing machine.

Recovery under Stress, and Effects of Hysteresis.

Experiments were carried out to test the effect of keeping an overstrained specimen loaded, instead of allowing it to rest in an unstressed condition, and it was found that the material, whether kept stressed or unstressed, recovered at practically the same rate.

In the following table extensometer readings are given, which show the gradual recovery from overstrain of two specimens, A and B. A was kept loaded to the maximum stress employed to produce overstrain, while B was allowed to rest free from load.

TABLE comparing Recovery under Stress and Recovery when no Load was Acting.

Load in lbs./in^2.	Extensometer readings.					
	Immediately after overstrain.		10 days after.		40 days after.	
	A.	B.	A.	B.	A.	B.
500	0	0	0	0	0	0
10,000	137	135	121	129	129	129
20,000	286	287	269	270	265	269
30,000	446	448	426	427	410	410
40,000	610	612	587	589	560	561
50,000	795	805	759	760	717	719
55,000	901	925	842	859	795	799

This experiment was carried out during a vacation in the Engineering Laboratory of Glasgow University (Professor BARR having kindly granted the use of apparatus and laboratory), so that the conditions of experiment are somewhat different from those explained at the beginning of this paper. A 10-ton single-lever testing machine was employed, and the load applied by thousands of pounds, instead of by tons. Professor EWING'S extensometer was still used. The material tested was a half-inch rod of fairly mild steel. It gave an ultimate strength of about 32 tons per square inch, with an elongation of 26½ per cent. on an 8-inch length. The primary yield-point occurred at a stress of 48,000 lbs., or about 21½ tons per square inch. It will be noticed from the table of figures that this material recovered very slowly, for even after forty days' rest recovery was by no means complete. Specimen A of the above table was further tested after about three and a-half months, and considerable imperfection of elasticity still found.

Although the table given above shows close agreement between the elastic states of the material in the two cases, there was really an interesting difference between them. This is clearly shown by Diagram No. V. Curves A and B in that diagram illustrate

Diagram No. V.—(Recovery under stress.)

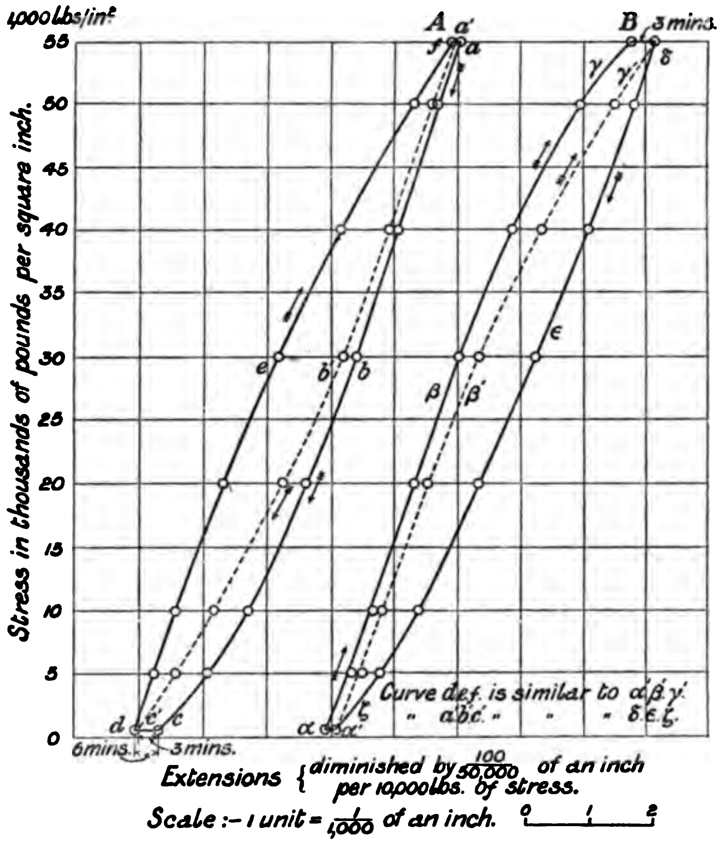


Fig. A represents the elastic condition of an overstrained specimen, which had been resting for forty days at a stress of 55,000 lbs./in^2.

Fig. B represents the elastic condition of an overstrained specimen, which had been resting for forty days free from stress.

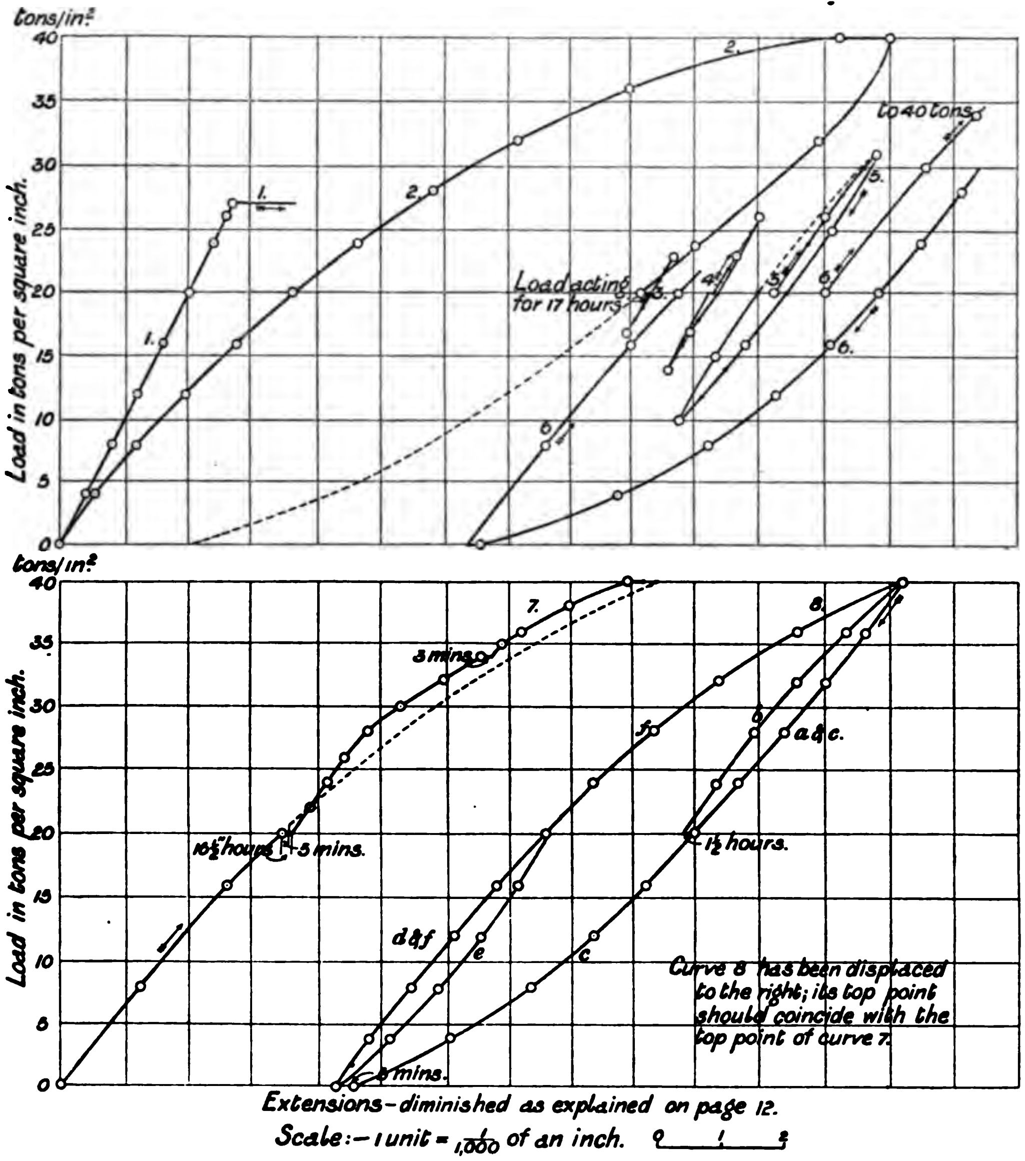
the testing of the two specimens after the forty days' rest referred to above; the last thirty of these days were uninterrupted by any intermediate testing, and during that time A was in the testing machine, with a load of 55,000 lbs. per square inch applied to it. The extensometer having been applied to A, the load was gradually removed, and readings taken, from which the part of Curve A lettered *abc* was plotted. This shows that, while a considerable proportion of the load was being removed, contraction occurred quite elastically, a straight line being first obtained at an inclination giving the Young's modulus for the material. Latterly, however, as more load was withdrawn, the retraction became more rapid, and after all load had been removed, creeping was observed to continue in a marked fashion for a few minutes, as is shown by the horizontal line *cd*. The load was now increased, and the curve *def* obtained. At the maximum stress slight creeping occurred, and then the load was once more removed, and curve $a'b'c'$ was plotted from the extensometer readings taken. This curve differs distinctly from *abc* (obtained on first removing the load), the material behaving less elastically during the early part of this second removal of the load. The curve $defa'b'c'$, however, represents an approximately cyclic state, which illustrates the imperfectly elastic condition of the material of specimen A at the time in question. When such a cycle—due to hysteresis in the relation of extension to load—is performed, work is done on the specimen, and the energy so spent is no doubt dissipated as heat.

Specimen B, which had been resting for forty days free from load, was next put into the testing machine, and the load was gradually applied. The result of the first loading is shown by the part of fig. B, Diagram V., lettered $\alpha\beta\gamma$. This curve is straight for a considerable portion ($\alpha\beta$)—the material at first approximately obeying the elastic law—but latterly greater extension occurred, and at the highest load creeping continued very obviously for a short time. The load was then removed, and curve $\delta\epsilon\zeta$ obtained. This curve resembles closely the part $a'b'c'$ of Curve A, not the part *abc*. Specimen B was next reloaded, and curve $\alpha'\beta'\gamma'$ illustrates the manner of yielding. This curve differs from $\alpha\beta\gamma$ just as $a'b'c'$ differed from *abc*. A cyclic state has now been attained, and the cycle $\alpha'\beta'\gamma'\delta\epsilon\zeta$ closely resembles that got with specimen A, which had been allowed to rest under high stress. The readings for curves *def* and $\alpha'\beta'\gamma'$ of these cycles were compared in the table given above.

It will have been noticed that it is not only the cycles ultimately obtained which are analogous in the two figures of Diagram V., but that, if one of the figures, say A, be turned upside down, then the three curves of that figure closely resemble the three curves of the other figure, B. Considering in particular the curves first obtained in the two cases, viz., *abc* and $\alpha\beta\gamma$, it will be seen that they consist of two parts. There is first a range of almost perfect elasticity, then an elastic limit is passed, and greater extension or retraction obtained, according to the curve in question. The breaking up in the structure of the material, which occurs after this elastic limit has been passed (by a decreasing or an increasing load, as the case may be), is probably analogous to the much greater breaking up which occurs on the passage of a yield-point.

In Diagram No. VI., there is illustrated the effect, on an overstrained specimen, of keeping an intermediate load acting for some time; and it will be seen that the process of recovery tends to produce an elastic range about the position of continued stress. The experiments illustrated by this diagram were carried out in the Cambridge Engineering Laboratory, hence the loading is performed in tons (not in pounds) per square inch. The material used is very similar to that employed to obtain Diagram No. III., so that the rate of recovery from overstrain is much quicker than with the material of Diagram V.

Diagram No. VI.—(Hysteresis).



Curves 4, 5 and 6 are displaced to the right—they should be continuous with one another and with Curves 2 and 3.

Curve No. 1 of Diagram VI. shows the primary elastic properties of the steel rod considered. After this first test, the specimen was largely overstrained and allowed to recover. On testing, it was then found to give a yield-point at about 40 tons per square inch, and immediately after this yield-point had been passed, Curve No. 2 was obtained.

It will be noticed from the diagram that in Curve No. 2, the removal of the load was stopped midway and the stress of 20 tons per square inch allowed to act over a night. Had the load been entirely removed, then Curve No. 2 would have continued in some such fashion as is illustrated by the dotted line in the diagram.

The continued action of the load of 20 tons was found in the morning to have produced the slight extension shown. On then testing the specimen by first increasing and then decreasing the load for a short range on either side, Curve No. 3 was first obtained, and then Curves Nos. 4, 5 and 6.

Curve No. 3 shows a short range of nearly, though not quite, perfect elasticity. A slight discrepancy of $\frac{1}{50000}$th of an inch was obtained on each side of the starting position.

Curves Nos. 4, 5 and 6 show how elastic behaviour is departed from, and greater and greater indications of hysteresis obtained as the range of loading is increased.

Curve No. 7, drawn on the bottom half of Diagram VI., shows the effect of applying a load of 20 tons to this same specimen, and allowing it to act for over sixteen hours before proceeding with the loading. That is, the effect produced by a prolonged pause in the loading of an overstrained specimen is shown. After the pause, the curve starts off at a much steeper gradient; but shortly it falls back again to a rather less inclination than if there had been no interruption in the loading. The dotted line in Curve 7 shows the manner in which the curve would have continued had no pause occurred in the loading. This continuation was accurately known; for previously the specimen had been loaded to 40 tons, three times in succession, and the last two applications had given very accurately the same curve—no two readings differing by more than $\frac{2}{50000}$ths of an inch. At the stress of 34 tons, in Curve No. 7, the effect of a three minutes' pause is shown to be similar to, but of course much smaller than, the effect of the long pause at 20 tons. This slight effect at the higher load may, however, be explained by simply considering that if creeping be allowed to occur at any load, then a small increase of load cannot be expected to produce so great an extension as it otherwise would.

After Curve No. 7 was obtained, the complete cycle represented by Curve No. 8 was gone through. The part of this cycle lettered *a* represents the partial removal of the load from Curve No. 7. The stress was only reduced to 20 tons, and then one and a-half hours were allowed to elapse. Slight back-creeping occurred instead of forward-creeping, such as happened in Curve No. 2. The load was then increased and Curve *b* obtained. This curve arrived very accurately at the same top point as Curve *a*. The load was then entirely removed, and Curve *c*, which coincides so far with Curve *a*, was obtained. After a pause of five minutes under no load, during which back-creeping showed itself, the load was re-applied to 20 tons (Curve *d*), decreased to zero (Curve *e*), and then increased to 40 tons (Curve *f*), which completed the large cycle of loading. The hysteresis exhibited by a cycle such as this may be represented numerically by expressing the breadth of the cycle at any stress, as a fraction of the total elongation of the specimen. If this be done, the hysteresis, in the relation of strain to stress, which recently overstrained iron has just been shown to exhibit, may be compared with that observed with ordinary material by Professor EWING in experiments on very long wires.* In Professor EWING'S experiments, the wires were subjected many times to a certain range of stress, and the extension at half the range was observed both as the load was applied and as it was removed. The latter extension was found, due to hysteresis, to be greater than the former; and the difference being expressed as a fraction of the extension produced by the maximum load, values were obtained ranging from $\frac{1}{86}$ in the case of high carbon steel, to $\frac{1}{300}$ in the case of an iron wire in the hard-drawn state, or $\frac{1}{333}$ with mild steel wire annealed and then hardened by stretching. The hysteresis shown at half the range of stress in the cycle described above (Curve 8, Diagram VI.) is about $\frac{1}{12}$ of the extension produced by the maximum load of 40 tons per square inch.

In order to see how far the hysteresis in the relation of strain to stress exhibited by recently overstrained iron is statical in character, or how far it depends on the rate of loading, a cycle was performed allowing ten minutes to elapse after the addition of every 4 tons of stress. The only effect was to produce a series of little notches in the curve obtained, similar to the notch shown at the stress of 34 tons in Curve 7, Diagram VI. The area of the cycle was thus not appreciably affected. The time allowed after the addition of every 4 tons was ample so far as the amount of creeping was concerned, as is clearly shown by the creeping at the stress of 20 tons in Curve 7, Diagram VI. If a much longer time had been allowed to elapse, then recovery of elasticity would have taken place as in Curves 3 and 7, Diagram VI., and the question of the static character of the hysteresis would have become complicated.

The back creeping which occurs after the removal of the load from a specimen which has been several times overstrained (for example, the creeping shown to have occurred during 5 minutes in Curve 8, Diagram VI.) is not simply due to the immediately preceding loading, but to all previous loadings. It was often observed that if sufficient time were allowed to elapse after the removal of a load, the zero reading would become negative. That is, the bar would become shorter than it was before the loading was commenced—an effect which is no doubt to be ascribed to previous overstrains, and is analogous to phenomena which have been observed in the residual charge of dielectrics, and in the torsional strains of glass and other materials.

Before leaving this section of the paper, attention should be called to the close analogy between the hysteresis effects shown in Diagram VI., and the known characteristics of magnetic hysteresis in iron.†

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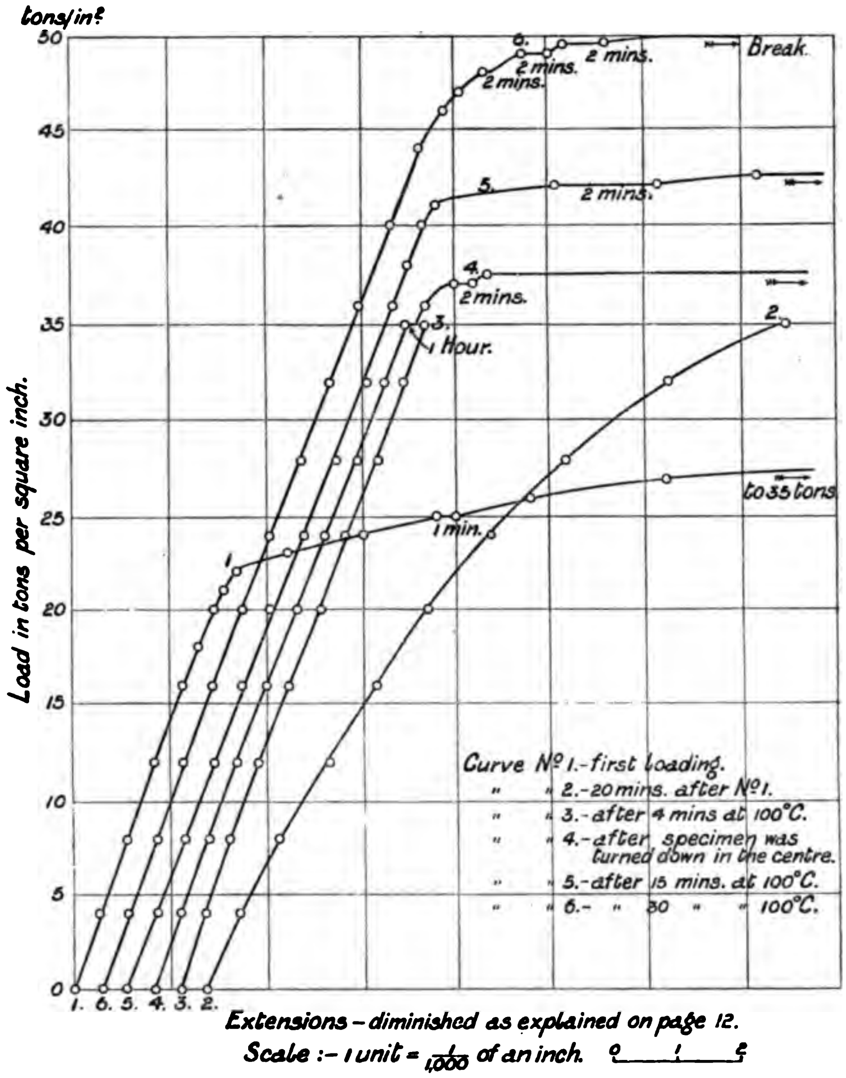
* "On Hysteresis in the Relation of Strain to Stress," 'B.A. Report,' 1889, p. 502.

† EWING. "Experimental Researches in Magnetism," 'Phil. Trans.,' 1885, or book on "Magnetic Induction in Iron and other Metals."

Effect of Moderate Temperature on Recovery from Overstrain.

The slow recovery of iron from tensile overstrain which has been illustrated by the examples already given, was found to be hastened to a remarkable extent by a slight increase of temperature. Three or 4 minutes at 100° C. were found sufficient to bring about a complete restoration of elasticity; and this, at the normal temperature, could not be effected in less than a fortnight, with the material considered.

Diagram No. VII.—(Recovery at 100° C.)



Before describing experiments which show this effect, it may be stated that experiments were made (though perhaps they might be considered unnecessary) to show that the temperatures to be dealt with could in no way alter the elastic properties of the material in its primitive condition. A virgin specimen was kept immersed in boiling water for many hours, and another was kept in a sand bath at about 250° C. for half an hour or so ; in neither case was there found, on cooling, any change in the elastic condition of the material.

In Diagram No. VII. there is shown the history of a specimen which was dipped in boiling water, whenever an overstraining load had been applied and removed. By this means recovery from overstrain instead of taking days, as in Diagram No. III., was effected in a few minutes. The material and the primary overstrain given to it are exactly as in the second example of slow recovery with lapse of time given above ; so that Curves Nos. 1 and 2 of this diagram (No. VII.) should be practically the same as Nos. 1 and 2 of Diagram IV. In order to show that the two pairs of curves are really to a close approximation identical, some of the extensometer readings taken to obtain these curves are compared in the following table. Curve No. 2 in both cases represents the first loading performed after the overstrain which is illustrated by Curve No. 1.

Load in tons/in^2.	Curves No. 1.		Load in tons/in^2.	Curves No. 2.	
	Diagram IV.	Diagram VII.		Diagram IV.	Diagram VII.
0	0	0	0	0	0
8	241	240	8	259	258
16	482	483	16	539	536
20*	604	606	24	830	829
24	810	825	32	1171	1167
28	1440	1430	35	1349	1335
29	out of range	(say 6500)			
35					

The readings for the other curves of Diagram VII. need not be tabulated.

Immediately after the readings for Curve No. 2, Diagram VII., were obtained, the specimen was taken out of the testing machine and placed in a bath of boiling water for 4 minutes. It was then removed, cooled in cold water and re-tested by gradually applying a load of 35 tons per square inch. Curve No. 3 was plotted from the observations taken. This curve is very similar to Curve No. 5 of Diagram IV., and is distinctly straighter than Curve No. 4 of that diagram ; so that 4 minutes at 100° C. has sufficed to produce more perfect recovery than would have resulted from, say, a fortnight's rest at the ordinary laboratory temperature.

After having been thus recovered and tested, the specimen was turned down in the centre to avoid breaking in the machine grips. About 4 inches at each end were left at the full diameter of 1 inch, a central portion, fully 9 inches long, being turned down to about 0·8 of an inch, and gradually tapered out at each end to the full diameter. On now testing the specimen, no change was found in the behaviour of the material; Curve No. 4, which shows this test, agreeing very accurately with Curve No. 3 up to the stress of 35 tons per square inch. This maximum load of 35 tons to the square inch (now 17·50 tons total instead of 26·91 tons as before turning down) was kept on for 1 hour, with the result that slight creeping took place, as is shown in Curve No. 4. On augmenting the load a distinct yield-point was got at 37½ tons per square inch. This second yield-point happened at a lower stress than would naturally have been expected, for in Diagram No. IV. the same material is shown to have been subjected to a stress of over 40 tons per square inch without a second yield-point having been passed. The lowness of the yield-point in the present case, was probably due to an inherent weakness in the specimen, which was shown by the small flaw which ran up the centre of the bar.* Owing to the specimen having been turned down this weakness would exert a greater influence than when the bar was of the full diameter of 1 inch. Experiments on another specimen of the same material, in fact, directly showed that after turning down a yield-point was obtained, at a stress lower than that to which the specimen had already been subjected without other than elastic yielding resulting. Such behaviour was anomalous. With other material which exhibited no flaw, turning down was found to have no effect on the position at which subsequent yielding took place.

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* Here elastic behaviour may be said to end.

The second yield-point shown in Diagram VII. having been passed, the material was once more in the semi-plastic state, so to effect recovery it was placed in boiling water again for 10 or 15 minutes. On cooling and re-testing, a third yield-point was obtained at 41½ tons per square inch, as is shown by Curve No. 5, Diagram VII. The specimen was once more put in boiling water and then tested, with the result that fracture occurred at a stress of 50 tons per square inch. The break was outside the central 8-inch length, close to the tapering neck joining turned and unturned portions.

A short virgin specimen of the same rod as the above was tested (after being turned down in the centre to a diameter rather smaller than in the last case), in order to find the ordinary ultimate strength of the material. The result of this test has already been given on p. 4; local extension set in at a stress of 39 tons per square inch of original area, or about 45 tons per square inch of actual stress, and fracture was allowed to occur at that load.

The effect which a temperature of 100° C. had in hastening the recovery process was further strikingly shown by an experiment on one of the specimens employed to obtain Diagram V. This specimen had been allowed to rest for three and a-half months, and was even then found to exhibit considerable imperfection of elasticity. By heating to 100° C. for a few minutes, a marked improvement was made in the elastic behaviour of this specimen.

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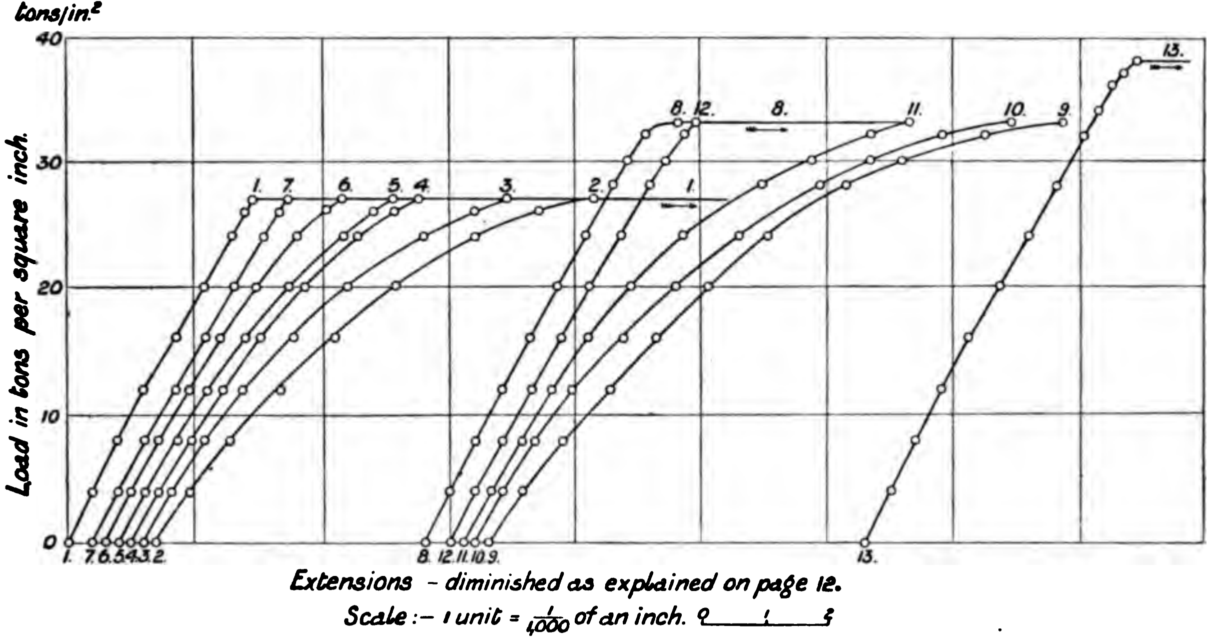
* See pp. 4 and 14.

Effect on Recovery of Temperatures below 100° C.

Diagram No. VIII., which is in two parts, gives the complete history of a specimen which was allowed to recover its elasticity at various temperatures, after having been overstrained. It shows, among other things, the very considerable hastening produced in the process of recovery from overstrain by even such a moderate temperature as 50° C. (120° Fahr.). A lengthy description of this diagram need not be given, as the side-notes accompanying the diagram give all necessary details. The tables which follow give most of the readings from which the curves of this diagram have been plotted. The material employed differed slightly from that considered last, particularly as regards the position and character of the yield-point; it resembled more closely, perhaps, the material of Diagram III.

Diagram No. VIII. (First Part).—(Recovery at 50° C., &c.)

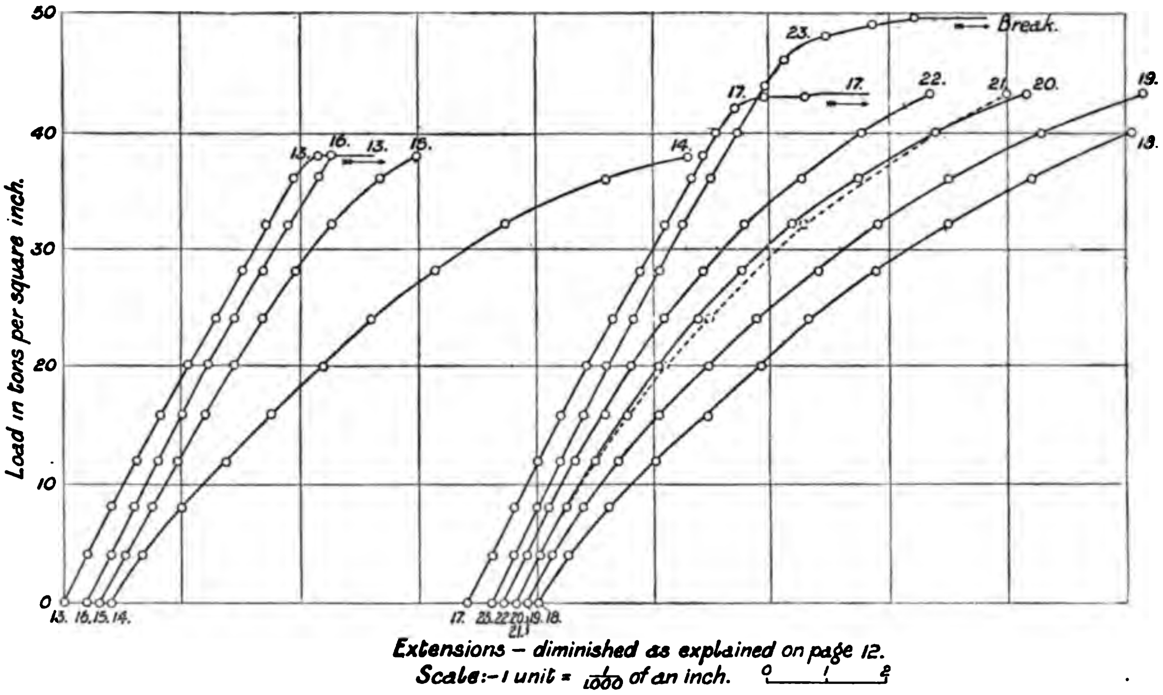


Curve No. 1.—Primary test.
„ „ 2.—30 minutes after No. 1.
„ „ 3.—After 5 minutes at 50° C.
„ „ 4.— „ 15 „ more at 50° C.
„ „ 5.— „ 17 hours at normal temperature (say 13° C.).
„ „ 6.—After 15 minutes at 50° C.

Curve No. 7.—After 5 minutes at 95° C.
„ „ 8.—3 days after No. 7.
„ „ 9.—20 minutes after No. 8.
„ „ 10.—4 hours after No. 8.
„ „ 11.—After 15 minutes at 50° C.
„ „ 12.— „ 15 „ „ 70° C.
„ „ 13.— „ 5 „ „ 95° C.

A comparison of the distances at the top between Curves 3, 4, 5, and 6, strikingly indicates the large effect of a small increase in the temperature of the restoring bath. The distance between 3 and 4 (after subtracting $\frac{1}{5}$th of a unit for change of origin), *i.e.*, $1\frac{1}{5}$ units, may be taken as a measure of the recovery due to 15 minutes at 50° C. Similarly, only $\frac{1}{5}$th of a unit (the distance between 4 and 5) measures the recovery due to 17 hours at the normal temperature (about 13° C.), while $\frac{3}{5}$ths gives the recovery due to a further 15 minutes at 50° C.

Diagram No. VIII. (Second Part).—(Recovery at 60° C., &c.)



Curve No. 13.—See first part of diagram.
„ „ 14.—20 minutes after No. 13.
„ „ 15.—After 15 minutes at 60° C.
„ „ 16.— „ 10 „ „ 95° C.
„ „ 17.— „ specimen turned down.
„ „ 18.—20 minutes after No. 17.
„ „ 19.—After 5 minutes at 60° C.

Curve No. 20.—After other 15 minutes at 60° C.
„ „ 21.—After 16 hours at normal temperature.
„ „ 22.—After 15 minutes at 60° C.
„ „ 23.— „ 10 „ „ 95° C.

Comparison of Curves Nos. 19, 20, 21, and 22, which illustrate the process of recovery after the passage of the fourth yield-point, shows a similar large difference between recovery at the ordinary temperature and that at 60° C. (140° Fahr.). In this case Curve No. 21 (obtained 16 hours after No. 20) shows that the material has yielded more, after its long rest, except for the higher loads. This apparent weakening is, of course, not due to the resting, but to the fact that the re-application of the load, necessary to obtain the readings for Curve No. 20, has had the effect of further overstraining the material to a slight extent. A curve obtained immediately after No. 20 would have fallen below that curve and also below Curve No. 21, while reaching approximately the same top point as No. 20. It may here be remarked that all the curves of this diagram have been obtained from first loadings

READINGS for Diagram No. VIII. (First Part.)

Load in tons/in².	Curve 1 (first test).	Curve 2. (30 minutes after 1.)	Curve 3. (5 minutes at 50° C.)	Curve 4. (15 minutes at 50° C.)	Curve 5. (17 hours at 13° C.)	Curve 6. (15 minutes at 50° C.)	Curve 7. (5 minutes at 95° C.)
0	0	0	0	0	0	0	0
4	119	126	122	122	122	120	120
8	239	258	248	248	247	242	241
12	360	397	377	371	370	366	366
16	486	539	517	500	499	489	489
20	608	688	660	635	633	619	611
24	729	850	820	776	777	750	734
26	789	952	910	857	849	822	798
27	820 and then off scale	1019	960 965	901 905	889 890	860	828
20	...	800	745	685	670	646	612
10	...	459	405	360	348	330	308
0	...	63	27	19	4	1	-1

Load in tons/in².	Curve 8. (3 days after 7.)	Curve 9. (20 minutes after 8.)	Curve 10. (4 hours after 8.)	Curve 11. (15 minutes at 50° C.)	Curve 12. (15 minutes at 70° C.)	Curve 13. (5 minutes at 95° C.)
0	0	0	0	0	0	0
4	120	128	120	122	120	120
8	240	260	248	248	241	240
12	361	398	379	371	365	360
16	482	534	518	500	489	482
20	605	675	661	636	611	607
24	729	822	811	779	738	729
28	850	986	974	940	860	850
30	911	1079	1064	1029	924	910
32	978	1195	1170	1126	989	971
33	1028 and then very large yielding	1290	1250	1180	1022	1001
						tons/in². 34 1032
20	...	869	829	761	627	36 1094
						37 1128
10	...	520	478	419	318	38 1163
0	...	99	79	35	10	and then large yielding

READINGS for Diagram No. VIII. (Second Part.)

Load in tons/in².	Curve 13. (See last table.)	Curve 14. (20 minutes after 13.)	Curve 15. (15 minutes at 60° C.)	Curve 16. (10 minutes at 95° C.)	Curve 17. (After turning down.)
0	0	0	0	0	0
4	120	128	122	120	120
8	240	260	245	240	240
12	360	398	368	360	360
16	482	536	490	480	481
20	607	678	613	602	601
24	729	820	739	726	724
28	850	973	865	849	849
32	971	1133	997	970	970
36	1094	1320	1138	1094	1092
38	1163 and large yielding	1440 } 1450 }	1219	1157	1152
					tons/in².
					40 1212
30	...	1190	970	910	42 1280
20	...	853	662	608	43 1330 }
10	...	492	347	303	1365 }
0	...	78	17	−3	43¼ large yielding

Load in tons/in².	Curve 18. (20 minutes after 17.)	Curve 19. (5 minutes at 60° C.)	Curve 20. (15 minutes at 60° C.)	Curve 21. (16 hours at 15° C.)	Curve 22. (15 minutes at 60° C.)	Curve 23. (10 minutes at 95° C.)
0	0	0	0	0	0	0
4	128	122	122	122	120	120
8	261	249	247	247	240	239
12	402	380	370	371	363	360
16	547	515	498	499	488	480
20	691	657	624	629	610	600
24	832	798	759	768	741	722
28	989	950	895	907	871	845
32	1150	1100	1039	1049	1009	966
36	1320	1260	1195	1207	1158	1089
40	1508	1439	1359	1360	1308	1212
43¼	1712	1612	1518	1500	1445	...
						tons/in².
....						44 1336
20	911	820	758	739	700	46 1402
						48 1488
0	100	38	24	19	8	49 1550
						49½ 1598
						and then large yielding and fracture

of the specimen at the various stages, so that, as explained when Diagrams Nos. V. and VI. were described, cyclic conditions of material are not represented. The difference between the behaviour of the material when a gradually increasing load was applied for a first time, and when the same load was applied for a second time, was, however, not usually so great as that shown in Curve B, Diagram V., at least with regard to the yielding at the higher loads. At early stages in recovery slightly smaller elongations were obtained on a second loading, but at intermediate stages greater extensions were obtained at the lower loads, and approximately the same extensions at the higher. The following table of extensometer readings, obtained from a specimen very similar to the last, may be taken as showing maximum differences, for the material commonly employed in these experiments, between the elongations produced at intermediate stages in recovery by a first and by a second loading. Curves Nos. 6 and 6′ of Diagram IX. also show in a striking fashion this difference in elastic condition.

Load in tons/in^2.	Extensometer readings.					
	1st application.	2nd application.	Difference.	1st application.	2nd application.	Difference.
0	0	0	0	0	0	0
4	120	120	0	120	120	0
8	246	246	0	240	246	+ 6
12	368	370	+ 2	362	370	+ 8
16	490	501	+11	488	500	+12
20	619	638	+19	618	637	+19
24	759	779	+20	760	776	+16
28	930	936	+ 6	917	925	+ 8
28½	961	959	− 2	939	945	+ 6

The effect produced by a third loading of a specimen usually differed from that produced by a second, but the difference was comparatively very slight.

To return to Diagram No. VIII., in the last test of the specimen (illustrated by Curve No. 23 in the second part of the diagram), the load was increased by a quarter of a ton to the square inch at a time; a gradual falling away from elastic behaviour was recorded, and finally local extension and fracture occurred at a stress of 49½ tons per square inch. This corresponded to about 46 tons per square inch of primitive area. The total elongation which the specimen had received was estimated to be 12 per cent. on an 8-inch length.

A fresh specimen from the same rod as the above, broken in a single test without allowing intermediate recoveries to take place,* gave an ultimate strength of rather under 40½ tons per square inch of original area. The total elongation in this case was found to be fully 16 per cent. on the 8-inch length.

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* Owing to the specimen breaking in the machine grips (at a stress of 38 tons to the square inch) a partial recovery took place while the specimen was turned down in the centre. The strength given above may therefore be a little too great and the elongation a little too small.

In order further to call attention to one or two features of this recovery from overstrain and the effect of temperature on it, the history of another specimen is given in Diagram No. IX.

The steel rod from which this specimen was cut differed but slightly from preceding ones. The rod was 1 inch in diameter, but the specimen was turned down, except at the ends, to a diameter of about 0·8 of an inch. The yield-point occurred at a stress of 23 tons per square inch, and was well defined like those shown in Diagrams III. and VIII., unlike those in Diagrams IV. and VII. The position of the yield-point was, however, sometimes found to vary even with specimens taken from the same rod. Thus, the specimen of the present diagram (No. IX.) gave apparently a perfectly steady extensometer reading after 22 tons per square inch had been applied—steady for, say, half a minute. The addition of the next half-ton produced rather greater elongation than was in accordance with the elastic law, but the reading was still steady. With 23 tons, however, creeping set in shortly after the extensometer reading—a rather large one—had been observed. This yielding continued, becoming greater and greater, and the skin of oxide began to spring off in the manner characteristic of the yield-point. Another specimen taken from the other end of the same bar (a 10-foot one) showed creeping and the springing off of the oxide after 22 tons of stress had been applied.

After the passage of the yield-point, illustrated in Curve No. 1, Diagram IX., the specimen was put into boiling water, and kept there for over 12 hours. This was to see if the position of the second yield-point would be affected by such prolonged treatment. As was expected, on cooling and re-testing the specimen a yield-point occurred at a load which agreed with that obtained from an adjacent specimen of the same rod, which had been immersed in boiling water for 3 minutes only. A third specimen from this same rod, after being overstrained, was put in a sand-bath, and kept at 250° C. for half-an-hour. On slowly cooling and then re-testing, the material behaved exactly as in the case of the comparison specimen, which had been restored by 3 minutes' immersion in boiling water. Had the specimen been annealed by heating to redness and slowly cooling, then, of course, the effect of overstrain would have been entirely annulled, and a yield-point obtained at a load corresponding to the stress at which the primary yield-point occurred.* It was found, however, that no effect (other than the recovery from the temporary effect of overstrain) was produced, until a fairly high temperature was attained.

To return to Diagram No. IX. After the second yield-point had been passed, the bar was re-measured and re-tested in the usual manner, Curve No. 3 being obtained. In this test the maximum load was kept on over night, and the creeping which occurred during the first 5 minutes is shown to have been considerable; for the next 15 hours it was perhaps not so great as might have been expected. This creeping went to the production of permanent set. Curve No. 4, Diagram IX., was obtained

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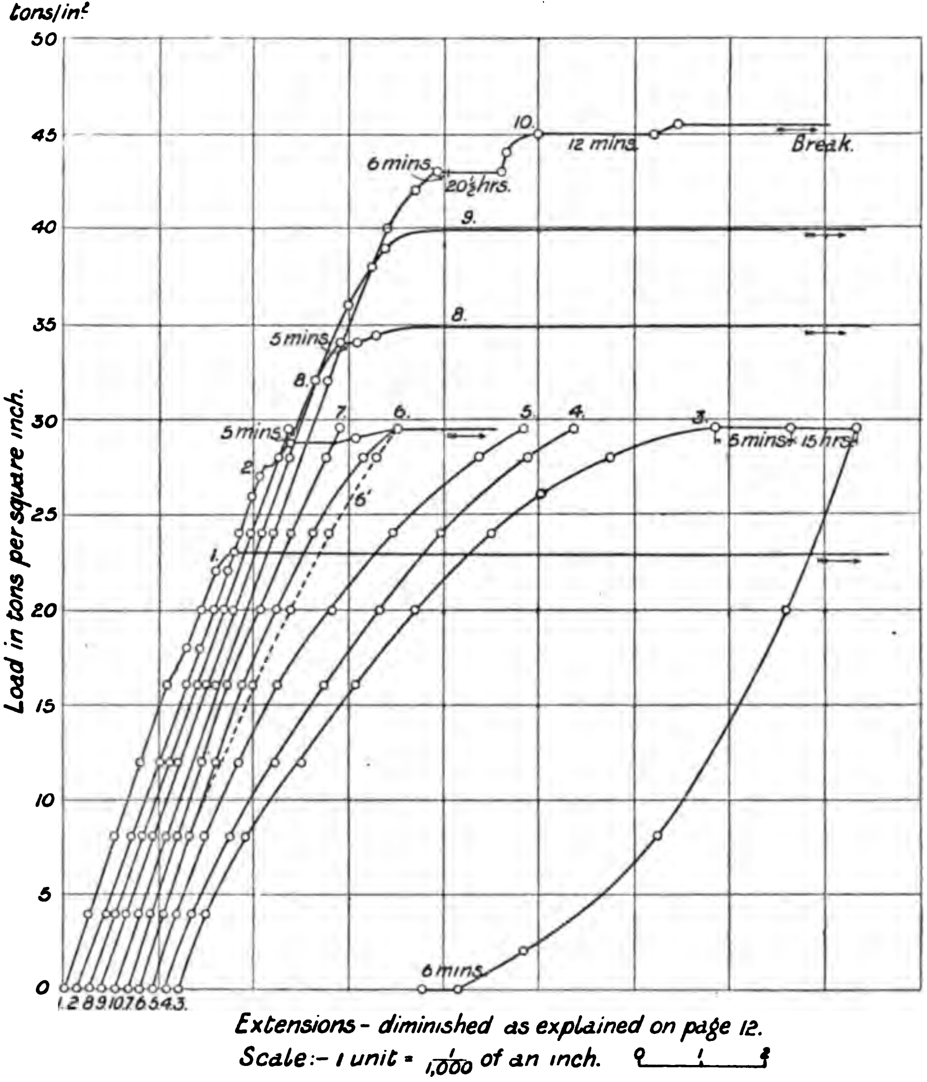
* See paper by UNWIN, "On the Yield-point of Iron and Steel, and the Effect of Repeated Straining and Annealing," 'Roy. Soc. Proc.,' vol. 57, 1895.

Diagram No. IX.—(Temperature effects.)



Curve No. 1.—Primary test.
" " 2.—After 12 hours at 100° C.
" " 3.—15 minutes after No. 2.
" " 4.—After load removed from 3.
Specimen now at 45° C., see table, p. 32.
Curve No. 5.—After 3 minutes at 60° C.

Curve No. 6.—After 4 minutes at 70° C.
" " 6'.—Immediately after No. 6.
" " 7.—After 4 minutes at 70° C.
" " 8.— " 3 " " 100° C.
" " 9.— " process A, diagram X.
" " 10.— " " B, " X.

immediately after the removal of the load from the test illustrated by Curve No. 3, and then the specimen was kept at 45° C. for 5 minutes, and afterwards for 15 minutes. The effects produced—which were slight—are shown in the following table. The second column under each heading in that table gives the extensometer readings for a test performed immediately after that given in the preceding column.

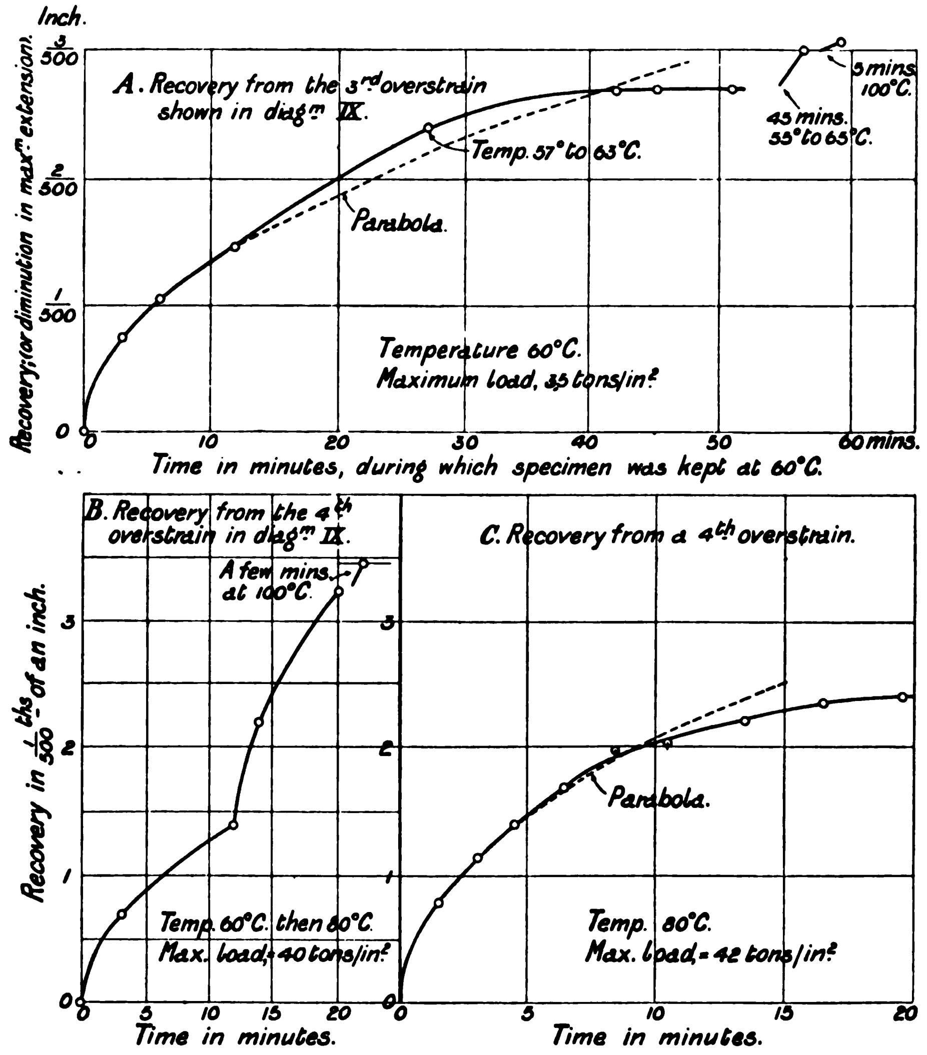
Load in tons/in².	Extensometer readings.					
	Curve No. 4, Diagram VIII.		After 5 minutes at 45° C.		After 15 minutes at 45° C.	
	1st.	2nd.	1st.	2nd.	1st.	2nd.
0	0	0	0	0	0	0
4	121	122	120	121	120	120
8	251	251	242	249	241	245
12	387	388	378	384	369	378
16	525	521	519	520	503	511
20	670	662	661	661	650	658
24	818	812	818	814	802	808
28	987	979	988	981	978	969
29½	1059	1048	1061	1049	1050	1039
¼ minute (say)	1061	1050	1070			
20	762	749	766	751	755	739
8	348	336	350	339	339	324
0	22	14	25	20	29	14
½ minute (say)	19	10	...	...	20	

These figures show that an immediate re-application of the load has produced in each of the three cases less total elongation than the first application, in consequence of the bar's gradual settlement into a cyclic state through successive loadings. After the recovery has become fairly perfect, this considerable diminution in the total elongation obtained by a second application of the testing load does not occur. This was clearly shown in the table on p. 29. In the present diagram Curve No. 6′ (shown dotted) was obtained immediately after No. 6, and it shows almost no change in the total elongation produced.

The tests made immediately after treatment at 45° C. are shown by columns 2 and 3 of the table above to have given slightly greater total elongations than those obtained from the loadings performed immediately before warming. This is perhaps contrary to what might have been expected, since increase of temperature has been shown to hasten recovery. But although the total elongations are greater, the process of recovery really has been aided. This is shown by the fact that distinctly smaller yieldings are obtained with low loads after the bar has been heated to 45° C. But though the bar is more perfectly elastic under low loads, under higher ones the yielding which occurs has not been decreased ; so that, as the specimen is subjected to a gradually increasing load, there is a transition from more elastic behaviour to less, which is suggestive of a yield-point.

To return again to Diagram No. IX., the specimen, after being warmed to 45° C. in the manner just explained, was subjected to the treatment recorded in the notes accompanying the diagram, and finally, as shown by Curve No. 8, complete restoration of elasticity was effected. The load was therefore increased until a yield-point was obtained at a stress of about 35 tons per square inch. The recovery from the overstrain produced by the passage of this third yield-point is shown by means of a curve at A in Diagram No. X. This curve was obtained by plotting amounts of recovery—measured in the manner described on page 25 (that is, by the diminutions in the extensions produced by the maximum load)—against the time taken at 60° C. to produce the recoveries. This method of measurement is of course faulty, for it has been shown above that a slight recovery may have occurred, although a greater total elongation has been obtained. But if the recovery is tolerably rapid, the method may be justified, for the sake of comparing the rates of recovery at different stages of completion, by means of a time-recovery curve. At C, Diagram X., there is shown a

Diagram No. X.—(Time-recovery Curves.)



curve of this kind obtained from a specimen whose history will not be given. This curve was more fully and carefully determined than those obtained from the specimen of Diagram No. IX.; but any of the curves in Diagram No. X. show that in the earlier stages the amount of recovery is approximately proportional to the square root of the time.

The heating of the specimens was accomplished by immersing in a hot water-bath at the required temperature for the required time, and then cooling by at once dipping in cold water. Had the specimen been allowed to cool slowly in the air, then greater recovery, due to a long and indefinite time at lower temperatures, would have been obtained.

Curve A, Diagram X., shows that at 60° C. a long time would have been required to produce perfect recovery, so the specimen (of Diagram IX.) was finally put in boiling water for 5 minutes. After cooling, a gradually increasing load was applied, and a fourth yield-point obtained at a stress of 40 tons per square inch. This test is shown by Curve No. 9, Diagram IX.

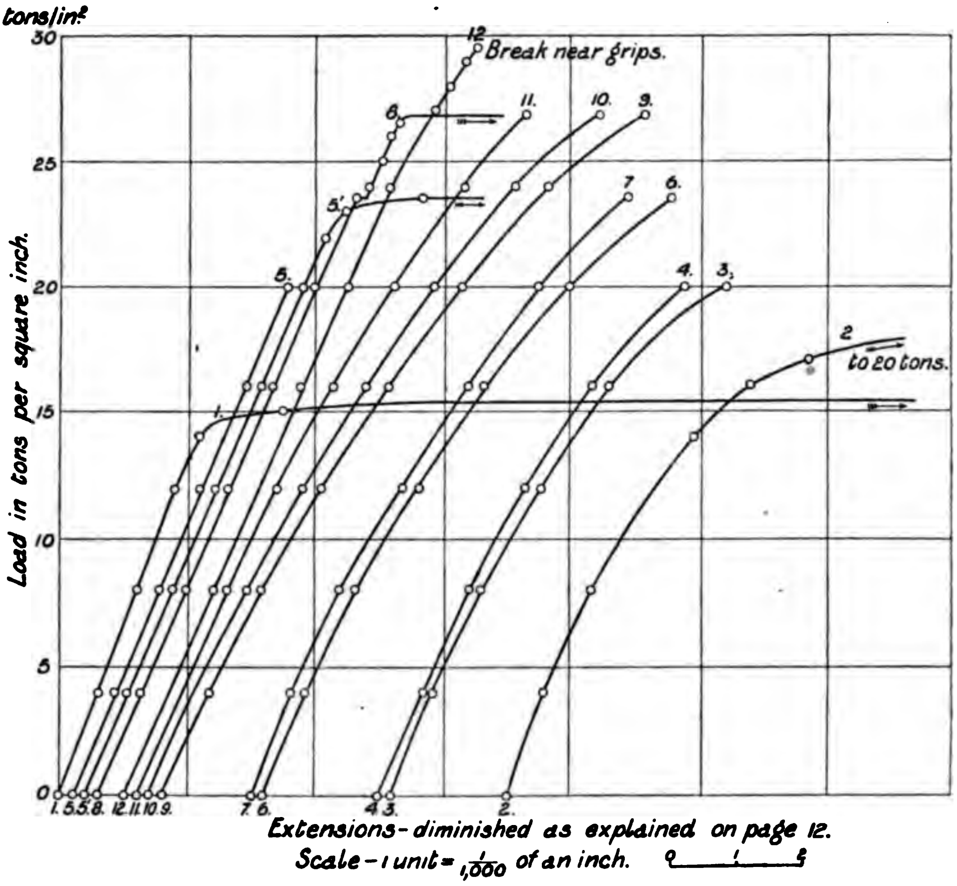
The recovery from this fourth overstrain is illustrated by Curve B, Diagram X. First 60° C. and then 80° C. were employed, perfect recovery being again obtained by bringing the piece to 100° C.

On load being once more applied to this specimen, elastic behaviour was shown up to the stress of 40 tons per square inch. At the 43rd ton, creeping was detected, and this load was allowed to remain on for 20½ hours. Considerable extension resulted, as is shown by Curve 10, Diagram IX. On now further increasing the load, the yielding was found for the subsequent three half-tons to be in close accordance with the elastic law. With the fourth half-ton (*i.e.*, at 45 tons per square inch) creeping was again detected. After 12 minutes, however, this creeping became very slow, so another half ton was applied, with the result that local extension and fracture occurred at that load of 45½ tons per square inch. This stress was equivalent to rather over 42½ tons per square inch of primitive area; the total elongation was about 0·81 of an inch, or rather over 10 per cent. on the 8-inch length.

A virgin specimen from an adjacent portion of the same bar as the above, gave, when tested at once to breaking, an ultimate strength of 36½ tons per square inch of original area; the total elongation in this case was 1·82 inches or nearly 23 per cent. on the 8-inch length.

It will be noticed that the distance in tons between the successive yield-points shown in Diagram IX. is roughly constant, and further that fracture has occurred where a yield-point (if not a fracture) would naturally have been expected. More correctly, it is the distance between a yield-point and the previous maximum load that is the same throughout. Thus a specimen from the same bar as that employed for this diagram, No. IX., was overstrained primarily by 27 tons to the square inch, and the subsequent yielding was obtained at 33 tons. That is at about 4 tons higher than the second yield-point shown in Diagram IX., when the primary loading was only carried to 23 tons per square inch. This regularity in the raising of the yield-point was also shown in Diagram No. VIII., the material being slightly different from that which has just been considered; and, further, it will be shown in Diagram No. XI., which gives the history of a specimen of unhomogeneous wrought iron. The distance between the yield-points is 3 to 3½ tons with the common wrought iron and 5 or 6 tons with the semi-mild steel usually employed in these experiments. In Diagram No. VIII. yield-points were obtained at loads of about 27, 33, 38, 43¼, and 49½ tons per square inch,—fracture occurring at the last mentioned stress. With another specimen from the same steel rod the primary loading was carried to 30 tons per square inch, and after recovery of elasticity a 2nd yield-point was obtained at about 35 tons per square inch. On again restoring elasticity a 3rd yield-point was found to occur at a stress somewhat under 40 tons,

Diagram No. XI.—(Common iron.)



Curve No. 1.—Primary test.
„ „ 2.—Immediately after No. 1.
„ „ 3.— „ „ „ 2.
„ „ 4.— „ „ „ 3.
„ „ 5.—16 hours after No. 4.
„ „ 5'.—After a few minutes at 100° C.
„ „ 6.—Immediately after No. 5.

Curve No. 7.—Immdiately after No. 6.
„ „ 8.—After a few minutes at 100° C.
„ „ 9.—Immediately after No. 8.
„ „ 10.—¼ hour after No. 9.
„ „ 11.—After 5 minutes at 55° C.
„ „ 12.— „ 10 „ „ 100° C.

and when recovery from overstrain had once more been effected fracture took place at 45 tons per square inch. It is however probable since this material is the same as in Diagram VIII., that had the primary loading in the present case been carried only to 29 tons per square inch, then a yield-point would have been obtained at a stress of 44 tons, and fracture would not have taken place until a load of over 49 tons per square inch had been applied. A yield-point obtained at a high stress is thus a crisis in the history of the specimen under test; the material is in danger of giving way, but if it does not, then, after recovery it will stand, before fracture occurs, a stress 5 or 6 tons higher than that at the critical yield-point.

It should, perhaps, be pointed out that in Diagram No. VII. no uniformity exists in the position of the yield-points. In this case the specimen cannot, perhaps, be taken as illustrating the behaviour of a certain material, for it will be remembered that a small flaw ran through the centre of the bar from which this specimen was taken, and probably this flaw had a considerable influence in determining the position of the yield-points.* Chemically this material differed only slightly from that of the other steel rods used, as is shown by the analyses given on page 4.

Before concluding this section of the paper, attention should perhaps be directly called to Diagram No. XI., which has already been incidentally referred to. It gives the history of a specimen of common wrought iron, the diameter of the specimen being 1 inch. Curve No. 1 illustrates the primary loading and shows that the yield-point has occurred at a stress of 15½ tons per square inch. After the large stretching had ceased, and the load had been removed, the 8-inch length of the specimen was found to have been stretched about 0·20 of an inch. On re-loading, the material exhibited comparatively little semi-plasticity, as is shown by Curve No. 2. The load was, therefore, increased until a stress of 20 tons per square inch was attained, the specimen being thereby stretched further by about a quarter of an inch on the 8-inch length. On re-testing, the curve obtained was still found to agree closely with Curve No. 2 up to the stress of 15 tons, but as the loading was now continued to 20 tons the semi-plasticity was more clearly shown. Curve No. 5 shows that a night's rest at the ordinary temperature has been sufficient to produce complete recovery of elasticity; so common iron recovers much more quickly than the semi-mild steel employed for the most part in the course of these experiments. It may be of interest here to recall that the half-inch specimens of comparatively mild steel, employed for Diagram No. V., recovered at a very much slower rate than the harder steel usually employed in these experiments.

After Curve No. 5 was obtained the specimen was put in boiling water for a few minutes to ensure perfect recovery. On testing, Curve No. 5 was repeated, and on increasing the load a yield-point was got at 23½ tons per square inch, as shown by Curve 5′. Curve No. 8 shows that a few minutes in boiling water has effected perfect recovery from this second overstrain. The maximum load of 23½ tons was kept on in this test for 45 hours, and only the slight creeping shown in the diagram occurred.

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* See p. 24.

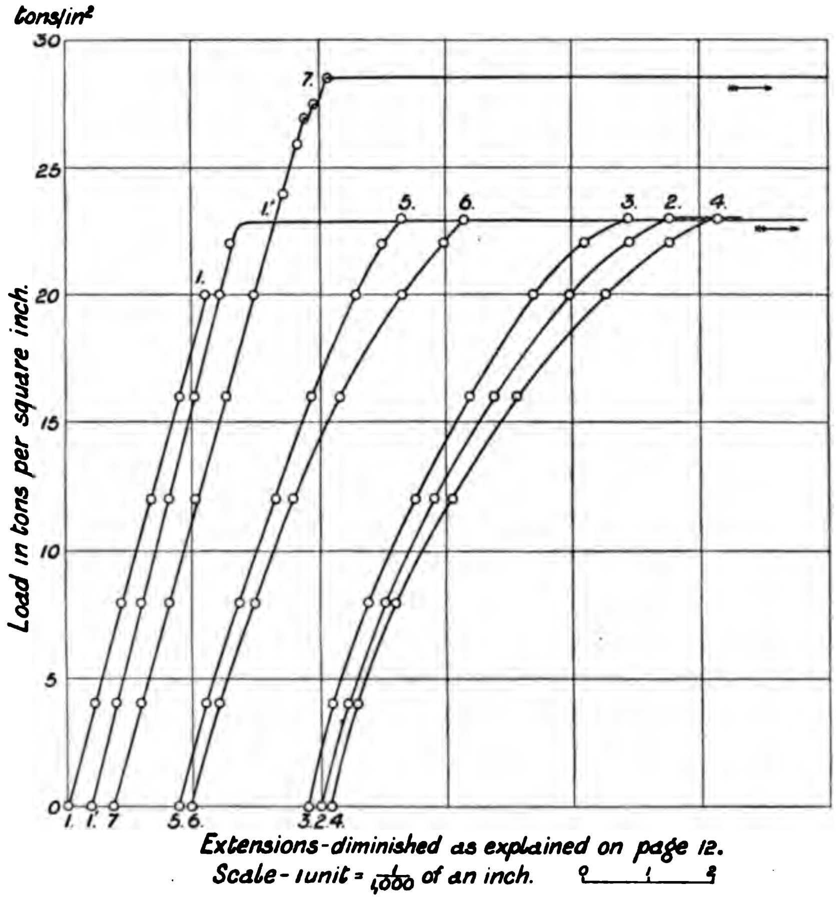
On increasing the load, a well-defined yield-point was now got at a load of 26¾ tons per square inch. Comparison of Curves Nos. 9, 10, and 11 shows the remarkable hastening in the recovery from this overstrain, produced by a temperature of 50° C.; while Curve No. 12 shows the material once more in the perfectly elastic condition. On now carefully increasing the load a fracture was obtained close to the upper machine grips, at a stress of 29½ tons per square inch. The specimen was gripped and loaded again, with the result that fracture occurred close to the lower grips at 29¾ tons per square inch. This was repeated a third and a fourth time, so that the occurrence of the fracture close to the grips was not due to a primary effect of the gripping, that is, it was not due to the gripping having prevented the material from becoming hardened by overstrain. The breaking load, of over 29½ tons per square inch, was equivalent to a stress of fully 27 tons per square inch of the original area of the specimen. Another specimen of the same rod was found to give a yield-point at 14 tons per square inch, and on steadily increasing the load fracture occurred, near the centre of the specimen, at slightly under 23 tons per square inch of original area. The elongation was about 21 per cent. on an 8-inch length. Common iron thus exhibits the same features as steel in respect of recovery from overstrain and the effect of temperature on it; but in the case of common iron recovery is comparatively rapid.

The Effect of Mechanical Vibration on Recovery from Overstrain.

Diagram No. XII. illustrates the effect of mechanical vibration on recently overstrained iron, and shows that such treatment has an opposite effect to that of increase of temperature—instead of the recovery process being hastened, the material is made distinctly less elastic. The following table gives most of the figures from which various curves of this diagram have been plotted. The material employed is the same as that used for Diagram No. IX., but the specimen in this case was not turned down, and so was of the full diameter of 1 inch throughout its length.

Load in tons/in^2.	Curve No. 1. (2nd—after vibrating.)		Curve No. 2.	Curve No. 3. (½ hour's rest.)	Curve No. 4. (After vibrating.)	Curve No. 5. (16½ hours' rest.)	Curve No. 6. (After vibrating.)
	1st	2nd				(2nd loading)	
0	0	0	0	0	0	0	0
2	60	60	61	60	59	60	60
4	122	120	122	120	119	121	120
6	182	181	188	187	185	185	185
8	245	240	251	249	251	248	249
10	307	300					
12	368	361	390	387	396	374	379
14	429	421					
16	489	482	539	529	545	502	518
18	549	544					
20	609	601	699	680	718	639	665
22		659	797	771	818	709	749
23		Off the scale	851 1008 } 10 mins.	825 840 } 3 mins.	868 919 } 10 mins.	745 750 } 3 mins.	790

Diagram No. XII.—(Mechanical vibration.)



Curve No. 1.—Illustrates primary loading.
„ „ 1'.—Is after mechanical vibration.
„ „ 2.—Immediately after No. 1'.
„ „ 3.—½ hour after No. 2.
Curve No. 4.—After mechanical vibration.
„ „ 5.—A 2nd loading, 16½ hours after No. 4.
„ „ 6.—After mechanical vibration.
„ „ 7.— „ 4 minutes at 100° C.

Before the experiment corresponding to Curve No. 1 was performed, a test was made to ensure that such vibration as was contemplated would have no effect on the elastic properties of the primitive material. The specimen was loaded till a stress of 20 tons per square inch was attained, and the load was then removed. The extensometer readings obtained are shown in the first column of the table given above. The specimen was then taken out of the testing machine and vigorously tapped with a hammer, so as to make it ring in various modes. It was then re-tested and the second column of readings shown above was obtained. These readings are slightly less than those obtained during the first loading, but this was to be expected on a second loading, though, perhaps, to scarcely so great an extent. Large yielding occurred during this test at 23 tons per square inch, which is the known yield-point of the material. Hence violent vibration may be said to have had no effect on the primitive material, or if it had a slight effect, it was shown in the annihilation of the causes of small departures from accurate obedience to the elastic law.

Curve No. 2 of Diagram No. XII. shows the specimen to be in the ordinary semi-plastic condition produced by overstrain, and Curve No. 3 the condition of the material after half-an-hour's rest. After this test the specimen was taken out of the testing machine and vigorously tapped with a hammer. On re-testing Curve No. 4 was obtained, which shows that not only has the effect of the half-hour's rest been annulled by the vibration, but that the material was rather more plastic than it had been immediately after overstrain. The specimen was next allowed to rest for 16½ hours and was then re-tested, Curve No. 5 showing the progress made towards recovery. The specimen was then taken out of the testing machine, and once more struck with the hammer so as to make it ring. On again testing, the elasticity was found just as before to have been made more imperfect, Curve No. 6, which illustrates this test, lying below No. 5. The specimen was then put in boiling water for a little, and Curve No. 7 shows that recovery was complete. Hammering was found to have no appreciable effect on the elastic condition of material whose elasticity had been thus restored.

In concluding this section it may be of interest to state that the effect of turning down the diameter of a recently overstrained specimen was to produce partial recovery of elasticity. This was in all probability due to the warming which accompanied the cutting action—the bar being heated by conduction, and only the surface subjected to severe mechanical vibration.

The Influence of Magnetic Agitation in Hastening or Retarding the Recovery of Elasticity.

The experiment which is now about to be described was made with the object of finding the effect on recovery, of magnetising and de-magnetising an overstrained specimen.

A coil (1¼ inch diameter × 7½ inches long) was made which gave a field strength at the centre of about 140 C.G.S. units, when a current of 10 amperes was passing. This coil was put round a specimen and supported at the 8-inch length, to which the extensometer was to be applied.

The material used was the same as that of Diagrams IV. and VII.; the specimen, however, was not in its virgin condition, it had been largely overstrained and had recovered its elasticity again, so that a yield-point was not expected till a stress of about 40 tons was attained. During the loading of the specimen, a current of 10 amperes was passed at intervals through the coil, and it was found that the extensometer could clearly detect (when the current was passed) the slight elongation due to magnetisation. This elongation occurred only at the lower loads; at the higher ones the slight contraction, which is known to occur, was quite readily observed.*

At a stress of 40¼ tons per square inch a yield-point was obtained, and while the bar was stretching rapidly at this load the current was put on and off several times, its direction being constantly reversed. The contraction, which had been noticed just before the yield-point had been reached, was still clearly shown at each "make," by a temporary check in the rate of extension. Time readings—of a few minutes duration—taken while the bar was stretching, detected no change in the rate of extension when the current was allowed to pass for some time, and so the bar for that time kept magnetised.

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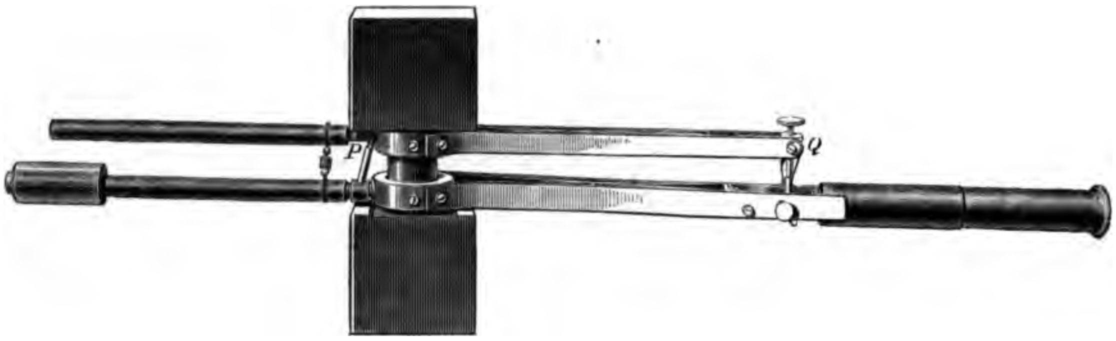
* For the change in length caused by magnetisation, when iron is under various stresses, see papers by SHELFORD BIDWELL, 'Phil. Trans.,' A, 1888, and 'Roy. Soc. Proc.,' 1890.

When the stretching at the yield-point had practically ceased, the specimen was re-measured and the curve showing semi-plasticity obtained. The specimen was then allowed to rest for two hours, and the recovery effected was recorded by a curve. The current was next passed through the coil for periods of from 10 to 15 minutes, and was reversed all the time rapidly by hand, so that the bar was subjected to considerable magnetic agitation. Such treatment was found to have no appreciable effect on the recovery of the specimen, the curve obtained on re-testing being almost exactly the same as that obtained after the 2 hours' rest.

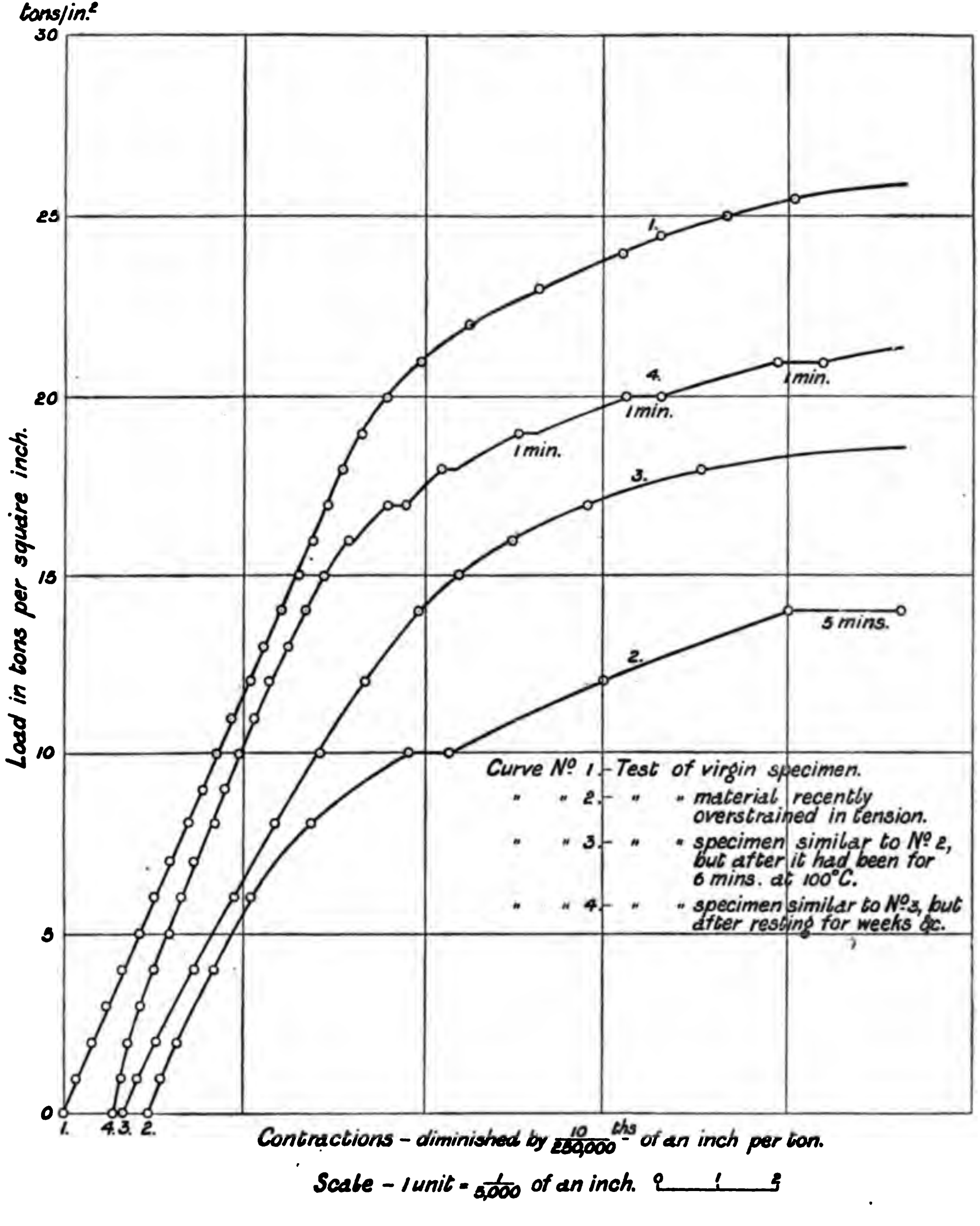
Compression Experiments.

The experiments which are now about to be described illustrate the recovery of iron from tensile overstrain by means of compression tests. These tests were carried out on small cylindric blocks, 1⅛ inches diameter by 1⅞ inches long, compression being applied by means of the 50-ton testing machine. The small compressional strains obtained were measured by an instrument specially designed by Professor EWING. This instrument resembles in principle Professor EWING's extensometer, especially a more recent form of that instrument, and like it is self-contained, and is entirely supported by the specimen under test. A detailed description of this instrument, which is shown attached to a compression specimen in the following illustration, need not be given here, but it may be stated that with a mechanical multiplication of 10, and further optical magnification, a contraction of $\frac{1}{250000}$th of an inch can be measured. This corresponds to a compressional strain of $\frac{1}{312500}$th, since the length of specimen actually tested is only 1¼ inches. It may be recalled that the unit of the readings of Professor EWING's extensometer represented an elongation of $\frac{1}{400000}$th, the length of specimen tested being 8 inches.



The following two series of readings, obtained with the new compression instrument, clearly show by comparison the semi-plasticity which is induced in iron by tensile overstrain. The curves which have been plotted from these readings are shown in Diagram XIII. The first series was obtained from a virgin specimen of $1\frac{1}{8}$ inch round steel rod, very similar in quality to the 1-inch rods usually employed in the tension experiments. The total length of the specimen was only $1\frac{2}{3}$ diameters, and the ends were carefully faced, so that "buckling" may be said to have been avoided, and the compressional stress applied in as uniformly distributed a manner as was practicable.

Diagram No. XIII.—(Compression experiments.)



FIRST Series of Compression Instrument Readings. (Test on a Virgin Specimen illustrated by Curve 1. Diagram XIII.)

Load in tons per square inch.	Contractions in $\frac{1}{250000}$ths of an inch.	Differences.
0	0	
1	20	20
2	43	23
4	88	45
6	134	46
8	183	49
10	227	44
12	277	50
14	321	44
16	367	46
18	410	43
20	468	58
21	505	
	time 510	
22	555	
	time 560	87
23	625	
24	705	150
25	800	
26	1000	295
	1 min. 1075	
The load was now removed and the following readings taken :–		
20	990	
15	915	75
10	815	100
5	705	110
1	622	
0	588	117

The difference column given above shows that the material has behaved elastically until a load of 20 tons per square inch was attained. Beyond that load there is shown a gradual but tolerably rapid departure from HOOKE's law ; there seems to be, however, no very definite yield-point. In a tension test of this material creeping was first noticed at 23 tons per square inch, and at 24½ tons a very large yielding occurred. YOUNG's modulus, as calculated from the compression readings given above, was found to agree with that obtained from tension experiments to two significant figures; in both cases the third figure was rather doubtful. Thus, the modulus as got from a first loading in tension to 20 tons per square inch, was 13,100 tons per square inch, while from a second loading to 10 tons per square inch it was found to be 13,300 tons per square inch. The modulus, as calculated from the contraction shown to have occurred in the table above, between 4 and 18 tons per square inch, is 13,600 tons per square inch.

The second series of compression instrument readings was obtained from a specimen of the same $1\frac{1}{8}$-inch rod, but after the rod had been overstrained largely in tension by a load of 33 tons per square inch. The compression specimen was cut from the overstrained bar immediately after the large stretching load was removed, care being taken to prevent warming during the cutting and mechanical manipulation necessary to the making of the small compression block. To test the effect of such mechanical treatment on the elastic condition of the material, a tension specimen was overstrained, tested, immediately turned down to a smaller diameter and tested again. As has already been recorded on page 39, considerable, though by no means perfect, recovery of elasticity was found to have been produced. Owing to the precautions taken in making the compression specimen used for this second series of readings, the mechanical manipulation may be assumed to have produced no effect on the elastic properties of the material.

SECOND Series of Compression Readings. (Material freshly overstrained, Curve 2, Diagram XIII.)

Load in tons per square inch.	Contractions in $\frac{1}{250000}$ths of an inch.	Differences.
0	0	
1	20	20
2	43	23
4	94	51
6	145	51
8	217	72
10	315	98
	time 350	
12	500	185
14	670	170
	5 mins. 768	
16	948	278
	3 mins. 1010	
18	1260	312
20	1650	390
	2 mins. 1750	
	20 mins. 1835	
Load was now removed and the following readings taken:—		
10	1615	
4	1466	
2	1415	
0	1360	

A comparison of the difference column of the present series of readings with that of the last very clearly shows the change in the elastic condition of the material, produced by tensile overstrain. There is now not even approximate conformity with HOOKE'S law at the lowest loads.

The recovery of elasticity, which is brought about either by prolonged rest at normal temperatures, or by keeping the piece for a few minutes at a temperature such as 100° C., is shown in the following series of compression instrument readings. This third series of readings was obtained from a compression specimen taken from the same overstrained rod as the last; but in the present case the specimen was boiled in water for 6 minutes before being tested. This test is illustrated by Curve No. 3, Diagram XIII.

THIRD Series of Compression Readings. (Showing Recovery of Elasticity produced by 6 Minutes' Boiling.)

Load in tons per square inch.	Contractions in $\frac{1}{250000}$ths of an inch.	Differences.
0	0	
1	22	22
2	48	26
4	99	51
6	152	53
8	208	56
10	262	54
12	319	57
14	382	63
15	428 (creeping noticed)	
16	482	100
17	558	
18	660	178
19	795	
20	970	310
	1 min. 1020	
On removing the load the following readings were obtained:—		
15	912	
10	788	
4	639	
2	589	
0	540	

Comparison of the differences in this table and those in the last, or comparison of Curves 2 and 3 of Diagram XIII., clearly shows the large effect produced by the 6 minutes at 100° C. Comparison of the first and third series of readings, or of Curves 1 and 3, Diagram XIII., seems to indicate that the 6 minutes' boiling has not sufficed to produce quite perfect recovery of elasticity. In Curve No. 4 of this diagram—the readings need not be tabulated—there is shown the testing of a specimen very similar to that employed for Curve No. 3, but in this case it was certain that recovery was complete. The specimen was not taken from the same overstrained portion of a bar as that from which Curves Nos. 2 and 3 were obtained, but from another portion of the same material, which had been similarly overstrained. Before the compression specimen was cut off, the overstrained tension specimen had been allowed to rest for many weeks, and had been tested and found to be quite elastic up to a stress of 35 tons per square inch. The tension specimen was, however, boiled for some time as a further precaution, and then the compression specimen was cut from it, and the test illustrated by Curve No. 4 was performed. The modulus given by this curve agrees very well with that obtained for the virgin material from Curve No. 1. The marked discrepancy shown in this curve, No. 4, at the lowest loads may evidently be discarded; it was probably due to imperfect facing of the ends of the specimen, or some such cause.

Curve No. 4, further, very clearly shows that tensile overstrain—which raises the yield-point in tension—lowers that in compression, or, it may be more definite to say, lowers the load at which any arbitrary amount of plastic contraction occurs. This is in agreement with Professor BAUSCHINGER'S conclusion with regard to the elastic limits, viz., "that the elastic limit in tension cannot be raised without lowering the limit in compression, and *vice versâ*."* Professor BAUSCHINGER draws a further conclusion from his experiments, namely, that when the elastic limits of a material are varied by overstrain, the range of perfect elasticity seems to remain constant, so that, if the elastic limit in tension be raised, then that in compression is lowered by an equal amount. The author's experiments do not bear this out. They show that such a proposition cannot be applied to the yield-points, for the yield-point in tension of the material in the condition whose compression properties are illustrated by Curve 4, Diagram XIII., was found to occur at a stress between 12 and 13 tons per square inch, above the yield-point of the material in the primitive condition, and no matter where the yield-points in Curves 1 and 4 be supposed to exist, the lowering, which is the result of the tensile overstrain, cannot be greater than 4 or 5 tons of stress.

The characteristics of overstrained iron in respect of hysteresis and imperfect elasticity may be considered as illustrating MAXWELL'S views on the 'Constitution of Bodies,' as set forth by him in the 'Encyclopædia Britannica.'† In that article all bodies are assumed to be composed of groups of molecules oscillating about more or less stable configurations. If the oscillations are such as to cause all the groups to be continually breaking up, then we have a viscous fluid. But if "groups of greater stability are disseminated through the substance in such abundance as to build up a solid framework, the substance will be a solid, which will not be permanently deformed. except by a stress greater than a certain given stress." A solid, however, is not assumed to be entirely composed of these stable groups of molecules, or say of sensible particles, but to contain groups of less stability, and also groups which break up of themselves. When a solid has been permanently deformed or overstrained then "some of the less stable groups have broken up and assumed new configurations, but it is quite possible that others more stable may still retain their original configurations, so that the form of the body is determined by the equilibrium between these two sets of groups; but if, on account of rise of temperature, increase of moisture, violent vibration, or any other cause, the breaking up of the less stable groups is facilitated, the more stable groups may again assert their sway, and tend to restore the body to the shape it had before its deformation."

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* See UNWIN'S book on 'Testing of Materials of Construction,' p. 386, or BAUSCHINGER, "Ueber die Veränderung der Elasticitätsgrenze und die Festigkeit des Eisens und Stahls," 'Mittheilungen aus dem Mech. Techn. Laboratorium in München,' 1886.

† Or see the 2nd volume of CLERK MAXWELL'S 'Collected Papers.'

The semi-plasticity exhibited by recently overstrained iron may thus, on the above theory, be attributed to the less stable groups, which after overstrain are in comparative abundance. And since these less stable groups will tend to break up of themselves, there will be a slow recovery through lapse of time towards elastic behaviour which is associated with the idea of stable groups.

Increase of temperature has been shown in the present paper to hasten recovery from overstrain to a remarkable extent. This, as indicated by the quotation given above, may be ascribed to a greater facility given by slight warming, to the breaking up of the less stable groups, and possibly to the re-formation of more stable groups.

Violent mechanical vibration, however, seems to break up the rather more stable groups, rendering the material more semi-plastic and hindering the recovery process.

That recovery from overstrain, or more generally, that the phenomenon of "elastic after-action," is associated with complexity in the physical structure of the material, is further borne out by the fact that a crystalline body, such as a quartz torsion fibre, exhibits little or no after-action (in the form of zero-creeping); while a complex body like glass shows such action in marked degree. An analogy to this difference in the behaviour of material, according as it is simple or complex, is found in the phenomenon of the residual charge in the Leyden jar. Condensers with pure dielectrics such as sulphur, quartz, air, exhibit little or no residual charges; while with complex substances like glass, gutta-percha, caoutchouc, the phenomenon is particularly observable.

INDEX SLIP.

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Phil. Trans., A, vol. 193, 1899, pp. 47–87.

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INTRODUCTION.

THE phenomenon of surface tension exhibited at the surface of separation between two homogeneous liquids may be regarded as arising from such a variation in the distribution of the matter composing each of the liquids, in the immediate neighbourhood of the surface of separation, that the energy of given quantities of the two liquids is greater when these are in the neighbourhood of the surface than when each is in the homogeneous interior of its corresponding liquid.

On such a view, the tension per unit length in the surface will be equal to dE/dS, where dE is the increase in the potential energy of the system of two liquids, resulting from an increase, dS, of the surface of separation between them. The distribution of energy here referred to may be considered independent of possible electrostatic effects at the surface of separation.

There is little doubt, however, that there is frequently a potential difference of considerable amount at the surface separating two such liquids as mercury and a solution of a salt in water. There must, therefore, be a corresponding separation of electricities of opposite sign at the surface, and we may regard these as forming a condenser-like "double-layer." This double-layer will give rise to an electrostatic surface energy, whose value we may write as $E' = \frac{1}{2}cS\pi^2$, where c is the capacity of the double-layer per unit surface and π is the potential difference across it. S is, as before, the area of the surface separating the two liquids. Now, if a small change of this surface, dS, be supposed to take place while the potential difference across the double-layer is kept constant by an external electromotive force, we get

$$dE'/dS = \tfrac{1}{2}c\pi^2.$$

This increase in the potential energy of the system, with increase in the surface of separation between the two components, will be an effect that tends to take place under the influence of the external electromotive force, and will be equivalent to a force per unit length tending to *increase* the surface of separation between the two liquids.

The observed surface tension will thus be

$$\gamma = dE/dS - dE'/dS = \gamma_0 - \tfrac{1}{2}c\pi^2,$$

where γ_0 is the surface tension arising from the non-electrical distribution of energy first mentioned. The equation will give the relation between the observable surface tension and the potential difference at the mercury surface.

The above may be regarded as the Helmholtz theory of electro-capillary phenomena. LIPPMANN found that the curve showing the relation between the surface tension and the E.M.F. applied between the terminals of a capillary electrometer was (for a particular solution of sulphuric acid) approximately parabolic through a considerable portion of its course, and it appeared from this that c was a constant and that γ_0 was independent of π.

It is not a necessary consequence of considerations such as the above, that c should be constant; but if the only effect of the potential difference is to produce an electrostatic surface energy represented by $\frac{1}{2}c\pi^2$ per unit surface, then the observed surface tension should have a maximum value when $\pi = 0$, even although c may be variable. The assumption, that the maximum surface tension corresponds to zero potential difference between the mercury and the solution, has been much employed in recent years in the deduction of values for the contact potential difference between various electrodes and electrolytes. It must be remembered, however, that the observed variation in the surface tension need not be due solely to variation in the quantity $\frac{1}{2}c\pi^2$. The non-electrical surface energy, represented by γ_0, may vary with the potential difference. A variation in the potential difference at the surface of separation between the mercury and the solution may be accompanied not only by a variation in the electrostatic surface energy, but also by a variation in the distribution of the matter in the neighbourhood of the surface.

If a variation of the kind just mentioned can be traced, it is clear that we cannot consider the phenomena as if they were due to a certain non-electrical distribution upon which is superposed an electrostatic double-layer, producing no other effect than that represented by its electrical energy. Hence it need not happen that the maximum surface tension corresponds to zero potential difference, for the maximum surface tension may arise from the fact that non-electrical effects, accompanying the change in the potential difference and tending to reduce the surface tension, pass through a minimum value as the potential difference changes. This minimum value need not necessarily correspond to zero potential difference. The possible nature of non-electrical effects which may accompany changes in the potential difference is discussed later. The first part of the paper contains an experimental analysis of the Lippmann-Helmholtz theory.

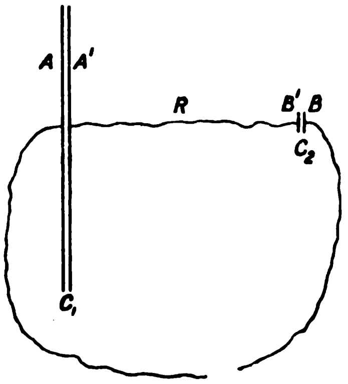
THE LIPPMANN-HELMHOLTZ THEORY OF ELECTROCAPILLARY PHENOMENA.

In the Lippmann-Helmholtz theory of the capillary electrometer there are in reality two distinct hypotheses, either of which may be separately justifiable. The first concerns the manner in which the potential difference at the capillary varies with the electromotive force applied between the terminals of the electrometer. The second deals with the relation between the above potential difference and the tension of the surface separating the mercury and the solution.

The First Hypothesis of the Lippmann-Helmholtz Theory.

The first hypothesis would apply to any electrolytic cell consisting of two polarizable electrodes placed in a conducting solution. When an E.M.F. (of which the value is kept within certain limits depending on the nature of the electrodes and of the solution) is applied to such a cell there may be a considerable current for a very short time; but the system almost at once assumes a practically steady state in which there is only a very feeble continuous current through the cell. The value of this current can in general be neglected in comparison with the current value found by dividing the E.M.F. applied by the calculable resistance of the electrolyte. It is therefore considered that the potential fall within the liquid can be neglected in comparison with the sum of the potential changes in the neighbourhood of the electrodes, and that this sum is equal in value to the applied E.M.F. The system is, in fact, considered equivalent to a pair of condensers (supposed existent at the surfaces of separation between electrode and solution) connected in series by a resistance (represented by the resistance of the electrolyte). For electrodes of the same nature in the same solution the respective capacities are taken to be proportional to the areas of the surfaces in contact with the solution. In the capillary electrometer, therefore, the capacity of one electrode would in general be indefinitely small compared with that of the other.

Let AA′ and BB′ represent condensers (of capacities C_1 and C_2) of which the plates A′ and B′ are connected by a resistance R, and of which the "external" plates A and B are at first also connected.



Suppose the condensers are charged, and let A and B be at zero potential while A′ and B′ are at the potential $-\pi_n$. Let now an E.M.F. π_e be introduced in the external circuit connecting A and B, and let the resistance of this circuit be R′. Let π, π' and π'' be the final potentials of A, A′ and B′ respectively, — B being supposed kept at zero potential. Then it is easy to show that

$$\pi_e = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\int_0^\tau \iota\, dt,$$

in which the integral represents the quantity of electricity that has passed in the circuit during the time τ taken by the system to acquire its steady state. Also

$$\pi = \pi_e$$

$$\pi' = (\pi_e - \pi_n) - \frac{1}{c_1}\int_0^\tau \iota\, dt$$

and

$$\pi'' = \frac{1}{c_2}\int_0^\tau \iota\, dt - \pi_n = \pi'.$$

Now if the condenser AA′ is very large compared with the condenser BB′, we may neglect $\frac{1}{c_1}$ in comparison with $\frac{1}{c_2}$, so that we get

$$\pi'' = \pi_e - \pi_n$$

and

$$\pi - \pi' = \pi_n.$$

Thus the effect of introducing the E.M.F. π_e is that the potential difference at the small condenser is changed from $-\pi_n$ to $\pi_e - \pi_n$, while the change at the large condenser is negligible.

If the supposed analogy were complete, we should, therefore, have the result that in the capillary electrometer the variation of the potential difference at the capillary electrode is the same as the variation of the E.M.F. applied between the terminals. The analogy between the condenser system and the electrolytic cell cannot, however, be complete. In the latter case the original potential difference (corresponding to $-\pi_n$) is not arbitrary, but represents one of the conditions of the equilibrium at the electrode. Any cause which tends to alter the "natural" potential difference $-\pi_n$ at the small electrode—the nature of the solution in the neighbourhood of the electrode remaining sensibly constant—must in general be accompanied by a "depolarization" current representing the continual tendency of the "polarized" electrode to revert to the original potential difference. We cannot, therefore, have $\iota = 0$ in the final steady state as in the condenser system.

The Effect of Depolarization.

If we assume, however, that the effect of the depolarization is to produce a fall of potential within the electrolyte according to Ohm's law, the nature and magnitude of the effect can be readily specified. Thus, taking the symbols as above to be applicable to the capillary electrometer, we shall have

$$\pi_c - \pi = \mathrm{R}' . f(\pi'')$$

and

$$\pi' - \pi'' = \mathrm{R} . f(\pi''),$$

in which the continuous depolarization current is written as a function of π'', because of the assumption that if the area of the capillary electrode is kept constant, the magnitude of the current will depend only on the potential fall at that electrode. It is not to be expected that the depolarization current will remain constant for an indefinite time. Owing to the variation of the concentration of the solution in the neighbourhood of the electrode, which must necessarily accompany the passage of the current through the electrolyte, the relation between π'' and the depolarization current will alter; but an effect of this kind will be gradual, and we may consider the depolarization current to be constant for some time after the introduction of π_e. In like manner, the accumulated effect of the continuous current upon the potential fall at the large electrode will only be gradually perceptible.

Adding the two above equations, and putting $\pi - \pi' = \pi_n$, we get

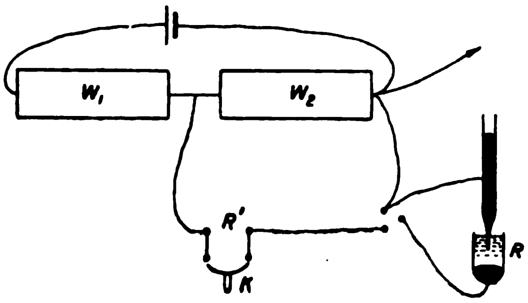
$$\pi'' + (\mathrm{R} + \mathrm{R}') f(\pi'') = \pi_e - \pi_n.$$

It therefore follows (as is otherwise obvious), that the effect of depolarization would be to cause the potential difference at the small electrode to change less rapidly than the applied E.M.F. Hence, before one can proceed beyond the first hypothesis to examine quantitatively whether the second hypothesis, concerning the relation between the potential difference and the surface tension, is true, it is necessary to determine whether the effect of the depolarization can under any circumstances become appreciable.

The magnitude of the effect will depend upon the value of $(\mathrm{R} + \mathrm{R}') f(\pi'')$. The internal resistance, R, will, of course, depend upon the nature of the electrolyte employed, upon the internal cross-section of the capillary tube and upon the distance between the mercury meniscus and the point of the capillary tube. Its value may range from something like 50,000 or 100,000 ohms to a million ohms or more. So that, under usual circumstances, the external resistance, R′, can be neglected in comparison with R. The value of $f(\pi'')$ will depend upon the area of the mercury meniscus. It will, therefore, be possible in a given electrometer to vary the value of $\mathrm{R}f(\pi'')$ for a given solution, and (by comparison of the curves for two different positions of the meniscus) to determine whether the form of the curve is appreciably affected by the change. On the other hand, the effect may be rendered directly evident and measurable by interposing a very high resistance in the external circuit, so that, although $f(\pi'')$, and even $\mathrm{R}f(\pi'')$, may be very small, $\mathrm{R}'f(\pi'')$ will have an easily observable magnitude.

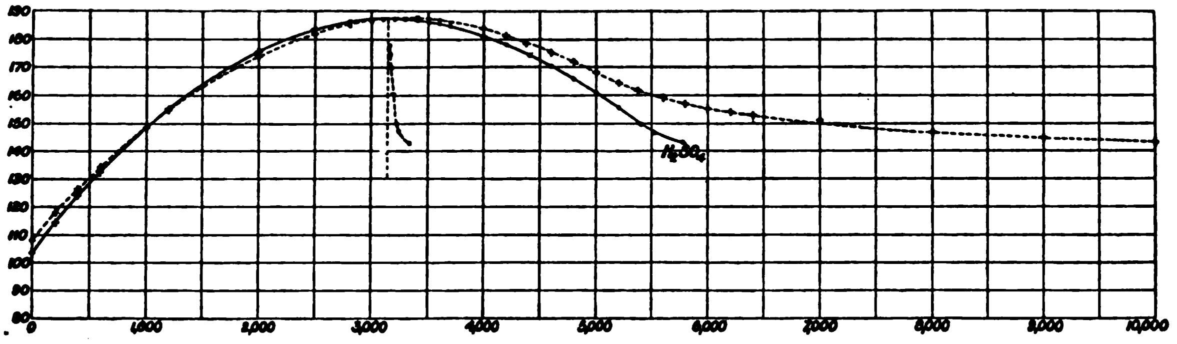
Experimental Determination of the Magnitude of the Depolarization Effect.

I have used this latter method, and the following experiments may be given in illustration of it. The high resistance consisted of graphite rulings upon ebonite, and in the experiments in question had an approximate value of 10 megohms. The arrangement was as in the diagram :—



The graphite resistance could be cut out of the circuit by means of the key K. For each E.M.F. applied, the direct reading with R′ cut out of the circuit (as in the ordinary method of determining capillary curves), was taken ; the "indirect" reading with R′ in the circuit was then observed. The results for a solution of sulphuric acid are shown in fig. 1. The ordinates are the scale readings of the summit of the mercury column of the electrometer, and the abscissæ denote the values of the resistances unplugged in the resistance box W_2. 1000 ohms correspond approximately to 0·28 volt. These depolarization experiments were made in June, 1896, with an electrometer slightly different from that described later in the paper.*

Fig. 1.



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* These experiments were described in a Dissertation presented at Trinity College, Cambridge, in August, 1896. I have since found that WIEDEBURG has also indicated, *theoretically*, the effect of depolarization. 'Wied. Ann.,' 59, 1896 (October). WIEDEBURG'S conclusion concerning the *possible* magnitude of the depolarization effect is not supported by the experimental results contained in this paper. For example, when the surface tension has its maximum value at the capillary electrode in an electrometer containing a normal solution of potassium iodide, he suggests a possible potential fall of about 0·25 volt within the electrometer solution (due to a depolarization current) as a means of reconciling certain results, mentioned later, with the ordinary Lippmann-Helmholtz theory. The actually observed depolarization current for a KI solution is far smaller than that required to substantiate WIEDEBURG'S suggestion, and, apart from this, the relations established later are at variance with his view.

The method of taking the electrometer curves and their degree of accuracy is discussed later.

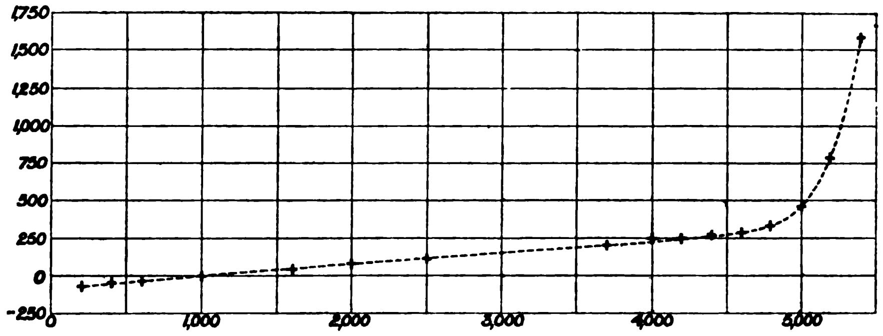
The dotted curve represents the indirect curve, while the continuous one represents the ordinary capillary curve. In this particular experiment it apparently happened that for some accidental reason (such as difference in purity of the mercury of the large and small electrodes) the "natural" potential difference at the small electrode was appreciably different from that at the large. In consequence of this, the surface tension of the small electrode began to increase immediately the connection between it and the large electrode was broken. As I have subsequently observed, this is an effect which can be obviated when due care is taken, with pure mercury and a solution of uniform concentration. The phenomenon does not affect the conclusions in the present case, but in fact rather increases the interest attaching to the observations. When the electrodes were joined by a short wire, the surface tension was different from what it was when they were joined by the graphite resistance. The reading in the latter case was not appreciably affected by reversing the resistance, so it could be assumed that the graphite did not introduce any appreciable E.M.F. into the circuit.

At first the indirect readings for a given external E.M.F. give higher values for the surface tension than the direct. The curves cut one another at an E.M.F. corresponding to about 0·28 volt, so that for this E.M.F., the surface tension assumed by the mercury is the same whether the E.M.F. is applied directly or through a very high resistance. Hence, when the surface tension at the capillary has this particular value, there can be no appreciable continuous current through the electrometer. It was found that the surface tension in question was practically identical with that assumed by the capillary, when the electrometer was disconnected from the rest of the apparatus—the electrodes being also unconnected. From this it is highly probable that this surface tension corresponded to the natural potential difference at the small electrode, and the significance of the disappearance of the depolarization current becomes immediately clear.

The horizontal distance between two points corresponding to the same surface tension, one on each curve, is a measure of the depolarization at the small electrode when the potential difference there has the value corresponding to the given surface tension. The actual value of the current is equal to the above horizontal distance (expressed in volts), divided by the value of the graphite resistance in ohms.

The curve in fig. 2 shows how the depolarization current varies with the externally applied E.M.F. The depolarization, apparently for a considerable range, is nearly proportional to the extent by which the potential difference at the small electrode has been displaced from its natural value. The ordinate of any point on this curve is the horizontal distance between a given point on the direct curve and the corresponding point on the indirect curve; the abscissa is the same as that of the given point on the direct curve. Since each of the curves has a maximum ordinate, they cross one another a second time. In the neighbourhood of the maxima the horizontal distances between the curves are not very accurately determinable; but the general nature of the depolarization curve is obvious.

Fig. 2.



The vertical dotted line in fig. 1 is drawn as nearly as possible through the highest point of the capillary curve. Immediately to the right of this line there is a dotted curve. The points required to determine this curve were obtained by bisecting the horizontal lines drawn between points on the direct curve corresponding to the same surface tension. Considering any horizontal line, therefore, the intercept made on it between the vertical line and the above dotted curve represents the extent by which a point on the descending branch departs from symmetry with the corresponding point on the ascending branch, with respect to the vertical axis. The whole curve, therefore, represents the manner in which the descending branch of the capillary curve departs from symmetry with the ascending branch, with respect to a vertical axis through the point of maximum surface tension. The internal resistance of the electrometer used in the experiment above described was certainly less than 100,000 ohms, and probably not much above 50,000 ohms. Hence the depolarization effect upon the form of the curve, due to the internal resistance of the electrometer, lay between 0·5 per cent. and 1 per cent. of the corresponding effect due to the external resistance of 10,000,000 ohms. From an examination of the horizontal distance between the ascending branches of the direct and indirect curves we therefore see that the depolarization effect (due to the internal resistance) upon the ascending branch of the direct curve is negligible within the limits of observation.

The ascending branch of the curve can, therefore, be taken to be sensibly independent of the depolarization. It is seen from the indirect curve, however, that the depolarization current continues to increase after the maximum is passed, and that, eventually, the rate of increase becomes very rapid. It is also evident from the curves that the rate at which the descending branch of the direct curve departs from symmetry with the ascending branch is very similar to the rate at which the depolarization current increases. The departure from symmetry might, therefore, be very well due to the depolarization.

In order to determine whether the depolarization is the sole cause of the observed flattening of the direct curve, an accurate knowledge of the relation between the graphite resistance and the internal resistance of the electrometer would be necessary. But an approximate knowledge of this relation is quite sufficient to show that the depolarization must soon affect the form of the curve. Thus, to take an example, the horizontal distance between the direct and indirect descending branches for a surface tension corresponding to 150 is not less than 64 horizontal divisions (see fig. 1), equivalent to a potential difference of 1600 ($= 1{\cdot}6 \times {\cdot}28$ volt), and even if the electrometer resistance is not above 50,000 ohms, the effect of the latter will be a displacement of the direct curve to the right to the extent of one-third of a horizontal division—a quite perceptible amount—due to the potential fall within the electrometer. As the actually observed displacement is considerably greater than the above, it seems probable that the flattening in the upper portion of the curve is not due to depolarization alone. This is rendered still more probable when the curve for such a substance as hydrochloric acid (in which the ascending branch is very much steeper than the descending) is considered.

Fig. 3.

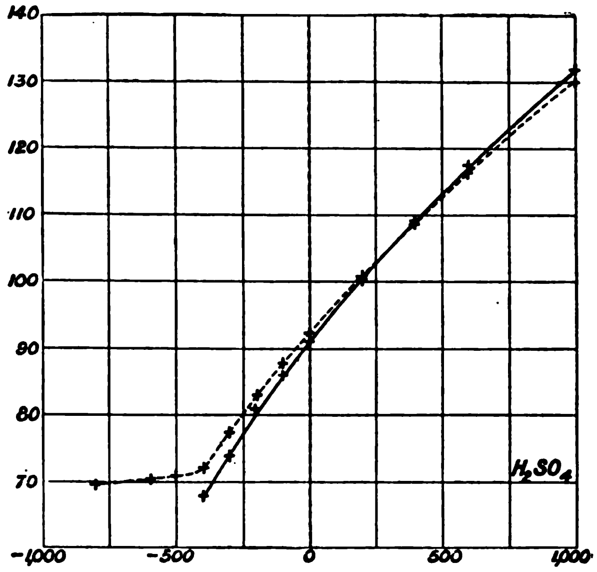
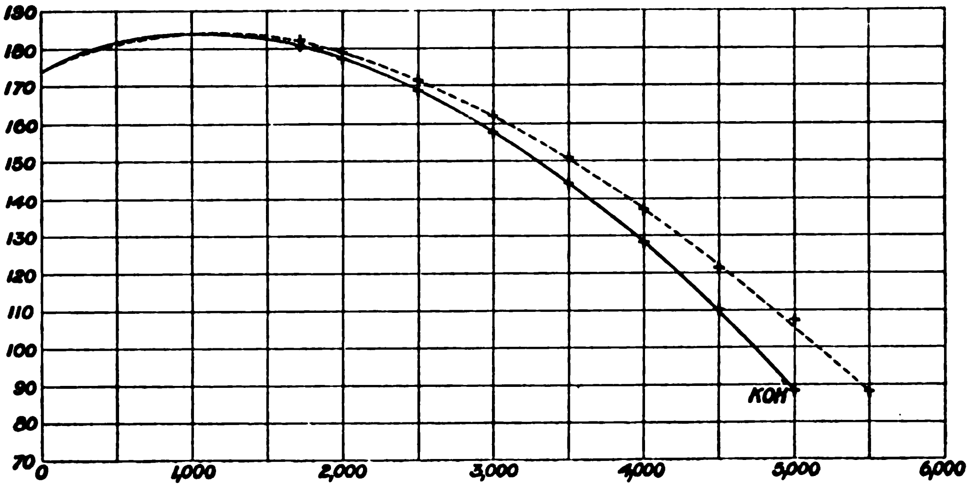


Fig. 3 gives a portion of another pair of direct and indirect curves, showing how the depolarization increases when the potential difference applied between the terminals of the electrometer is reversed in sign.

In order to show how the depolarization for kathodic polarization of the small electrode varies with the nature of the kation, curves (corresponding to those above described) are given for a dilute solution of caustic potash (fig. 4).

The curves show that in this case, for a given surface tension, the depolarization is very much less than when the kation is hydrogen of the same order of concentration. As is well known, the magnitude of the depolarization in an electrolytic cell depends not only upon the chemical nature of the ions, but also upon their concentration in the solution. This can be readily rendered obvious by the above method.

Fig. 4.



For the above solution of potash it is clear that within the range of the experiment the effect of depolarization upon the direct curve can be considered negligible. The same can be said of most of the solutions considered later.

The Second Hypothesis of the Lippmann-Helmholtz Theory.

The second hypothesis in the Helmholtz explanation of electro-capillary curves is that the electrical effect upon the surface tension is a purely electrostatic effect depending at a given potential difference upon the capacity of the electrode per unit area. It supposes that, in general, the capacity per unit area is independent of the chemical nature of the solution employed in the electrometer. From the approximately parabolic nature of some of the curves through a considerable portion of their course, the Helmholtz theory leads not only to the view that through the range considered the capacity per unit area is constant, but also makes it possible for the value of this capacity to be calculated. Assuming the inductive capacity of the dielectric of the double layer to be unity, the theory further allows an estimate to be formed of the distance between the parallel charges forming the double layer. That the distance so calculated is of the same order of magnitude as molecular distances calculated in other ways is, however, no proof of the validity of the Helmholtz view. For the distance between the layers, as so calculated, might have amounted to something very much larger without standing in opposition to other known data concerning molecular distances. In other words, there is no *à priori* objection to a view which supposes that the capacity per unit area of the common surface may really be much smaller than the Helmholtz view requires.* If such a view as this were true, the electrostatic effect would be insufficient to account for the observed variation in the surface tension. Assuming the potential difference, the existence of the electrostatic effect is scarcely open to doubt, it is only the relative importance of the effect that may be questioned. The electrostatic effect apart, the Helmholtz view assumes that the nature of the transition from the solution to the mercury is (through a considerable range) independent of the potential difference and of the nature of the solution.

Several published observations show that there are cases for which this assumption cannot be true. There are many facts in favour of the view that for a given potential difference there is a corresponding condition (of the partly physical, partly chemical kind, pictured by WARBURG) of the space bounded on the one side by the mercury, and on the other by the sensibly homogeneous solution. Obviously the surface tension will depend upon the nature of the transition through the surface layer. The mode of transition may depend only on the chemical nature of the solution and the potential difference across the space in which the transition takes place. On this view the electrostatic effect and the mode of transition for a given solution will be determined by the potential difference, and therefore the surface tension will be fixed by the potential difference. It remains to determine how the relation between the surface tension and the potential difference depends upon the chemical nature and concentration of the solution.

It is scarcely necessary now to set forth the arguments against the second hypothesis of the Helmholtz theory of the electrometer; but I shall endeavour to show by consideration of observations of the type held to throw greatest doubt upon the theory, that the first hypothesis gives results in close accord with the facts, and need not therefore be abandoned, even if the second should be proved untenable.

RELATION BETWEEN THE LIPPMANN-HELMHOLTZ THEORY AND OTHER THEORIES OF ELECTROCAPILLARY PHENOMENA.

It may be well to point out the relation such results bear to the theory of WARBURG, which is, perhaps, the most strongly advocated in opposition to the Helmholtz theory. Strictly speaking, the Warburg theory deals only with the ascending branch of the curve. It ascribes the increase in surface tension to the diminution in the concentration of a mercury salt in the neighbourhood of the capillary meniscus. According to WARBURG the effect of an E.M.F. established between the terminals of the electrometer is to convert the latter into a kind of concentration cell. Part of the E.M.F. of this cell will presumably be due to a potential difference within the electrolyte. The conclusions drawn later depend upon observations of the descending portions of capillary curves, which are usually much more definite and more accurately measurable than the ascending portions ; but unless the concentration E.M.F. within the electrolyte is supposed to be the same for quite different liquids subjected to very different degrees of polarization, there does not seem to be any simple method of reconciling the results with the Warburg theory. G. MEYER* has attempted to complete WARBURG'S theory of the phenomena by supposing that the descending portions of the curves are produced by formation of an amalgam between the mercury and the element forming the kation of the solution, while LUGGIN† has endeavoured to show that the descending branch of the curve is absent when the solution does not contain hydrogen. The experimental evidence adduced in favour of these views is mainly qualitative in nature. An extended examination of the quantitative relation between the capillary curves for differently concentrated solutions of the same salt shows that difficulties arise in the quantitative application of the idea that the surface tension in the descending branch depends only upon the concentration of the amalgam upon the electrode surface.

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* *Cf.* WARBURG, 'Wied. Ann.,' 1890, vol 41.

While it is unnecessary to deal with the nature of these difficulties at present, since they do not immediately concern the experiments first discussed, it may be pointed out that if the first hypothesis of the Helmholtz theory be true, it is possible to trace (by means of the capillary curves) the relation between the variation of the potential difference at the capillary electrode, the surface tension and the nature and concentration of the electrolyte. Probably it is only by the investigation of the relation between these quantities that the value of the capillary curves, as a method of determining "single potential differences" in voltaic phenomena, can be definitely fixed.

THE DISCREPANCY BETWEEN THE LIPPMANN-HELMHOLTZ THEORY AND THE NERNST-PLANCK THEORY OF THE POTENTIAL DIFFERENCE BETWEEN SOLUTIONS.

The result, derived from the Helmholtz theory, that the E.M.F. which must be applied between the terminals of the electrometer to cause the capillary electrode to assume its maximum surface tension, is equal to the natural potential difference between the large electrode and the solution, is so important, if true, that this E.M.F. has been observed for a large number of solutions. It is, however, impossible to test directly the validity of the numbers so found, since no other independent means of determining single potential differences has, up to the present, been discovered—if we except the dropping electrode method (which will be referred to later).

We may set up and measure the E.M.F. of a cell of the type

$$\mathrm{Hg} \;\Big|\; M_1X_1 \;\Big\vdots\; M_2X_2 \;\Big|\; \mathrm{Hg},$$

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* MEYER, 'Wied. Ann.,' 45, 1892.

† LUGGIN, 'Zeits. f. Physik. Chemie.,' 16, 1896.

where M_1X_1 and M_2X_2 signify two solutions which have been examined in the capillary electrometer; but the E.M.F. found cannot be applied to test the Helmholtz theory of the electrometer unless we know the value of the potential difference between the two solutions.

The state of our knowledge of the potential differences between liquids is not satisfactory. Within recent years, however, NERNST,[*] starting from the dissociation hypothesis, has given a theoretical investigation for the case in which the two liquids are solutions of the same salt, but of unequal concentration. PLANCK[†] has extended the investigation to the case in which the liquids are solutions of different salts. In many cases the values found experimentally agree very closely with those calculated; but it must be borne in mind that in the experiments the potential difference between the liquids only formed part of the E.M.F. actually measured. For example, in testing the formula as applied to two solutions of potassium chloride of different concentrations, it is necessary to introduce a fresh hypothesis in order to calculate the difference between the potential difference between mercury, covered with calomel, and the stronger solution, and the potential difference between mercury, covered with calomel, and the weaker solution.[‡] In order to show the nature of the agreement between the calculated values and those found experimentally, some results for KCl solutions are given below. In the first column are given the concentrations in gram equivalents per litre; in the second the observed E.M.F.s of cells of the type;

$$\mathrm{Hg}\;\Big|\;\underset{\text{dilute}}{\mathrm{HgCl}\;\;\mathrm{KCl}}\;\vdots\;\underset{\text{concentrated}}{\mathrm{KCl}\;\;\mathrm{HgCl}}\;\Big|\;\mathrm{Hg},$$

and in the third the calculated values of these E.M.F.s.

Concentrations. Gram equivalents per litre.	Observed E.M.F.	Calculated E.M.F.
	volts.	volts.
3·0 and 0·5	·0443	·0402
1·0 „ 0·1	·0533	·0525
0·5 „ 0·1	·0359	·0367
0·1 „ 0·05	·0162	·0162
0·1 „ 0·02	·0387	·0380
0·1 „ 0·01	·0545	·0548
0·05 „ 0·01	·0387	·0384

These observations form part of a series of experiments which will be described later.

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[*] NERNST, 'Zeits. f. Physik. Chemie,' 4, 1889.

[†] PLANCK, 'Wied. Ann.,' 39, 1890, and 40, 1890; *cf.* also NEGBAUR, 'Wied. Ann.,' 44, 1891.

[‡] NERNST, 'Zeits. f. Physik. Chemie,' 4, 1889.

In a cell of the type

$$\mathrm{Hg} \,\Big|\, \mathrm{M_1X_1} \,\vdots\, \mathrm{M_2X_2} \,\Big|\, \mathrm{Hg}$$

let π_{n_1} and π_{n_2} be the respective potential differences between mercury and the solutions (calculated on the Helmholtz theory of the capillary electrometer), and let π_{12} be the potential difference between the liquids and π_e the observed E.M.F. of the cell, then we should have

$$\pi_e = \pi_{n_1} - \pi_{n_2} + \pi_{12}.$$

Rothmund has made observations upon cells of this type and has found values for π_e, π_{n_1} and π_{n_2} for a number of different solutions. The following table gives the results of some of his experiments :—

M_1X_1.	M_2X_2.	π_e.	π_{n_1}.	π_{n_2}.	$\pi_e - (\pi_{n_1} - \pi_{n_2})$.
nKCl	nKI	·349	·560	·437	·226
nKCl	nKCNS	·172	·560	·534	·146
nNa$_2$S	nKCl	1·006	·560	− ·030	·416

He estimates the possible error in the determination of π_{n_1} and π_{n_2} from the curves to be not greater than 0·01 volt.*

Referring to such experiments, **Nernst** says :—“While, therefore, the Helmholtz hypothesis concerning electrocapillarity is found to be in good agreement with the osmotic theory . . . as far as the qualitative side of the phenomena goes, we come upon serious contradictions so soon as we proceed to a quantitative comparison. As results of the electrocapillary method of measuring contact potentials, we obtain the following table :—

HCl | H_2SO_4	·025	·010
KCl | HCl	·022	·028
KCl | KCNS	·161	·000
KCl | KI	·247	·000
KCl | Na_2S	·419	·000

Column 1 gives the symbols of the solutions examined ; column 2 contains the values of the potential differences between them deduced from the Helmholtz theory of the electrometer ; and column 3 gives the values of the same potential differences calculated according to the osmotic theory. The differences are great, and not explicable as errors of observation.”†

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* Rothmund, ‘Zeits. f. Physik. Chemie,’ 15.

† Nernst, ‘Wied. Ann.,’ 58, Beilage, 1896.

SOLUTIONS OF POTASSIUM CHLORIDE AND POTASSIUM IODIDE.

1. *The Potential Difference between Equally-concentrated Solutions.*

The experiments show that the Helmholtz theory of the electrometer and the Nernst-Planck calculations of the potential differences between solutions cannot both be true. While there are many facts in favour of the view that the Nernst-Planck hypothesis gives the quantitative expression for the potential difference between two solutions, there is one result calculated from the hypothesis which seems to possess greater weight than any of the others, since it would seem to be a consequence of almost any form of diffusion hypothesis. This is the result that the potential difference between equally-concentrated solutions of potassium chloride and potassium iodide is so small that in measurements of the type with which we are concerned it can be taken to be zero.

KOHLRAUSCH has investigated the electrolytic conductivity of solutions of KCl and KI for different degrees of dilution, and an examination of his numbers shows the relative amount of ionization in equally-concentrated solutions of the two salts may be considered identical when the solutions do not contain more than 0·01 gramme molecule per litre ($\frac{1}{100}$th normal). Even when the strengths correspond to a gramme molecule in 2 litres ($\frac{1}{2}$ normal) the coefficients of ionization only differ by about two per cent.* Again, according to the most recent values, the ionic velocities of chlorine and iodine are practically identical.† When, therefore, dilute solutions of KCl and KI of equal strength are brought into contact there can be no tendency of the potassium ions to diffuse, while the chlorine and iodine ions will tend to diffuse with equal velocities across the common surface. Granting the ionic hypothesis we may, therefore, safely assume that no forces which tend to alter the quantity of electricity in the unit of volume act across the surface of separation between the liquids, and that, therefore, no potential difference will arise between the liquids.

2. *The Nature of the Electrocapillary Curves for the same Solutions.*

For the reason given above I have carefully examined the relation between the capillary curves for KCl and KI in order to determine further the result of the hypothesis that the potential difference between equally-concentrated solutions of KCl and KI is zero.

(a.) *General character of electrocapillary curves.*

The behaviour of the meniscus in the capillary electrometer is in general very different in the "ascending" portion of the curve from what it is in the "descending" portion. In the ascending portion the mercury is often apparently sluggish, and the surface tension becomes difficult to measure. Further, the surface tension may take up a certain value immediately after a given potential difference is established between the terminals of the electrometer, and then fall gradually as the time of contact continues. In contrast with this, the surface tension in the descending portion of the curve is almost always very definite. Moreover, while the form of the ascending portion is widely different for solutions with chemically different anions, and even noticeably different for unequally concentrated solutions of the same salt, the form of the descending portion, for a considerable part of its course, is the same (within the limits of experimental error) for equally-concentrated solutions of quite different salts, and only varies very slightly for unequally concentrated solutions of a given salt.

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* KOHLRAUSCH, 'Wied. Ann.,' vol. 26, 1885.

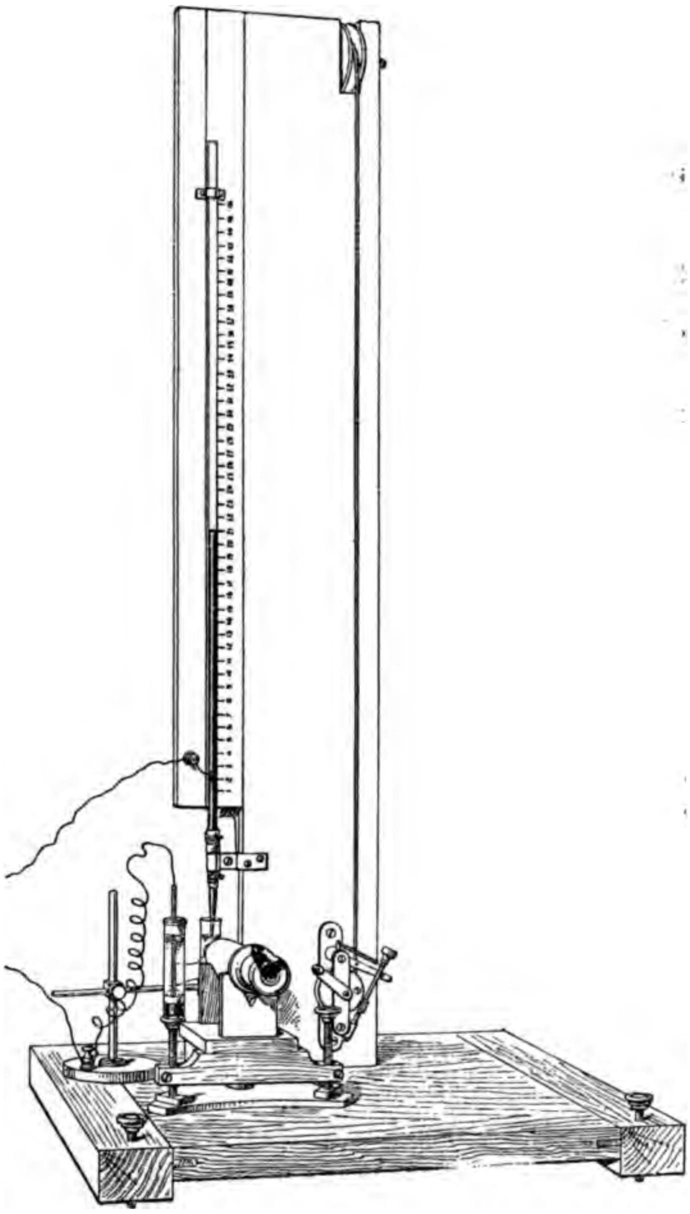
† KOHLRAUSCH, 'Wied. Ann.,' vol. 50, 1893.

(b.) *Definite nature of the descending branches.*

While therefore both the ascending and descending branches have been observed, the first conclusions are based upon observation of the descending portions of the curves. These were definite and amenable to quantitative treatment.

(c.) *Method adopted in examining the electrocapillary curves, and discussion of the degree of accuracy attainable in the experiments.*

The form of electrometer that I have used has a movable mercury reservoir in direct communication with the mercury column supported by the surface-tension effect at the capillary electrode. The mercury reservoir can be raised or lowered by means of a flexible cord, wound upon a bobbin, having a tangent-screw fine adjustment. By means of this arrangement the small electrode can be maintained at a constant position in the capillary tube. The diameter of the capillary—the same one was used in determining the curves for a large number of solutions—was about 0·003 centim., and the usual length of the column of solution between the meniscus and the point of the capillary was 0·057 centim. The resistance of such a column (supposed cylindrical) would, if the solution were normal KCl at 18°, be approximately 83,000 ohms. The position of the capillary meniscus was fixed by means of a scale within the microscope. The apparent magnitude of a division of this scale is about 1·5 millim. Every tenth division of the scale is marked. When the position of the microscope is so adjusted that the zero of the micrometer scale coincides with the end of the capillary, the fortieth division (marked 4) of the scale is practically in the centre of the field. The meniscus was always made to coincide as nearly as possible with the central division of the scale. The height of the mercury column, supported by the capillary electrode, when possessing its maximum surface tension in such a solution as dilute sulphuric acid, was about 440 millims. The variation of the surface tension was observed by means of a scale divided into millimetres placed directly behind the mercury column. The greatest error in the scale division was about 1 part in 500. The conclusions first drawn from the curves are practically independent of the accuracy of the division of the scale. The zero of this scale was about 12·4 centims. above the capillary point, so that the reading corresponding to the maximum surface tension, in dilute sulphuric acid, was about 31·6 centims. The position of the summit of the mercury column relative to the scale could be determined to within about a tenth of a millimetre. It would not be difficult to arrange for a greater degree of accuracy in this measurement; but, apart from the increase in the time occupied by the observations, which more delicate determination would involve, the above error does not exceed that introduced from other causes. Usually the solution to be examined was placed in a Clark cell tube of the Rayleigh H form.



The mercury forming the large electrode was placed at the bottom of one limb; the capillary electrode dipped into the other limb. In order that the definition of the capillary meniscus might be as good as possible, the capillary was placed close to the side of the limb that received it. After the curve for the solution had been determined, the H tube, containing it, was removed and replaced by another containing a different solution. It would have been very inconvenient to have worked always with the same H tube, as in many cases it was desirable to allow the solution to stand some considerable time over mercury before examining it in the electrometer. A number of H tubes were used, and naturally these were not precisely similar. In every case the definition of the capillary meniscus, when at the fortieth division of the scale, was made as good as possible; but it did not always happen that the definition of the capillary point was then correspondingly good. However, the maximum error in the setting of the capillary point was not greater than about one of the small scale divisions, while it could usually be set at the zero within two-tenths of a small scale division. The capillary meniscus could be readily set at the fortieth division with an error of less than one-tenth. An error of a scale division in the setting of the meniscus produced a maximum error of about ·4 millim. in the reading of the summit of the mercury column. Under ordinary circumstances, therefore, the error introduced into the surface tension observations by the necessary replacement of one H tube by another was not greater than ·08 millim.

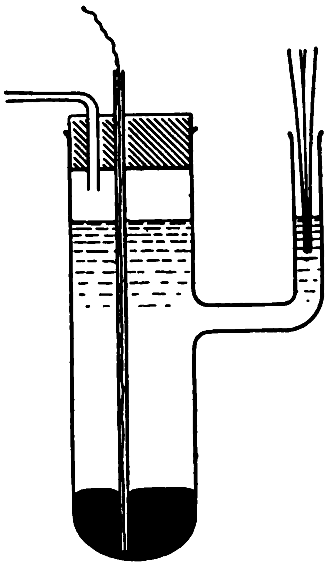
In order to remove one solution from the interior of the capillary tube before the introduction of the succeeding one, the following method of procedure was adopted. The capillary tube was immersed in a beaker of distilled water, and by alternately lowering and raising the mercury reservoir, the water could be drawn into the tube and then again expelled along with a little mercury. This process was repeated several times, the excursions of the meniscus in each case being very much longer than any that occurred during the experiments, as the result of electrical effects. The beaker was then withdrawn, and as much of the water as possible was removed from the capillary. A similar process was then adopted in order to fill the capillary with the solution next to be examined. To test the sufficiency of this treatment, a second set of observations upon a given liquid was made, several other solutions having been used in the electrometer during the interval. The first and second sets were found to agree within the limits of experimental error. In a case where one solution was followed by a more concentrated one of the same salt, the intermediate operation with distilled water was, of course, unnecessary.

The potential difference between the terminals of the electrometer was varied by means of an ordinary potentiometer arrangement. The potentiometer circuit consisted usually of a secondary cell (E.M.F. about 2·03 volts) and two resistance boxes in series. The sum of the resistances introduced into the circuit by these boxes always amounted to 10,000 ohms. Usually the resistance in each box was altered by 500 ohms at a time, so that the corresponding change in the potential difference applied between the electrometer terminals was about a tenth of a volt. The probable error of each of the box resistances was less than a tenth per cent., and the constancy of the potentiometer current was tested by means of a standard Clark cell, of which the E.M.F. at 15° was known to be within one-tenth per cent. of 1·434 volts. The accuracy of the potential measurement was therefore considerably greater than that which could be conveniently given to the surface tension observations.

(d.) *The Electrocapillary Curves for* KCl *and* KI.

1. *Preliminary Experiment.*—Before proceeding to the experiments of the ordinary capillary electrometer type, mention may be made of a simple means by which very suggestive results as to the relation between capillary curves may be obtained.

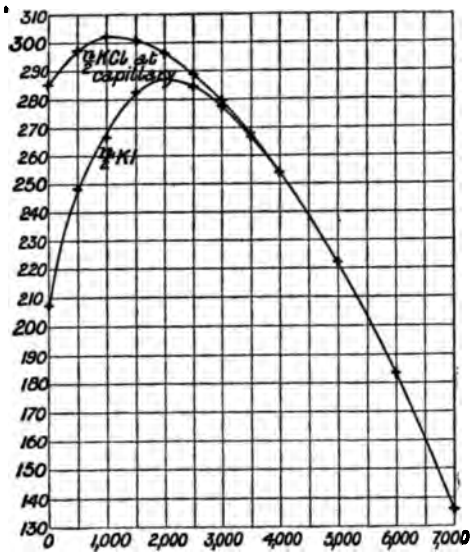
A vessel for containing the solution is constructed of the shape shown in the figure.



The mercury forming the large electrode is placed at the bottom of the main tube. The point of the capillary is brought within the smaller tube. The vessel is first filled with a solution of one of the salts (say $\frac{1}{2}n$ KI), and the capillary curve is determined. Withdrawing air from the apparatus by means of the side tube at the top of the main tube, the liquid rises in the latter and falls in the narrow tube. After the small tube has been nearly emptied, it is filled to the level of the liquid in the main tube with a solution of the other salt (say $\frac{1}{2}n$ KCl), and the capillary curve is again taken.

Fig. 5 shows the forms of the resulting curves for the solutions in question. The numbers from which the curves were constructed are as follow :—

Fig. 5.



Applied E.M.F.	$\frac{1}{2}n$ KI.	$\frac{1}{2}n$ KI with capillary in $\frac{1}{2}n$ KCl.
0	20·75	28·5 ?
500	24·9	29·75
1000	26·69	30·23
1500	28·3	30·1
2000	28·71	29·69
2500	28·47	28·88
3000	27·73	27·88
3500	26·65	26·7
4000	25·39	25·38
5000	22·2	22·2
6000	18·3	18·3
7000	13·52	13·55

The surface tensions in the ascending branches of the curves were somewhat uncertain and difficult to measure. The descending branches, however, were quite definite. It is seen that when the E.M.F. exceeded a value corresponding to the abscissa 4000 (= about ·8 volt) the curves were identical. This result can be very readily explained on the double layer view of polarization in an electrolytic cell considered at the beginning of the paper, if we assume that the potential difference between $\frac{1}{2}n$ KI and $\frac{1}{2}n$ KCl can be neglected, and that there is no appreciable concentration E.M.F. within the liquid. Since the potential difference at the large electrode has not been altered between the two sets of experiments, the potential difference ($\pi_e - \pi_n$) at the small electrode for a given applied E.M.F. will be the same for both curves. When the applied E.M.F. is less than ·8 volt., the surface tension does not depend merely upon the potential difference at the small electrode, but also upon the chemical nature of the solution. Now the solutions are the same in every respect except that the anion in one is iodine, and in the other chlorine. Until the potential difference reckoned from the solution to the electrode reaches a certain value the effect of the anion upon the surface tension (*i.e.*, in determining the mode of transition from the solution to the mercury) is appreciable; but this effect gradually diminishes and finally disappears, as is shown by the fact that the form of the curve (beyond 4000) is independent of the nature of the anion.

From this point of view it is obviously futile to consider that the highest point of the iodide curve corresponds of necessity to zero potential difference between the KI solution and the mercury electrode, since it might equally well be argued that the highest point of the curve obtained with KCl at the capillary corresponded to zero potential difference between the mercury and the KCl. If there is no appreciable potential difference between the KI and the KCl both results cannot be true. The potential differences in the two cases (maximum surface tension) must differ by about 0·2 volt, or else the potential difference between $\frac{1}{2}n$ KI and $\frac{1}{2}n$ KCl must be about 0·2 volt.

In the face of evidence that there is a chemical effect of the anion upon the surface tension, and that this effect increases as the potential of the liquid with respect to the electrode decreases, it does not seem advisable to say more than that the potential difference (reckoned from the solution to the electrode) is considerably less at the maximum surface tension when the solution is KCl than when it is KI. The marked depression of the maximum value of the surface tension observable in the case of potassium iodide solutions is one of the characteristic features of the curves dealt with by ROTHMUND—the actual fact of the depression was apparently first noticed by GOUY;[*] but the depression is really a perfectly general phenomenon. The amount of depression depends upon the concentration of the solution as well as upon its chemical nature. The depression for concentrated solutions of chlorides is very pronounced, and for dilute solutions it can readily be observed that the maximum value of the surface tension rises as the concentration diminishes. It is obviously an effect which does not depend upon the density of the solution. For example, a saturated solution of caustic potash (which is soluble in about half its weight of water) has as high a maximum surface tension as a half-normal solution of potassium chloride. The effect of the ions (apart from the electrostatic effect) upon the surface tension would appear to depend, for a given potential difference, upon their nature and concentration in the solution. Whether the surface tension in the neighbourhood of the maximum is ever controlled by the electrostatic effect alone, depends (on this view) upon whether, when the potential difference between the solution and the electrode is small, the nature and concentration of the ions is such that their non-electrical effect upon the surface tension can be neglected.

In the case above considered the curves are identical when the applied E.M.F. exceeds 0·8 volt. The subsequent variation of the surface tension is therefore presumably independent of the nature of the anion; but it is obvious that we obtain no information from the curves as to whether their subsequent course is free from any non-electrostatic influence depending upon the kation. For the kation is the same and of the same concentration in the two solutions. It is, however, easy to extend the above observations so as to show that (granting the Nernst calculation gives at least approximately the potential difference between unequally concentrated solutions of the same salt), although the *form* of the lower portion of the descending curve varies very little with the strength of the solution, yet the surface tension for a given potential difference depends upon the strength of the solution. From this it would appear that the surface tension does not depend upon the electrostatic effect alone even when the anion effect has presumably disappeared; but that, in fact, there is also a kation effect which becomes evident as the solution becomes increasingly positive with regard to the electrode.

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[*] 'Comptes Rendus,' vol. 114, 1892.

2. *Final experiments showing the agreement of the first hypothesis of the Lippmann-Helmholtz theory with the Nernst-Planck theory of the potential difference between* KCl *and* KI.—We may, however, first apply to the ordinary electro-capillary curves for equally concentrated solutions of KI and KCl, the result suggested by the curves already given, that ultimately the descending branch of either curve is practically unaffected by the nature of the anion, and that if it is then influenced by the kation, the nature of the influence is such that, in equally concentrated solutions of salts possessing the same kation, the potential difference for a given surface tension is the same in both solutions. It is found that the descending branches eventually approximate very closely to parallelism. Considering the parallel portions, let π_e be the E.M.F. required to be applied between the terminals to produce a given surface tension for the KCl solution, and let π_e' be the E.M.F. required to produce the same surface tension for the KI solution. Then $\pi_e - \pi_e'$ is very approximately constant. Let $-\pi_n$ be the natural potential difference between the KCl solution and mercury (the electrode being considered positive to the solution), and let $-\pi_n'$ be the corresponding quantity for the KI solution. Then on the first hypothesis of the ordinary electrometer theory (applicable to any electrolytic cell), the potential differences between the solution and the capillary for the two points of equal surface tension (one on each curve) are

$$\pi_e - \pi_n \qquad \text{and} \qquad \pi_e' - \pi_n'$$

respectively. Now if we suppose the potential difference is the same in the two cases because the surface tension is the same, we get

$$\pi_e - \pi_n = \pi_e' - \pi_n'$$

or

$$\pi_n - \pi_n' = \pi_e - \pi_e' = a,$$

where a, an observable quantity, is represented by the horizontal distance between the parallel portions of the curves. Let now a cell be constructed of the form

$$\mathrm{Hg}\,\Big|\,\mathrm{KCl}\,\Big\vdots\,\mathrm{KI}\,\Big|\,\mathrm{Hg}$$

and let its observed E.M.F. be b, and suppose π_l is the potential fall from the KCl to the KI solution. Then

$$\pi_n - \pi_n' + \pi_l = b.$$

If $\pi_l = 0$, we must have

$$a = b.$$

The following experiments show the observed relation between a and b.

E.M.F. applied to electrometer [1000 = ·202 volt].	Surface tension readings.		Horizontal distance between parallel portions of curves. (*a.*)	Calculated E.M.F. of cell $\mathrm{Hg}\,\vert\,\mathrm{KCl}\,\vdots\,\mathrm{KI}\,\vert\,\mathrm{Hg}$ assuming P.D. between KCl and KI negligible.	Observed E.M.F. of cell $\mathrm{Hg}\,\vert\,\mathrm{KCl}\,\vdots\,\mathrm{KI}\,\vert\,\mathrm{Hg}$. (*b.*)
	$\frac{n}{2}$ KCl.	$\frac{n}{2}$ KI.		volt.	
0	23·5	20·39			
500	26·0	24·8			
1000	28·0	27·1			
1500	29·21	28·29			
2000	30·0	28·71			
2500	30·49	28·5	1975 ± 6		1950 (max.) = ·3940 volt.
3000	30·6	27·81			
3500	30·29	26·78		·3990 ± ·0012	
4000	29·69	25·48			
4500	28·90	24·0			
5000	27·9	22·35			
5500	26·75	20·51			
6000	25·41	18·55			
6500	23·95				
7000	22·29				
				Calculated E.M.F. on ordinary Helmholtz theory, assuming P.D. between KCl and KI to be zero.	
				·162 volt (approx.)	

<table>
<tr><th rowspan="2">E.M.F. applied to electrometer [1000 = ·2027 volt].</th><th colspan="2">Surface tension readings.</th><th rowspan="2">Horizontal distance between parallel portions of curves.
(a.)</th><th rowspan="2">Calculated E.M.F. of cell Hg | KCl | KI | Hg assuming P.D. between KCl and KI to be negligible.</th><th rowspan="2">Observed E.M.F. of cell Hg | KCl | KI | Hg.
(b.)</th></tr>
<tr><th>$\frac{n}{10}$ KCl.</th><th>$\frac{n}{10}$ KI.</th></tr>
<tr><td>0</td><td>24·78</td><td>19·9 ?</td><td></td><td>volt.</td><td></td></tr>
<tr><td>500</td><td>27·40</td><td>25·6</td><td></td><td></td><td></td></tr>
<tr><td>1000</td><td>29·0</td><td>28·34</td><td></td><td></td><td></td></tr>
<tr><td>1500</td><td>30·2</td><td>29·73</td><td></td><td></td><td></td></tr>
<tr><td>2000</td><td>30·65 ?</td><td>30·3</td><td></td><td></td><td></td></tr>
<tr><td>2500</td><td>31·33</td><td>30·2</td><td></td><td></td><td></td></tr>
<tr><td>3000</td><td>31·41</td><td>29·57</td><td>1725 ± 6</td><td></td><td>1729 ± ·5
=
·3503 volt.</td></tr>
<tr><td>3500</td><td>31·2</td><td>28·6</td><td></td><td></td><td></td></tr>
<tr><td>4000</td><td>30·7</td><td>27·49</td><td></td><td>·3496
± ·0012</td><td></td></tr>
<tr><td>4500</td><td>29·99</td><td>26·13</td><td></td><td></td><td></td></tr>
<tr><td>5000</td><td>29·1</td><td>24·65</td><td></td><td></td><td></td></tr>
<tr><td>5500</td><td>28·0</td><td>22·99</td><td></td><td></td><td></td></tr>
<tr><td>6000</td><td>26·72</td><td>21·11</td><td></td><td></td><td></td></tr>
<tr><td>6500</td><td>25·33</td><td>19·13</td><td></td><td></td><td></td></tr>
<tr><td>7000</td><td>23·79</td><td>16·9</td><td></td><td></td><td></td></tr>
<tr><td>7500</td><td>22·0</td><td>14·43</td><td></td><td></td><td></td></tr>
<tr><td>8000</td><td>20·01</td><td>11·73</td><td></td><td></td><td></td></tr>
<tr><td>8500</td><td>17·9</td><td></td><td></td><td></td><td></td></tr>
<tr><td>9000</td><td>15·6</td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td colspan="2">E.M.F. for maximum surface tension</td><td></td><td rowspan="2">Calculated E.M.F. of cell assuming P.D. between KI and KCl zero, on ordinary Helmholtz theory</td><td></td></tr>
<tr><td></td><td>$\frac{n}{10}$ KCl.</td><td>$\frac{n}{10}$ KI.</td><td></td><td></td></tr>
<tr><td></td><td>2800 ± 50</td><td>2200 ± 50</td><td></td><td>·122 (approx.)</td><td></td></tr>
</table>

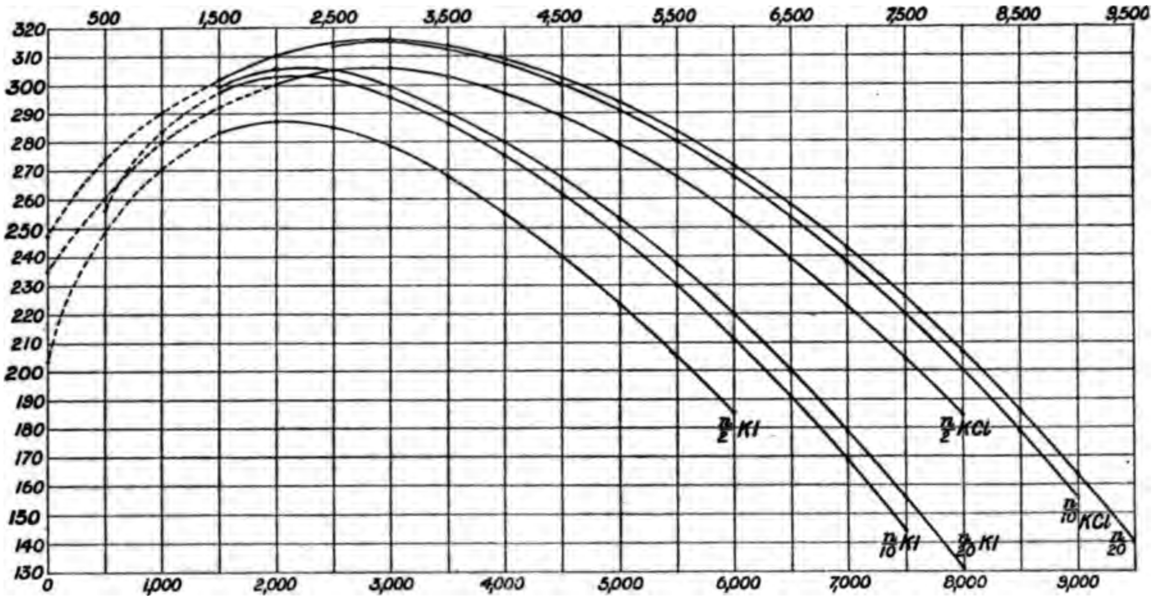
E.M.F. applied to electrometer [1000 = ·2027 volt].	Surface tension readings.		Horizontal distance between parallel portions of curves. (*a*.)	Calculated E.M.F. of cell Hg | KCl ⋮ KI | Hg assuming P.D. between KI and KCl zero.	Observed E.M.F. of cell Hg | KCl ⋮ KI | Hg. (*b*.)
	$\frac{n}{20}$ KCl.	$\frac{n}{20}$ KI.		volt.	
0	25·1 ?	19·95 ?			
500	27·4 ?	25·3			
1000	29·0 ?	28·35			
1500	30·18	29·92			
2000	30·99	30·57			
2500	31·45	30·49	1663 ± 6		1668 ± 0·5
3000	31·5	29·9			=
3500	31·3	29·0			
4000	30·85	27·98		·3371	·3381 volt.
4500	30·2	26·72		± ·0012	
5000	29·38	25·31			
5500	28·32	23·78			
6000	27·13	22·0			
6500	25·79	20·0			
7000	24·28	17·9			
7500	22·6	15·6			
8000	20·7	13·1			
8500	18·62				
9000	16·4				
9500	14·0				
	E.M.F. corresponding to maximum surface tension.			Calculated E.M.F. on ordinary Helmholtz theory, assuming P.D. between KCl and KI negligible.	
	$\frac{n}{20}$ KCl.	$\frac{n}{20}$ KI.			
	2950 ± 50	2150 ± 50		·162 (approx.)	

The corresponding curves are shown in fig. 6. In order to find *a*, the horizontal distances were measured directly from the curves, as it was more convenient to do this than to observe the difference between the E.M.F.s required to produce the same surface tension in the different solutions. The E.M.F. of the cell was determined immediately after the corresponding curves had been observed. The same potentiometer circuit was used in the determination of the curves and of the cell E.M.F. The constancy of this circuit was checked by means of a standard cell.

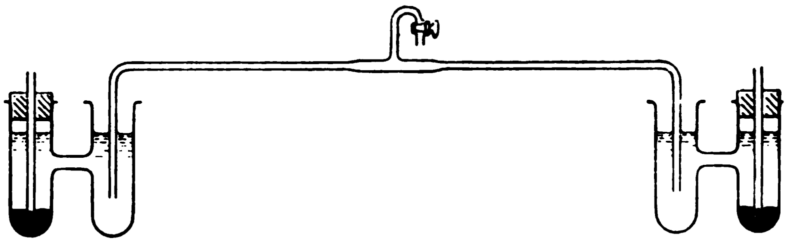
The curves for the half normal solutions are not strictly comparable with the others; they were taken (with the same electrometer) about a year before the latter, under slightly different conditions. Although in the case of the later KCl solutions the mercury was covered with a layer of calomel, it was not necessary to add a

" depolariser."* When any one of the above solutions has stood for several hours over mercury, the electro-capillary curve determined from it can be treated as independent of the time, since it remains constant for a much longer period than a complete set of observations occupies. The E.M.F. of the Hg | KCl ⋮ KI | Hg cell was determined by fixing the pair of H cells at a suitable distance apart, and adjusting so that the

Fig. 6.



solutions in the cells were at the same level. The tubes were then joined across by a capillary siphon (about 1 sq. millim. in cross-section) of which the total length was about 60 centims. Contact within the cell was made by carefully drawing the solutions to the centre of the siphon. Care was further taken to avoid any considerable shaking of the mercury surfaces. The diagram following represents the cell.



The numbers given show that, for dilute solutions at any rate, the double-layer theory gives results in complete accord with the Nernst-Planck view of the potential difference between KCl and KI. If it be considered that the former is *à priori* the more probable of the two hypotheses, the results may be taken to corroborate the views of NERNST, and the method may be further employed to test directly the potential differences between equally concentrated salts of the same metal. It will be shown below that there are apparently cases in which the method can still be applied even when the metals of the solutions are different.

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* *Cf.* NERNST, 'Zeits. f. Physik. Chemie,' 4, 1889.

The agreement for the half-normal solutions is less complete than for the more dilute solutions. This may be partly due to the fact that the observed E.M.F. of the cell was more uncertain in this case than in the others; but a much more probable explanation lies in the fact that the difference between the coefficients of ionization for KI and KCl respectively becomes appreciable when the solutions are not dilute. Before examining the effect of such a difference further, I will give the results ot experiments upon some of the other solutions mentioned by NERNST.

EXAMINATION OF OTHER KNOWN DISCREPANCIES BETWEEN THE LIPPMANN-HELMHOLTZ THEORY AND THE NERNST-PLANCK THEORY OF THE POTENTIAL DIFFERENCES BETWEEN SOLUTIONS.

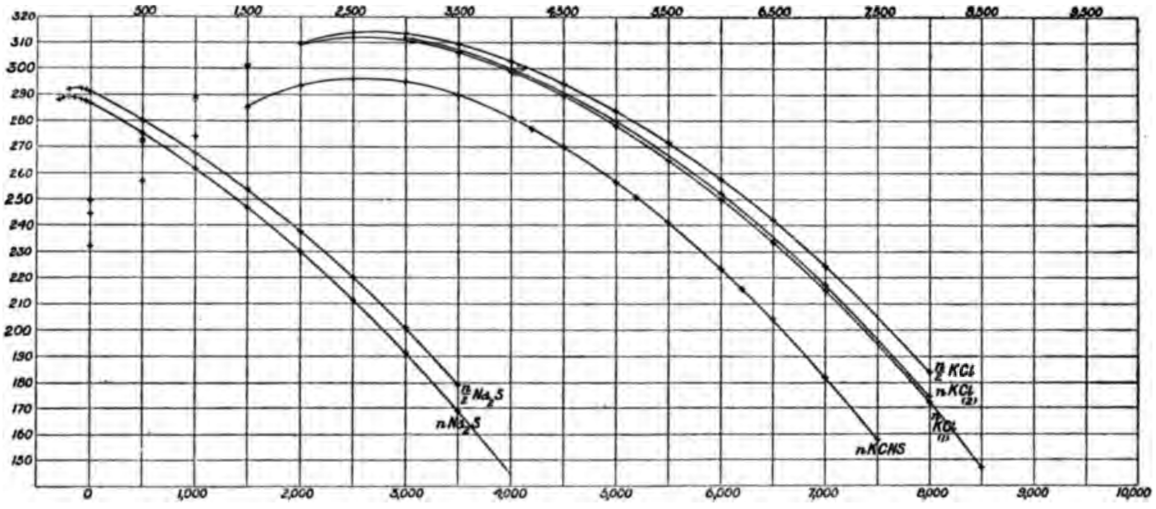
The tables which follow give the results of experiments conducted in a similar way to those already described.

E.M.F. applied to electrometer [1000 = ·2037 volt].	Surface tension Readings.		Mean horizontal distance between curves. (*a.*)	Calculated E.M.F. of Hg │ KCl ⋮ KCNS │ Hg assuming P.D. between KCl and KCNS to be negligible.	Observed E.M.F. of cell Hg │ KCl ⋮ KCNS │ Hg. (*b.*)
	n KCl.	*n* KCNS.			
0	24·75	23·2		volt.	
500		25·7			
1000	28·92	27·4			
1500		28·59			
2000	30·92	29·34			
2500		29·62			
3000	31·2	29·5			
3500		29·0			
4000	30·01	28·14			
4500		27·02			
5000	27·99	25·66	830 ± 6		820 ± 10
5500		24·1			=
6000	25·2	22·33		·1691	·1670 volt.
6500		20·4		± ·0012	± ·0020
7000	21·7	18·2			
7500		13·2			
8000	17·4	10·39			
8500					
9000		7·3			
4200		27·7			
5200		25·05			
6200		21·57			

E.M.F. applied to electrometer [1000 = ·204 volt].	Surface tension readings.		Mean horizontal distance between curves. (*a.*)	Calculated E.M.F. of Hg | KCl ⋮ Na_2S | Hg assuming P.D. between KCl and Na_2S to be negligible.	Observed E.M.F. of cell. Hg | KCl ⋮ Na_2S | Hg (*b.*)
	n KCl.	*n* Na_2S.			
- 300		28·8 ?		volt.	
- 250		28·89			
- 200		28·91			
- 150		28·9			
- 100		28·84			
- 50		28·78			
0	24·94	28·69			
500	27·3	27·5			
1000	28·95	26·18			
1500	30·1	24·68	4600 ± 20		4700 (max.)
2000	30·85	23·01			=
2500	31·21	21·15		·9384 ± ·0041	·9588 volt.
3000	31·11	19·15			
3500	30·63	16·89			
4000	29·9	14·4			
4500	28·97	11·75			
5000	27·81			Calculated E.M.F. of cell, on ordinary Helmholtz theory, assuming P.D. between liquids zero	
5500	26·5				
6000	25·01				
6500	23·39				
7000	21·5				
7500	19·41				
8000	17·19				
8500	14·7			·581 (approx.)	
9000	12·0				

E.M.F. applied to electrometer [1000 = ·204 volt].	Surface tension readings.		Mean horizontal distance between curves. (*a*.)	Calculated E.M.F. of Hg $\mid$ KCl ⋮ Na_2S $\mid$ Hg assuming P.D. between KCl and Na_2S to be negligible.	Observed E.M.F. of cell. Hg $\mid$ KCl ⋮ Na_2S $\mid$ Hg (*b*.)
	$\frac{n}{2}$ KCl.	$\frac{n}{2}$ Na_2S.			
				volt.	
− 200		29·19			
− 100		29·23			
0	24·43	29·1			
500	27·2	28·02			
1000	28·89	26·79			
1500	30·1	25·39			
2000	30·96	23·8	4625 ± 25		4695
2500	31·4	22·0			=
3000	31·33	20·08		·9435	·9578 volt.
3500	30·95	17·91		± ·0051	
4000	30·29	15·59 ?			
4500	29·41				
5000	28·39				
5500	27·18			Calculated E.M.F. of cell on Helmholtz theory, assuming P.D. between KCl and Na_2S zero.	
6000	25·78				
6500	24·2				
7000	22·43				
7500					
8000	18·37				
8500					
9000	13·41			·581 (approx.)	

Fig. 7.



The corresponding curves are given in fig. 7. In the case of the curves for potassium sulphocyanide and chloride it is clear that the results again agree with the hypothesis that in these solutions, when the potential difference between the solution and the meniscus exceeds a certain value, the surface tensions are the same when the potential differences are equal. Conductivity data for KCNS are not available, so that it is impossible to say whether the ionic concentration is strictly the same for the two solutions; but if we may argue from the values for corresponding solutions of HCl and HCNS,* it would appear probable that the degree of dissociation of equally concentrated solutions of the salts is practically the same. The ionic velocities of the anions would also seem to be about the same.

THE RELATION BETWEEN THE NATURE OF THE KATION OF THE SOLUTION AND THE FORM OF THE ELECTROCAPILLARY CURVE.

Before the argument previously employed can be applied in the case of salts like KCl and Na_2S possessing different kations, it is necessary to assume that the effect of the kation upon the electro-capillary curve is independent of its chemical nature. The observations upon KCl, NaCl and HCl described below show that this is probably true. But apart from this there are uncertainties in the above experiments with sodium sulphide from which the others are free. It is difficult to prepare a standard solution of sodium sulphide. The solutions used were only approximately normal and half-normal. But besides this, the degree of dissociation of a normal sodium sulphide solution is probably considerably less than that of a normal potassium chloride solution. Hence the potential difference between the solutions is uncertain. Further, if the Na_2S solution is less concentrated than the corresponding KCl solution, the Na_2S descending curve will be relatively nearer the KCl descending curve than it would be if the ionic concentrations were equal (see below). This in itself might suffice to explain why the horizontal distance between the curves is less than that corresponding to the observed E.M.F. of the cell.

While it seems reasonable to suppose that if the uncertainity concerning the sodium sulphide were eliminated, the agreement would be as close as in the other cases, the experiments seem to leave little doubt as to which of the numbers quoted by NERNST (0·416 volt and zero) is nearer the true value of the potential difference between equally concentrated solutions of KCl and Na_2S.

Experiments with equally concentrated Solutions of Potassium and Sodium Chlorides.

The following experiments show the probability that for equally concentrated solutions of potassium and sodium salts the kation produces the same effect upon the surface tension.

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* OSTWALD, 'Journ. f. Prakt. Chemie,' 32, 1885.

E.M.F. applied.	Surface tension readings.					
	$\frac{n}{2}$ KCl.	$\frac{n}{2}$ NaCl.	$\frac{n}{10}$ KCl.	$\frac{n}{10}$ NaCl.	$\frac{n}{20}$ KCl.	$\frac{n}{20}$ NaCl.
0	24·09	24·31	24·78 ?	24·99	25·1 ?	25·22
500	26·99	27·08	27·4	27·42	27·4 ?	27·36 ?
1000	28·71	28·76	29·0	29·0	29·0 ?	28·9 ?
1500	29·95	30·01	30·2	30·21	30·18	30·1
2000	30·85 ?	30·89	30·65	31·0	30·99	30·95
2500	31·25	31·31	31·33	31·44	31·45	31·41
3000	31·25	31·3	31·41	31·47	31·5	31·5
3500	30·9	30·93	31·2	31·2	31·3	31·27
4000	30·28	30·31	30·7	30·7	30·85	30·81
4500	29·42	29·5	29·99	30·0	30·2	30·19
5000	28·41	28·5	29·1	29·11	29·38	29·38
5500	27·27	27·3	28·0	28·10	28·32	28·35
6000	25·9	25·95	26·72	26·83	27·13	27·18
6500	24·32	24·4	25·33	25·43	25·79	25·83
7000	22·62	22·7	23·79	23·89	24·28	24·31
7500	20·7	20·82	22·0	22·15	22·6	22·7
8000	18·6	18·7	20·01	20·22	20·7	20·81
8500	16·33	16·5	17·9	18·15	18·62	18·79
9000	13·80	14·01	15·6	15·89	16·4	16·59
9500	...	...	...	...	14·0	14·2
Mean temperature.	17°	16°	17°	17°	16·7°	17°

Fig. 8 shows the electro-capillary curves for half-normal, one-tenth normal, and one-twentieth normal solutions of potassium and sodium chlorides. In every case the mercury of the large electrode was covered with a layer of calomel before the introduction of the solution into the electrometer cell. The appreciable departure of the curves from one another at the lower surface tensions may be partly due to the appreciable effects of depolarization, since the conductivity of a KCl solution is about 20 per cent. greater than that of the corresponding NaCl solution. Apart from this departure at lower surface tensions, it is seen that the curves for corresponding solutions coincide within the limits of experimental error.

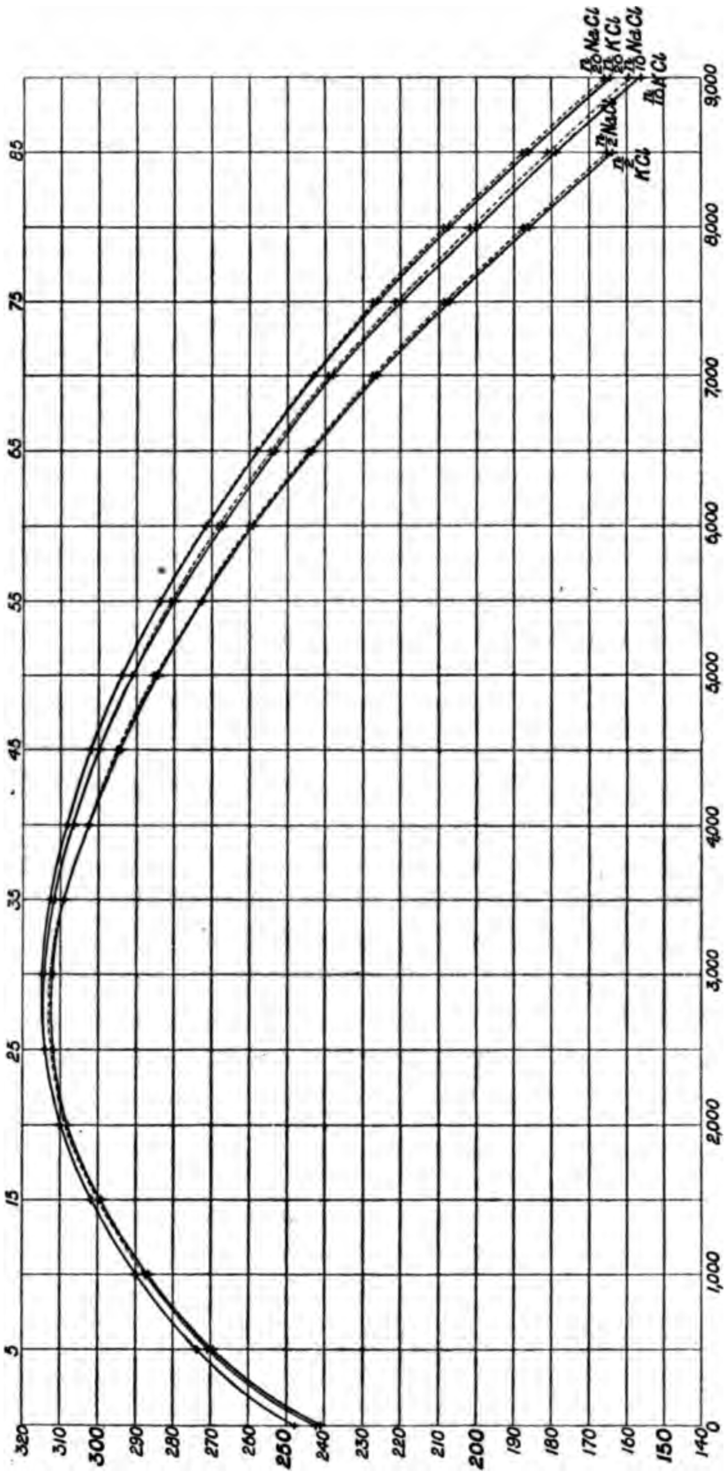
If we are to assume that the kations K and Na affect the surface tension in the same way, the E.M.F. of the cell of the type

$$\mathrm{Hg}\;\left|\;\begin{array}{rl} & \mathrm{NaCl} \\ \mathrm{HgCl} & \end{array}\;\vdots\;\begin{array}{lr} \mathrm{KCl} & \\ & \mathrm{HgCl} \end{array}\;\right|\;\mathrm{Hg}$$

should be merely the potential difference between the solutions.

The following are the calculated potential differences between the solutions and the observed E.M.F.s of the cells :—

Fig. 8.


$\frac{n}{20}$ NaCl
$\frac{n}{20}$ KCl
$\frac{n}{10}$ NaCl
$\frac{n}{10}$ KCl
$\frac{n}{2}$ NaCl
$\frac{n}{2}$ KCl
5
15
25
35
45
55
65
75
85
0
1,000
2,000
3,000
4,000
5,000
6,000
7,000
8,000
9,000
320
310
300
290
280
270
260
250
240
230
220
210
200
190
180
170
160
150
140

		Observed.	Calculated.
$\frac{n}{10}$ NaCl	$\frac{n}{10}$ KCl	·0050	·0045
$\frac{n}{20}$ NaCl	$\frac{n}{20}$ KCl	·0045	·0045

From these results we see that, assuming the potential difference between the liquids is as calculated, the respective effects of Na and K on the electro-capillary curves are practically identical. To examine the effect when hydrogen is the kation, I have also performed similar experiments with hydrochloric acid of different strengths. The curves are practically the same as those for NaCl and KCl, with the exception that the hydrochloric acid curves eventually flatten out in a similar way to the sulphuric acid curve described earlier, and doubtless for a similar reason. Subsequently I hope to describe how the surface tension for a given potential difference depends upon the nature and concentration of the kation in the general case.

Relation between the Surface Tension for a Given Potential Difference and the Concentration of the Solution Employed in the Electrometer.

Following the same hypotheses as have been adopted above, it is easy to see, from the curves just given, the nature of the effect of the kation upon the surface tension. The surface tension for a given potential difference would seem to increase as the concentration of the solution diminishes, or conversely, for a given surface tension the potential fall from the solution to the electrode would seem to increase with diminishing concentration. Thus, to take the potassium chloride curves, suppose π_{n_2}, $\pi_{n_{10}}$, and $\pi_{n_{20}}$ are the natural potential differences between the large electrode and the solution (reckoned from the electrode to the solution) in the $\frac{n}{2}$, $\frac{n}{10}$, and $\frac{n}{20}$ solutions respectively. Considering the parallel portions of the curves, let π_{e_2}, $\pi_{e_{10}}$, and $\pi_{e_{20}}$ be the E.M.F.s required to be applied between the terminals in the respective cases, in order to produce a given surface tension. If the surface tension depends only on the electrostatic effect we should have

$$\pi_{e_2} - \pi_{n_2} = \pi_{e_{10}} - \pi_{n_{10}} = \pi_{e_{20}} - \pi_{n_{20}},$$

assuming, as in the previous cases, that there are no appreciable potential falls within the solutions, or that, if present, they are the same for all.

Suppose, now, that we measure the E.M.F.s of the cells

$$\text{Hg} \,\Big|\, \text{HgCl}\,\tfrac{n}{20}\,\text{KCl} \,\vdots\, \tfrac{n}{10}\,\text{KClHgCl} \,\Big|\, \text{Hg}$$

and

$$\text{Hg} \,\Big|\, \text{HgCl}\,\tfrac{n}{10}\,\text{KCl} \,\vdots\, \tfrac{n}{2}\,\text{KClHgCl} \,\Big|\, \text{Hg}$$

and find them to be π_2 and π_1 respectively. Then

$$\pi_2 = \pi_{n_{20}} - \pi_{n_{10}} + \pi_{21},$$

and

$$\pi_1 = \pi_{n_{10}} - \pi_{n_2} + \pi_{51},$$

where π_{21} and π_{51} are the potential differences between the unequally-concentrated solutions of KCl. Calculating the values of these by NERNST'S formula we get

$$\pi_{21} = -\cdot 0003 \text{ and } \pi_{51} = -\cdot 0007.$$

The observed E.M.F.s of the cells were

$$\pi_2 = \cdot 0162 \text{ and } \pi_1 = \cdot 0359,$$

from which we see that (assuming NERNST'S formula)

$$\pi_{n_{20}} - \pi_{n_{10}} = \cdot 0165,$$

and

$$\pi_{n_{10}} - \pi_{n_2} = \cdot 0366.$$

Now, from the curves

$$\pi_{e_{20}} - \pi_{e_{10}} = \cdot 0331\ (\pm \cdot 0006)$$

and

$$\pi_{e_{10}} - \pi_{e_2} = \cdot 0670\ (\pm \cdot 0006).$$

Hence, subtracting corresponding equations,

$$(\pi_{e_{20}} - \pi_{n_{20}}) - (\pi_{e_{10}} - \pi_{n_{10}}) = \cdot 0166$$

and

$$(\pi_{e_{10}} - \pi_{n_{10}}) - (\pi_{e_2} - \pi_{n_2}) = \cdot 0304,$$

whereas, if the surface tension variation had been due to a purely electrostatic effect,* we should have expected

$$(\pi_{e_{20}} - \pi_{n_{20}}) - (\pi_{e_{10}} - \pi_{n_{10}}) = 0$$

and

$$(\pi_{e_{10}} - \pi_{n_{10}}) - (\pi_{e_2} - \pi_{n_2}) = 0.$$

Disagreements of a similar kind are found when solutions of other strengths are examined. The same result can be very easily illustrated by means of the apparatus first described. Fig. 9 shows a curve for $\frac{1}{2}n$ KI, and another for the same solution, with the exception that the $\frac{1}{2}n$ KI, in which the capillary was first immersed, was replaced by $\frac{1}{2}n$ KNO_3, the degree of dissociation of which is 16 per cent. less than that of $\frac{1}{2}n$ KI. The calculated potential difference between $\frac{1}{2}n$ KNO_3 and $\frac{1}{2}n$ KI is rather less than ·001 volt—an amount which would be barely perceptible on the scale to which the curves are drawn—so that the appreciable separation between the parallel portions of the curves is most probably due in part to the effect of the different concentrations of the potassium ions upon the surface tension.

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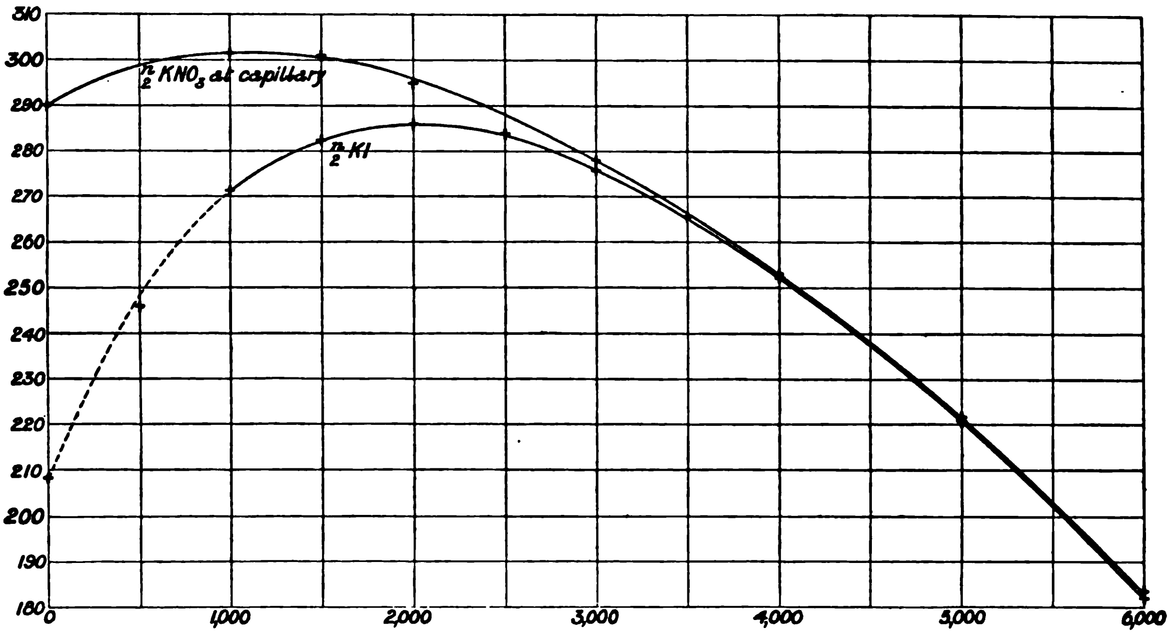
* It can also be shown from the above and similar experiments that the surface tension variation in the descending branch cannot, in general, be a purely amalgam effect.

Fig. 9.



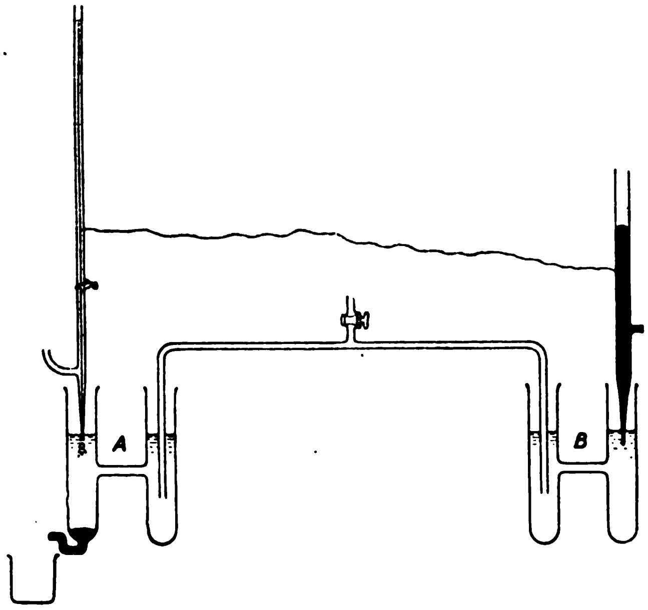
Since it is beyond the purpose of the present paper to deal with the quantitative nature of the possible ion effects upon the surface tension, it will be unnecessary to give further instances which show that the observed surface tension is, in general, partly determined by the chemical nature of the solution. It may, however, be remarked that there is nothing unusual in the supposition that as the potential difference reckoned from the solution to the electrode increases, the kation effect becomes pronounced, while conversely, when it diminishes, the anion effect becomes increasingly evident. For if the potential difference be altered in the above sense, through a sufficient range, we get an obvious combination of the mercury with the anion (and not simply a molecular layer of the compound) on the one hand, and with the kation (amalgam) on the other. Whether the formation of these compounds proceeds suddenly when the potential differences between the solution and the electrode reach certain amounts, or whether they are gradually led up to through intermediate stages in which the surface films contain only molecular quantities of the compounds, of gradually-increasing concentration, is a question concerning the answer to which we have imperfect knowledge.

Relation between the Electrocapillary Curves for KCl and KI and Dropping Electrode Measurements for Solutions of these Salts.

There is a point in connection with the explanation of dropping electrode phenomena which, as it is closely connected with the question of the potential difference between KCl and KI, may perhaps be mentioned here. If it be granted that the potential difference between $\frac{1}{2}n$ KCl and $\frac{1}{2}n$ KI can be considered negligible, it follows from the experiments described below that the potential difference between dropping mercury and KCl (under the conditions described by Paschen) is very different from that between dropping mercury and KI. And, at the same time, it follows that if the potential difference between a KCl solution and mercury is zero when the surface tension is a maximum, then, when the latter is a maximum in KI solution, the potential difference from the solution to the electrode has a considerable magnitude.

The result obtained by Paschen[*] (following up the experiments of Ostwald[†]) for chlorides and iodides, among other salts, that the E.M.F. required to produce the maximum surface tension for a given solution is practically identical with the E.M.F. of a cell containing the same solution and having as electrodes a similar large mercury electrode to that used in the electrometer, and a dropping electrode of which the jet becomes discontinuous in the surface of the solution seems, at first sight, a striking confirmation of the Helmholtz theory of the electrometer. But, in fact, it is sufficient for this result that the potential difference between mercury and the solution when the surface tension is a maximum is the same as the potential difference between the dropping electrode (of the Paschen type) and the solution.

A dropping electrode and a capillary electrode were connected up as in the diagram.



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[*] Paschen, 'Wied. Ann.,' 41, 1890.

[†] *Cf.* Ostwald, 'Lehrbuch,' 2, 938.

The mercury of the dropping electrode fell into the tube A, which had a vertical fine adjustment attached, so that the position of the liquid in it could be varied with respect to the point of the dropping electrode. The capillary electrode was immersed in the liquid of the vessel B. The two electrodes were directly connected by a wire, and A and B were connected by a siphon tube.

The surface tension of the capillary electrode was measured under four different conditions, viz., when—

1. A and B were filled with $\frac{1}{2}n$ KCl.
2. A and B were filled with $\frac{1}{2}n$ KI.
3. A was filled with $\frac{1}{2}n$ KCl and B with $\frac{1}{2}n$ KI.
4. A was filled with $\frac{1}{2}n$ KI and B with $\frac{1}{2}n$ KCl.

The capillary electrode formed part of the electrometer previously described, and the curves which it gave for the $\frac{1}{2}n$ KI and the $\frac{1}{2}n$ KCl respectively were first determined.

From the behaviour of the capillary meniscus it was easy to observe when the dropping electrode jet broke in the surface of the solution in B. If the jet were completely immersed in the liquid and if, then, the latter were gradually lowered, the surface tension of the capillary electrode in general increased correspondingly to a maximum value and then changed in the same way as it did when the dropping electrode was absent—showing that the communication between the electrodes had then ceased.

In cases 1 and 2 the mercury in the capillary electrode assumed (within the limits of experimental error) its maximum surface tension when the dropping electrode broke in the surface of the liquid in A. Thus, for $\frac{1}{2}n$ KCl the readings lay between 31·3 and 31·35; for $\frac{1}{2}n$ KI the reading for several observations was 29·41. The corresponding readings of maximum surface tension for the electro-capillary curves were 31·4 and 29·43.

In case 3 the capillary surface tension lay between 26·38 and 26·31, and from the behaviour of the capillary meniscus when the drop electrode jet broke above the liquid surface—the surface tension gradually decreased—it was evident that this corresponded to a point on the ascending portion of the $\frac{1}{2}n$ KI curve. The horizontal distance between surface tensions of 29·43 and 26·35 on the $\frac{1}{2}n$ KI curve corresponds to a difference in the applied E.M.F. of about ·28 volt. So that, if the potential difference between $\frac{1}{2}n$ KI and $\frac{1}{2}n$ KCl can be neglected, the conclusion from the above observations is that the potential difference between a Paschen electrode and the solution (reckoned from the latter to the former) is about ·28 volt. greater when the solution is $\frac{1}{2}n$ KI than when it is $\frac{1}{2}n$ KCl.

The observation in case 3 is corroborated by that in case 4. In this case, when the electric contact between the dropping electrode and the solution was on the point of breaking, the observed surface tension of the capillary lay between 30·48 and 30·5;

but the surface tension increased when the contact broke, showing that the number given corresponded to a point on the descending branch of the KCl curve. The horizontal distance between the maximum and 30·5 on the KCl curve is approximately the same as in the former case. It corresponds to a potential difference of about ·24 volt. Both results are somewhat uncertain, partly owing to the difficulty of determining the exact points of maximum surface tension on the capillary curves; but as the effects observed are so considerable, and as they are further corroborated by the results described below, there can be little doubt concerning their significance. The results can be further tested by experiments similar to Paschen's by measuring the E.M.F.s of cells of the type

$$\mathrm{Hg}\;\Big|\;\mathrm{KI}\;\vdots\;\mathrm{KCl}\;\Big\downarrow,$$

in which the vertical arrow signifies the dropping electrode. The following approximate results were obtained;—

$$\overset{+}{\mathrm{Hg}}\;\Big|\;\tfrac{n}{2}\mathrm{KCl}\;\vdots\;\tfrac{n}{2}\mathrm{KCl}\;\Big\downarrow^{-} = \cdot522 \text{ volt.}$$

$$\mathrm{Hg}\;\Big|\;\tfrac{n}{2}\mathrm{KCl}\;\vdots\;\tfrac{n}{2}\mathrm{KI}\;\Big\downarrow = \cdot784 \text{ ,,}$$

$$\mathrm{Hg}\;\Big|\;\tfrac{n}{2}\mathrm{KI}\;\vdots\;\tfrac{n}{2}\mathrm{KI}\;\Big\downarrow = \cdot383 \text{ ,,}$$

$$\mathrm{Hg}\;\Big|\;\tfrac{n}{2}\mathrm{KI}\;\vdots\;\tfrac{n}{2}\mathrm{KCl}\;\Big\downarrow = \cdot127 \text{ ,,}$$

The first and second results give ·262 volt and the third and fourth give ·256 volt as the difference between the potentials

$$\tfrac{n}{2}\mathrm{KI}\;\Big\downarrow \quad \text{and} \quad \tfrac{n}{2}\mathrm{KCl}\;\Big\downarrow.^{*}$$

The difference between the chemical effects during the dropping of mercury into

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* I have found, since this was written, that G. Meyer ('Wied. Ann.,' vol. 56, 1895) measured the E.M.F. of a cell of the type $\Big\downarrow\, n\mathrm{KCl}\;\vdots\;n\mathrm{KI}\;\Big\downarrow$ and found it to be ·284 volt. The observations described above are in accord with this result.

KI and KCl respectively is rendered evident if a drop of phenol phthalein solution is added to the solution in each experiment. The solution turns pink very much more rapidly in the case of the KI than in the case of the KCl, showing apparently that the combination of the mercury with the iodine is very much more rapid than with the chlorine, of the solution into which it is dropped, under the conditions described.

SUMMARY.

1. The Lippmann-Helmholtz theory of the capillary electrometer contains two assumptions.

2. The first assumption would apply to any electrolytic cell. A deduction from it, which would apply to any cell having a large and a small electrode, is that the variation of the potential difference at the capillary electrode of an electrometer is the same as that of the applied electromotive force.

In order to trace the relation between surface tension and potential difference on the view that this first assumption is correct, it is necessary to eliminate the possible effect of depolarization upon the form of the electrocapillary curve—*i.e.*, the curve which shows the relation between the surface tension and the applied electromotive force. A direct method of examining the depolarization current is described and applied. An estimate of the magnitude of the depolarization effect is given, and the circumstances under which the effect may become appreciable are discussed.

3. The second assumption of the Lippmann-Helmholtz theory, that the electrocapillary phenomena are controlled by a simple variation of the electrostatic surface energy, leads to two conclusions, each of which is beset with difficulties.

(*a*.) The form of the electrocapillary curve is remarkably dependent upon the nature and concentration of the electrolyte, and depolarization is quite insufficient to account for the dependence.

(*b*.) The conclusion that the potential difference between the solution and the capillary electrode is zero when the surface tension has its maximum value, leads to the necessity for assuming large potential differences between certain solutions.

4. The hypothesis that the potential difference between equally concentrated solutions of potassium chloride and iodide is negligible possesses a high degree of probability. It has been shown by previous observers that if this hypothesis be true the points of maximum surface tension on the electrocapillary curves for the above solutions cannot have the significance which HELMHOLTZ's theory gives them.

It is shown in the paper that the first hypothesis of the Lippmann-Helmholtz theory is in striking accord with this hypothesis concerning the potential difference between KCl and KI when the very definite "descending" branches of the electrocapillary curves are considered.

5. If both the hypotheses just mentioned be true, we get the result that the surface tension of mercury (for a certain range of potential differences) in two solutions

is the same for a given potential difference between the mercury and the respective solutions, if the solutions are equally concentrated and possess the same kation.

6. An extension of this result shows that it is indifferent whether the kation be K, Na, or H.

7. The relation found for the KCl and KI curves can be extended to the other known cases in which the electrometer curves and liquid potential difference calculations seem to be contradictory, in such a way as to account for the apparent contradiction. Several of the cases are examined.

8. The results in 4, 5, and 6 would give a direct and accurate method of finding the potential differences between equally concentrated solutions, and could be extended to the case of solutions of different concentrations.

9. The probability that the electrocapillary curves are never completely free from influences other than electrostatic is shown by an examination of the relations between the curves for unequally concentrated solutions of the same salt.

10. In confirmation of results obtained by G. MEYER, in a slightly different way, it is shown that if the potential difference between KCl and KI is very small, the potential fall from a half-normal solution of KI to a dropping electrode of the Paschen type is about a quarter of a volt greater than that from a half-normal solution of KCl to the same electrode.

In the same way the potential fall from KI to mercury when the surface tension is a maximum is about a quarter of a volt greater than that from KCl to mercury when the tension of the surface separating the solution from the mercury is a maximum.

These results follow from direct observations with dropping electrodes, and give further support to the view that the first assumption of the Lippmann-Helmholtz theory is true and that the second is not.

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III. *The Electrical Conductivity and Luminosity of Flames containing Vaporised Salts.*

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Communicated by Sir H. E. ROSCOE, *F.R.S.*

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THE colour imparted to flames by the salts of an alkali metal is generally considered to be due to the metal existing in the state of incandescent vapour, but there does not appear to be any settled opinion as to the process by which the metal is set free from its salts. It is frequently assumed that the high temperature reigning in the flame dissociates the salt. There is, however, little, if any, independent evidence in favour of this view. Another explanation ascribes the liberation of metal to chemical decomposition. Thus, in the case of sodium chloride introduced into the flame of a Bunsen burner consuming coal-gas, it would be supposed that in the first instance the water vapour present would act in accordance with the following equation :—

$$\mathrm{NaCl} + \mathrm{H_2O} = \mathrm{NaHO} + \mathrm{HCl}.$$

The sodium hydrate (or possibly oxide) so produced would then be deprived of its oxygen by reducing gases (hydrogen, hydrocarbon, carbon monoxide) existing in the flame. A somewhat similar explanation would have to be applied to the flames of hydrogen, carbon monoxide, and cyanogen, though, in the case of the last two gases, the steps of the processes are still more hypothetical.

It is a noteworthy fact that the coloration of flames by alkali salts extends up to, and even beyond, the outer margin of the visible region of combustion, where the flame gases are usually considered to be fully oxidised and where free atmospheric oxygen exists. In such parts of a flame metals, much less oxidisable than the alkali metals, are rapidly oxidised when in the massive state. A copper wire, for example, becomes incrusted with oxide in a region where yellow light is abundantly emitted from a salted flame. A superoxygenated oxy-hydrogen flame is also coloured yellow by salt.

In discussing these facts* one of us was led to consider whether an alternative explanation might not be sought from some conclusions, derived by Professor ARRHENIUS, from a study of the electrical conductivity of salt vapours in flames.†

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* 'Philosophical Magazine' (V.), vol. 37, 245, 1894.

† 'Wied. Ann.,' vol. 42, 18, 1891.

From the analogy stated to exist between dilute solutions of solids and matter in the gaseous state, and from his own theory that in dilute solutions electrolytes are in greater or less degree dissociated into their ions, ARRHENIUS was led to suppose that the vapour of electrolytes distributed in small concentration throughout a gas would likewise be electrolytically dissociated. He considers his experimental results to harmonise with this view. He supposes that when an alkali salt is introduced into a coal-gas flame the large excess of water vapour present converts the salt into a hydroxide, in accordance with the following equation :—

$$M'X' + H_2O = M'OH + HX'.$$

The hydroxide is then supposed to dissociate to a certain extent into its ions.

Now, according to the electrolytic theory of solution, a free ion may, in virtue of its electric charge, be characterised by properties totally different from those pertaining to the ordinary chemical atom. Thus, in a dilute solution of sodium chloride, sodium ions may persist in presence of water. If we accept the results and views of ARRHENIUS, we might suppose that in a flame coloured by an alkali salt, the metal is liberated as an ion, and as such may persist in a strongly oxidising medium of flame gases. Such an explanation would avoid the difficulties attending the more usually adopted views.

Another consideration appeared to favour the hypothesis suggested to us by the results of ARRHENIUS. According to him the conductivity of a salt vapour is proportional to the square root of its concentration in the flame. Now GOUY has shown ('Ann. Chim. Phys.,' 18, 5, 1879), that the luminosity of a flame coloured by an alkali salt is also within certain limits nearly proportional to the square root of its concentration in the flame.

The parallelism of these numerical relationships would obviously find a simple explanation in the event of the luminosity and electrical conductivity being both dependent on the presence of free ions.

The importance of these deductions as affecting spectrum analysis decided us to undertake an experimental investigation of the subject, and we were the more inclined to this from the belief that we had at hand an apparatus capable of giving accurate results. Besides this, a close examination of the results of ARRHENIUS revealed some apparent discrepancies that detract from the weight of his conclusions.*

The apparatus that we designed to employ was that used in other investigations of flame (SMITHELLS, 'Phil. Mag.' (V.), 39, 123 (1895)). This apparatus permits of the wide separation of the two cones that constitute the non-luminous flame of a Bunsen burner, and it was thought that the interconal space which, in the apparatus, is quite shielded from draughts, would afford a particularly favourable means of obtaining constant conditions during the experiments. In other respects it was proposed to use arrangements similar to those of Arrhenius.

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* In his paper (*loc. cit.*) on p. 33, ARRHENIUS, dealing with the relation of conductivity to electromotive force, gives a set of galvanometer readings for an electromotive force of ·2 Daniell. Later in the paper, on p. 36, when dealing with the relation of electromotive force to concentration, another set of readings,

Preliminary experiments with the apparatus showed at once that numerous precautions were necessary to secure constant results, and it was only after many months of trial and the accumulation and rejection of series of measurements that the apparatus was finally brought into a reliable working condition.

The experiments have been confined to salts of the alkali metals. The relative conductivities of flames into which the salts were projected were determined and the experiments were designed to show:

I. The relation between conductivity and electromotive force.
II. The relation between the conductivity and the amount of salt present.
III. The relation between the conductivities of equivalent quantities of various salts of the same metal.
IV. The comparative conductivity of equivalent quantities of the salts of different metals.
V. The behaviour of the same salts in different flames.

From these results, conclusions are drawn respecting the primary object of the investigation, which was to discover whether the coloration of flames, and their electrical conductivity when containing vaporised salts, are due to a common cause.

Description of Apparatus and Method of Working.

The apparatus employed in this investigation consisted essentially of an arrangement for producing a Bunsen flame with separated cones, into which salt solutions could be introduced as a fine spray along with the mixed coal-gas and air. A pair

also for an electromotive force of ·2 Daniell, is given. The following are extracts from the two sets of readings:—

	Normal NaCl.	$\frac{N}{4}$ RbCl.	$\frac{N}{16}$ CsCl.	Normal LiCl.	Normal Na$\overline{\text{Ac}}$.
From Table 1	25·8	215	152	5·4	30·0
„ Table 2	41·0	161	121	8·1	30·2
Per cent. difference . .	59	−26	−20	50	0·6

The fact that the readings are not identical is not important, as Arrhenius expressly states that his electrodes suffered change in course of the investigation; but it will be observed that the differences are quite capricious. Thus the second reading for sodium chloride is 60 per cent. greater than the first, whilst for rubidium chloride the second reading is 25 per cent. less than the first. We are unable to conjecture any satisfactory explanation of these discrepancies.

of electrodes was placed between the cones, and the electric currents between them measured under various E.M.F.s.

It was found that any unsteadiness of the flame produced considerable and irregular deflections of the galvanometer which prevented the current from being measured in a satisfactory manner. The steadiness of the flame could be judged readily by observing the lower cone, which, in consequence of the large amount of air mixed with the gas (about 5 vols. air to 1 of coal-gas) in order to produce separation of the cones, was very sharply defined. This cone in our earlier experiments could be seen to oscillate up and down and from side to side.

With a view to reducing this oscillation to a minimum, so as to obtain a steady galvanometer deflection, care was taken to regulate the quantity and pressure of the gas and air supplies with as much nicety as possible. This greatly diminished the oscillations, and a further great improvement was effected by admitting the gas close to the nozzle of the sprayer, so as to produce a more perfect admixture.

With the arrangements finally adopted, it was found that the values obtained for any particular solution remained approximately constant during the whole period covered by our experiments.

The air was supplied under high pressure by means of a Westinghouse air-pump, the amount of air used being but a small fraction of what the pump could supply. The air was filtered through a long cotton-wool plug, P (fig. 1), and a first adjustment of the supply was afforded by a tap close to a small mercury manometer, M. The excess of air thus diverted blew off through a tube dipping 160 centims. below the surface of water contained in the cylinder, C. The air to be used was next passed into an iron drum, D, of 500 litres capacity, in order to damp down any pulsations of pressure.

The coal-gas was regulated by admission into a large gas holder, G, whence it passed through a micrometer screw tap, S, to the flame tube.

The adjustment of the sprayer is a matter of great importance, as the constancy of its action determines mainly the accuracy of the experiments. To obtain constant action it is necessary to work the sprayer with a much greater air supply than is just sufficient to actuate it, otherwise small variations in the pressure produce considerable changes in the amount of spray produced. Besides this, a strong air supply, producing a large amount of spray, permits the use of more dilute solutions than would otherwise be necessary. The difficulty of obtaining reliable results increases rapidly as the concentration of the solutions is increased.

The arrangement of the sprayer finally adopted is shown in fig. 2. The outer tube, O, was blown like a test-tube, with a hole 2 millims. in diameter. The inner tube, I, which was made narrow, so as to leave as much free space as possible, was joined to a wider tube, and, after exact adjustment, this was cemented to the outer tube. The sprayer was fitted into a large paraffined cork, C, which closed the end of the tubulated

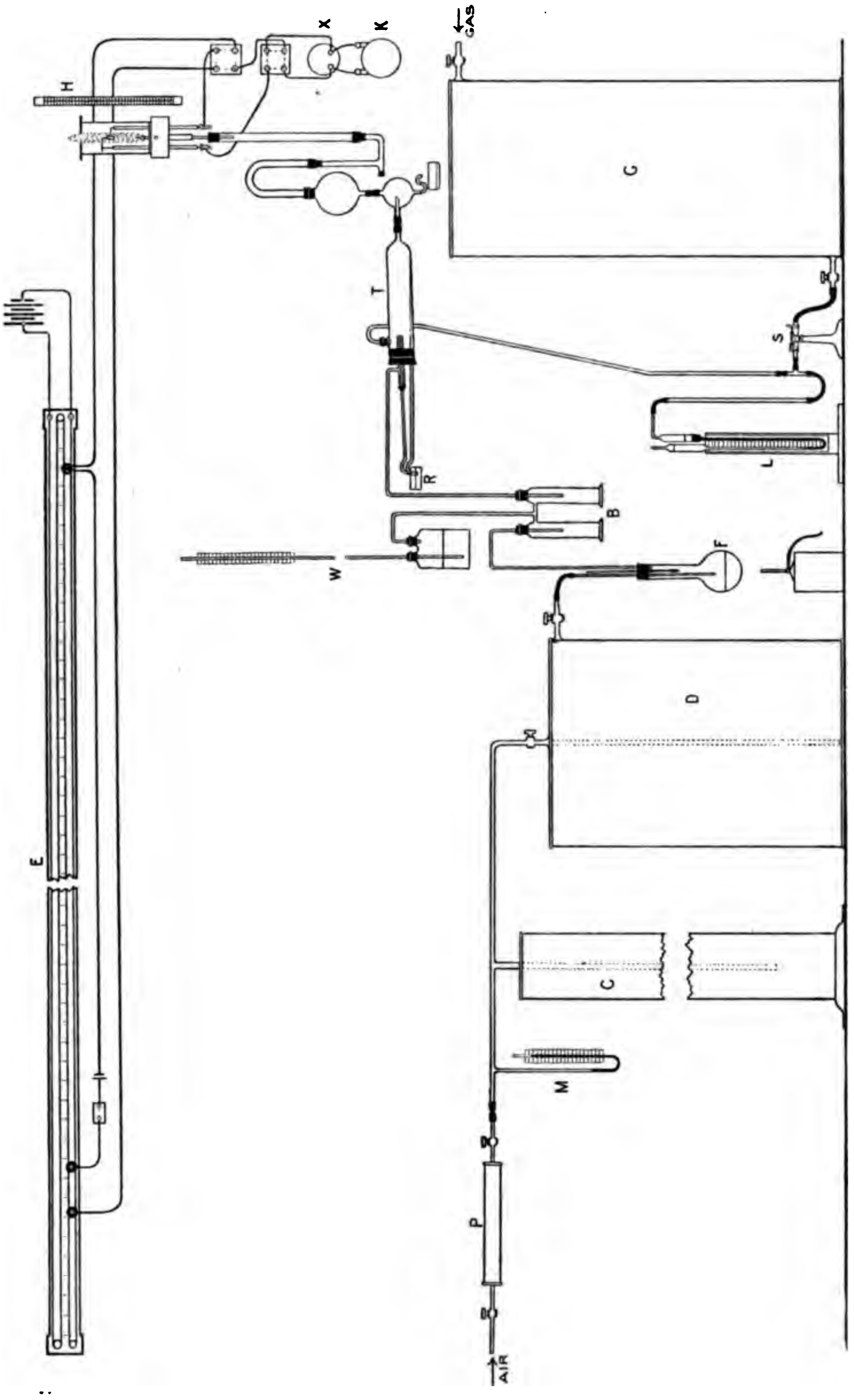


Fig. 1.

glass cylinder, and in this most of the coarse spray was deposited, running back into the reservoir of solution.

The solution in the reservoir, R, was always adjusted to a constant level, which was about 1 centim. below the nozzle of the sprayer.

The air supply from the iron drum, D (fig. 1), was passed through a flask, F, half full of water, which, being gently warmed, served to saturate the air with moisture. Any excess of moisture was condensed in the long tube and in two empty bottles, B. A water manometer, W, was connected to a branch between these bottles, and the air supply regulated so that this indicated 118·5 centims., the variations not exceeding 1 or 2 millims.

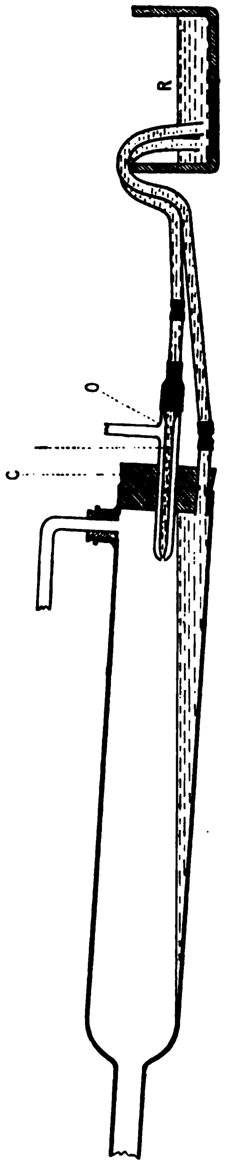
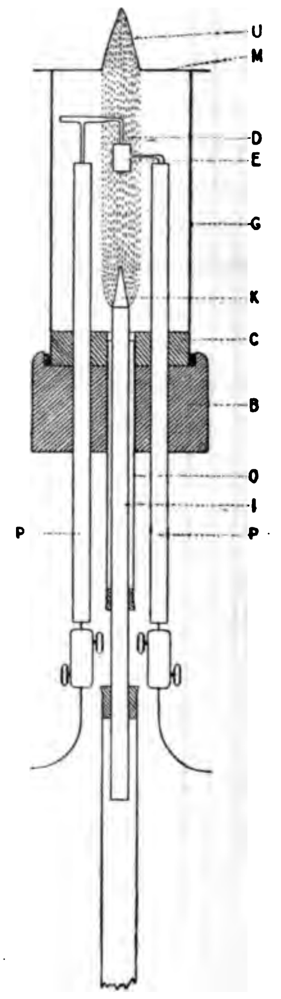


Fig. 2.

Fig. 3.



The gas pressure was measured by a multiplying differential manometer, L, containing two immiscible liquids of nearly equal specific gravity. A pressure of 1 centim. of water corresponded to a motion of 10 centims. in the surface of separation. The pressure actually employed produced a motion of the surface of separation of 18·65 centims. A constriction was made in the gas supply tube just before it joined the air, so that the pressure should be great enough to be measured with accuracy. The variation in the gas pressure during the experiments did not exceed ·25 per cent.

It was found advisable to pass the mixture of gas, air, and spray through a considerable length of apparatus, in order to allow only very fine spray to get to the flame, and to allow time for the thorough mixing of gas and air, on which the steadiness of the flame so greatly depends. The arrangement of this part of the apparatus is obvious from fig. 1.

The cone-separating apparatus and electrodes are shown in fig. 3. The gaseous mixture and spray pass up through the central tube, I, made of thin brass. On this tube the upper part of the apparatus could be slid up or down, so as to bring the electrodes to any desired height above the mouth of the tube. The hard wood block, B, was provided with three screws for centring the tube. A wider brass tube, O, was fixed into the block, and was provided at the lower end with a cork, through which the inner tube could slide.

A large cork, C, was cemented on to the wooden block in order to keep the glass cylinder, G, in position. The lower edge of the cylinder fitted into a groove in the block and was trapped with mercury.

A mica plate, M, kept in position by brass clips and pierced in the centre by a hole 2·2 centims. in diameter, was placed on the top of the glass cylinder. The electrodes were supported by two rigid porcelain tubes, P, fixed into the wood block by a packing of fusible metal. The electrodes consisted of two concentric cylinders of platinum iridium alloy. The outside cylinder, E, was supported by a wire of the same material, ·5 millim. in diameter, thickened after a distance of 1·5 centims. to 2 millims., and bent at right angles so that it reached for 3 centims. down the porcelain tube, where it was joined to a platinum wire. By means of a cross-piece fitting into a V-shaped groove at the top of the porcelain tube, this attachment of the electrode was kept in a fixed position. The inner electrode cylinder, D, was supported in a similar way. The lower end of this cylinder was provided with conical cap so as to avoid the formation of eddies in the gas stream.

The dimensions of the electrodes were—

Height of cylinders	1·575 centims.
Inside diameter of outer cylinder.	·875 centim.
Outside diameter of inner cylinder	·450 ,,

The electrodes were set up so as to be concentric and co-axial with the flame, so that a symmetrical region of flame gases was included.

In our preliminary experiments we tried electrodes consisting of platinum foil suspended by stretched platinum wires, according to the plan of ARRHENIUS. We found them unsatisfactory owing to the slackening of the wires after heating. We also tried thicker platinum plates attached to thin platinum rods and supported by porcelain tubes. We believe, however, that our final arrangement was preferable in point of rigidity and in including a symmetrical zone of gases for measurement.

A vertical millimetre scale, H (fig. 1), fixed behind the glass cylinder, enabled the height of the electrodes and of the inner cone of flame, K, to be read by means of a cathetometer telescope.

The source of electricity used by us consisted usually of three accumulators. A German silver wire, 1·5 millims. in diameter and 20 metres long, carefully calibrated, was stretched four times along a bench over a millimetre scale, E. By making contact with two heavy three-legged contact-pieces at two points, any E.M.F. up to 5·7 volts could be taken off this wire.

The current in the circuit through the flame was measured by a Kelvin astatic reflecting galvanometer, K, of 5600 ohms resistance, provided with a shunt box, X. In every case in taking a reading the current through the galvanometer was reversed and the mean deflection taken. The sensibility of the galvanometer, as we used it, was about $2·85 \times 10^{-9}$ ampere for one scale division. All connecting wires were supported on glass rods fixed in paraffin blocks.

The E.M.F. given by the stretched wire was compared on all occasions with that given by a standard Clark element, and the sensibility of the galvanometer was repeatedly determined.

In some earlier experiments higher E.M.F.s were obtained by means of Leclanché cells up to thirty in number.

In the following pages the measurement of current strength is always given, except where otherwise specified, in terms of 10^{-7} ampere unit.

Method of making an Experiment.

In beginning an experiment the tubulated cylinder, T (fig. 1), was removed, and, together with the sprayer, washed well with distilled water and dried by a current of air.

The cylinder was then replaced and the reservoir, R, filled with the solution to be investigated.

Any salt which in a previous experiment had deposited on the flame tube was removed, and the cylinder, G (fig. 2), and mica plate washed and dried. The gas was then turned on and, after an interval, lighted above the mica plate. Air was next supplied until the sprayer came into action, when the level in the reservoir, R, was adjusted. When the flame had become non-luminous the mica plate was removed, so that the whole flame descended and burnt at the mouth of the flame tube. As soon as the air had reached the right pressure the mica plate was replaced, whereupon the outer cone, U (fig. 3), of the Bunsen flame rose and burned above the mica.

When distilled water is sprayed the two cones of flame are blue in colour; the inner one is extremely thin and bright. The cones are free from the reddish tints (due to dust) seen in an ordinary Bunsen burner. The interconal space (dotted in fig. 3) emits no light.

When a salt solution is sprayed the characteristic radiation appears at the surface of the inner cone and extends over the dotted region shown in fig. 3, forming an approximately cylindrical column 3 centims. in diameter, in which the electrodes are symmetrically immersed. The electrodes become bright red, the inner one being rather brighter than the outer one is on the outside.

After about ten minutes spraying the measurements were commenced. The highest E.M.F. was usually applied first and lower ones substituted successively. At the end of a series the earlier observations were repeated.

The current was measured, and then reversed and measured again, and as the current was also taken in both directions through the galvanometer, four readings were obtained, and of these the average was taken.

During an experiment one observer watched the manometers on the air and gas supply, whilst the other took the galvanometer readings.

In working with strong solutions great care is necessary to keep the air in the right hygrometric state. If it is not moist enough, salt crystals deposit on the nozzle of the sprayer and impede the air supply; whilst if it is too moist, drops of water are deposited, with the same effect.

In our earlier experiments with sprayers, in which the outer tube was conical at the end, much trouble came from these sources; the trouble was much less with the form of sprayer described above.

To gauge the constancy of the apparatus we employed a $\frac{1}{10}$ normal solution of potassium bromide. The following readings for an E.M.F. of 5·6 volts, taken at intervals during three months, will give an idea of degree of constancy attained:—

21·1, 21·3, 20·7, 21·8, 21·6, 22·9, 22·1, 21·7: Mean 21·4.

Other solutions were used, from time to time, as a control. When abnormal values were obtained, an examination of the apparatus always disclosed some slight remediable defect.

Conductivity of the Free Flame.

To obtain the true conducting power of a vaporised salt it is necessary, in every case, to make a correction in order to eliminate that portion of the observed conductivity which is due to the flame gases. For this purpose we made, in the first instance, a series of observations on the conductivity of the flame, with the addition only of the spray of distilled water.

In making these experiments, it was found necessary to change the distilled water at frequent intervals, otherwise the apparent conductivity steadily increased owing, it would appear, to solution of the glass. The quantity dissolved is, of course, small, but still quite sufficient to make itself felt. The conductivity of the flame gases alone, is, it must be remembered, extremely small in comparison with that of a flame containing the spray of $\frac{1}{1000}$ normal solution of a potassium salt.*

The following table gives the conductivity of the flame gases with distilled water spray for several electromotive forces :—

E.M.F. in volts . .	5·66	2·52	1·09	·803	·521	·230.
Current	·547	·313	·226	·177	·118	·056.

These results are plotted in Curve V, p. 110, the ordinates being multiplied by 10.

The above values were checked from time to time during our work. The agreement was always well within the limits of accuracy required, and consequently we have used these values in all cases to represent the conductivity of the free flame, subtracting them from the gross readings given by salt solutions to obtain the conductivity of the salt vapour itself.

Arrhenius, in his experiments on this subject, noticed an increased conductivity of the free flame immediately after a salt solution had been sprayed. He attributed it to a deposition of salt on the electrodes, the deposit remaining on the electrodes for some time after the spray of salt solution had been stopped. The effect in question was not important in the case of salts of the alkali metals on account of their ready volatility.

In our experiments we noticed this effect, but we believe that in our apparatus with the salts used by us it was fully accounted for by the fact that after the spray of salt had been replaced by one of water, salt spray lingered for some time in the apparatus between the sprayer and the flame tube. We noticed that the flame remained coloured below the electrodes, and that as the colour faded the galvanometer deflection fell steadily to the normal value. We have therefore used the normal value as the true correction. Even if our explanation of the higher value, found immediately after stopping the salt spray, is incomplete, the arbitrariness in our case of taking any other value than the normal one as the correction, would forbid us attempting further refinement, where, as a matter of fact, the difference involved could not seriously affect our final results.

Unipolar Conduction.

It has long been known that unipolar conduction is shown to a marked extent in the case of flame gases, that is to say the current passes from one electrode to the other more easily in one direction than in the reverse direction. The following table gives the results obtained by us in an early series of experiments, where the electrodes consisted of two platinum plates equal in size and placed as symmetrically as possible in the flame. A $\frac{1}{16}$ normal solution of KCl was used in these experiments :—

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* Schaller has shown ('Zeit. Phys. Chem.,' 25, 497, 1898) that pure water acts so rapidly on glass vessels as to forbid their use in conductivity experiments, but that with salt solutions, even of very small concentration, the solvent action is not sufficient to introduce sensible error.

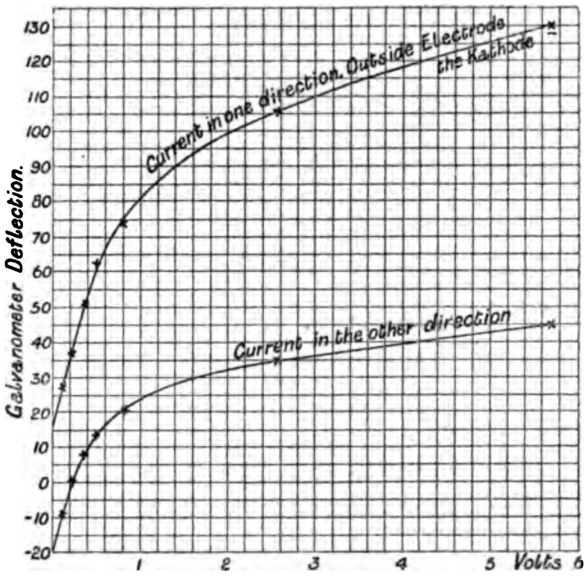
E.M.F. in volts.	Galvanometer deflection.	Galvanometer deflection (current reversed).
43·5	+ 39·7	− 48·8
29·0	+ 33·2	− 41·0
14·5	+ 24·7	− 29·3
7·3	+ 19·4	− 23·0
2·9	+ 15·1	− 16·9
1·5	+ 12·8	− 14·2

With the cylindrical electrodes used in our later experiments the electrode surfaces are of different size, and with these the unipolar effect is much greater. The following is a table for $\frac{1}{20}$ normal KCl solution :—

E.M.F. in volts.	Galvanometer deflection.	Galvanometer deflection (current reversed).
5·74	+ 81·3	− 22·5
2·56	+ 69·5	− 18·7
·815	+ 52·8	− 14·1
·378	+ 35·2	− 11·1
·116	+ 12·1	− 3·0

These results are plotted in Curve I.

Curve I.



Unipolar conduction $\left(\frac{N}{20} KNO_4 \text{ solution sprayed}\right)$.

We do not propose to enter into a discussion of the cause of this unipolar conduction, which has already been the subject of frequent, but not very fruitful, research. If it be due to the assymmetry of the electrode system, either in regard to their size or position in the flame, and if it be connected with the well-known influence of ultra-violet light or of a high temperature in facilitating the discharge of negative electricity, the circumstances of our experiments offer no obstacle to such an explanation. But they do not throw any fresh light on the phenomenon, and it is outside the scope of our enquiries to discuss it.

Concentration of Salt Vapour in the Flame.

To obtain an estimate of the quantity of salt vapour entering the flame in our experiments, we adopted the method used by ARRHENIUS.

The cone separating apparatus was removed, and the flame obtained at the orifice of the tube I (fig. 3). A bead of sodium sulphate was held in the flame for a measured interval of time, during which the light intensity was compared photometrically with that of a standard candle. The loss in weight of the bead was determined. By spraying a sodium sulphate solution of suitable strength in our apparatus, a flame of light intensity equal to that containing the bead was obtained.

The following numbers give the results. They afford, at the same time, a confirmation of the statement of GOUY, that above a certain limit the light intensity of a flame coloured by the vapour of an alkali salt varies approximately as the square root of the amount of salt introduced.

Experiment 1. Loss of weight of bead per minute, ·00161 gram. Intensity of light, 1·56.

Experiment 2. Loss of weight of bead per minute, ·00325 gram. Intensity of light, 2·00.

Experiment 3. Solution sprayed ½ normal sodium sulphate. Intensity of light, 1·53.

Experiment 4. Solution sprayed ⅕ normal sodium sulphate. Intensity of light, ·95.

From Experiments 1 and 2 we have—

Ratio of light intensities	1·3
Square root of ratio of concentrations	1·4

From Experiments 3 and 4—

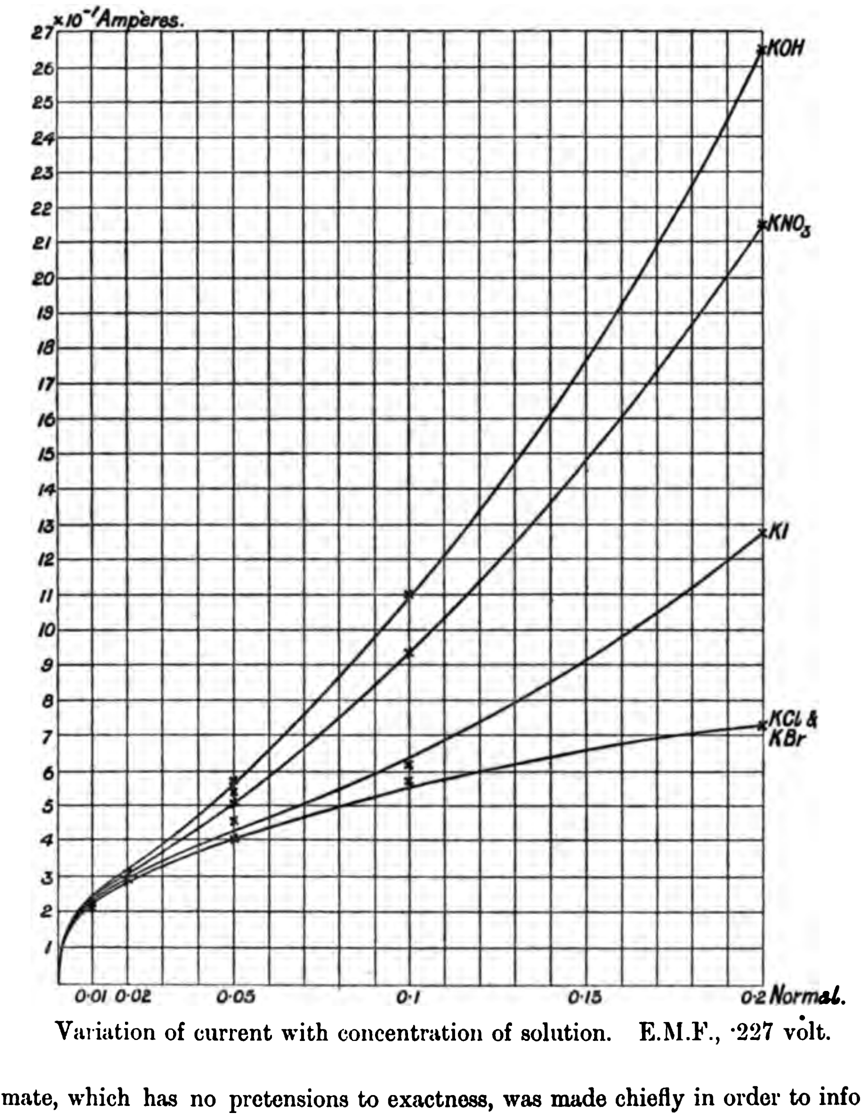
Ratio of light intensities	1·6
Square root of ratio of concentrations	1·6

The amount of salt supplied to the flame per minute by the half normal sodium sulphate solution may therefore be taken as ·0016 gram, and as a rough approximation we may conclude that the number of gram molecules supplied to the flame per minute is*

$$\frac{\cdot 0016}{71} \times 2 = 4{\cdot}5 \times 10^{-5}.$$

It is important to note that we were able to investigate the conductivity of salt vapours at much greater concentrations in the flame than was done by ARRHENIUS, and, as will appear in what follows, the results at high concentrations are very different from those obtained at the lower ones.

Curve II.



Variation of current with concentration of solution. E.M.F., ·227 volt.

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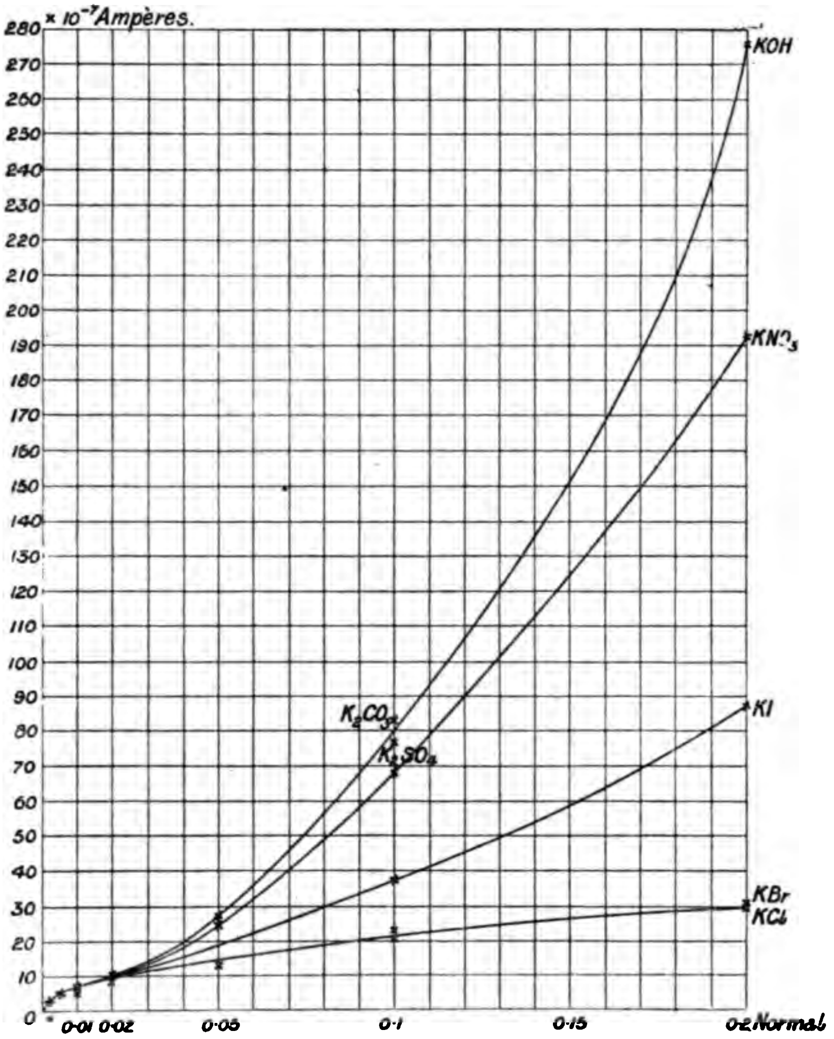
* This estimate, which has no pretensions to exactness, was made chiefly in order to inform us how our experiments compared with those of ARRHENIUS in respect to the concentration of salt in the flame. In his experiments a normal solution sent into the flame $\cdot 26 \times 10^{-5}$ gram molecule of salt per minute. In our apparatus an $\frac{1}{18}$ normal solution would yield this amount of salt. We confirmed this result by comparison of the conductivity numbers for salt solutions.

Relation of Conductivity to Concentration of Solution and to the Nature of the Salt.

Experiments with solutions of the various salts in different concentrations show that the relation between conductivity and concentration of salt vapour in the flame is not of a very simple nature.

Marked differences of conductivity are shown by salts according to their electropositive constituent, and under certain conditions, also, according to their electronegative constituent.

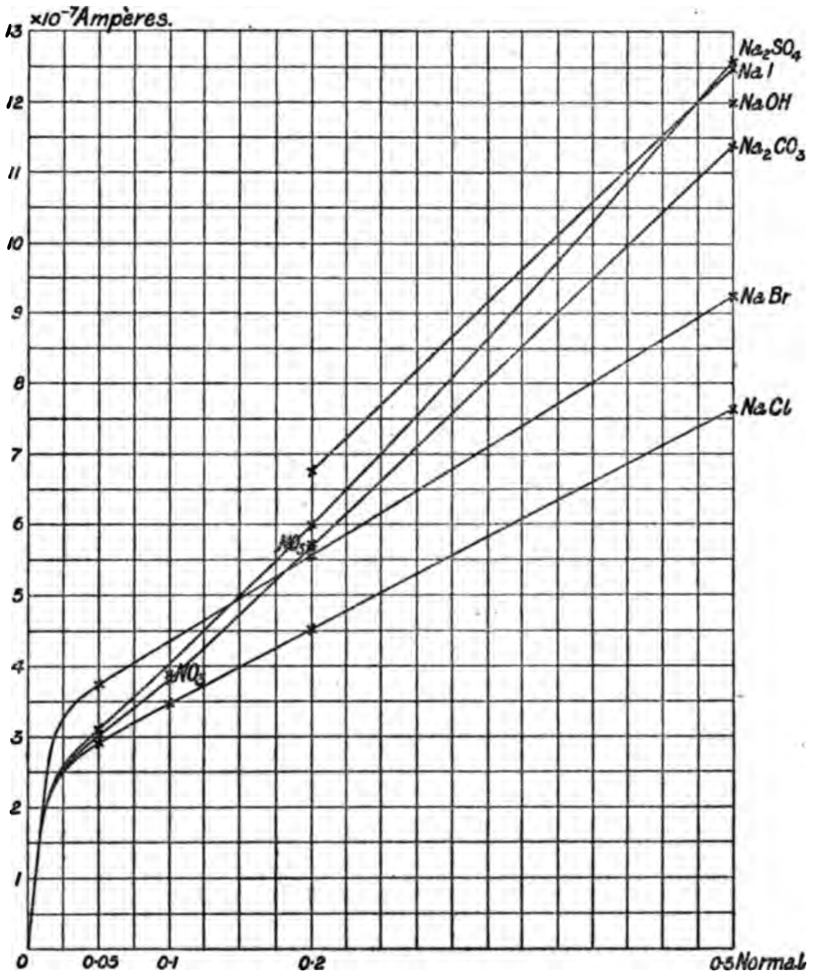
Curve III.



Variation of current with concentration of solution. E.M.F., 5·60 volts.

Curves II, III, and IV show clearly the variation of conducting power with concentration, the ordinates and abscissæ being respectively current strength and concentration.

Curve IV.



Variation of current with concentration of solution. E.M.F., 5·60 volts.

In the following table the experimental numbers are given for three different electromotive forces. In all cases the numbers have been corrected for the conducting power of the free flame.

Concentration of solution.	E.M.F.	KCl.	KBr.	$KClO_3$.	KI.	KNO_3.	$\frac{N}{2}K_2SO_4$.	$\frac{N}{2}K_2CO_3$.	KOH.
·2 normal	5·60	31·9	31·4	30·5	86·5	193	...	...	276
	·795	18·9	20·1	16·8	43·2	70·8	...	...	82·6
	·227	7·34	7·32	6·62	12·8	21·6	...	...	26·6
·1	5·60	21·0	21·4	...	37·8	68·3	83·3	76·4	
	·795	13·4	12·4	...	21·3	29·3	33·4	35·4	
	·227	5·75	5·74	...	6·2	9·35	11·0	11·2	
·05	5·60	14·1	14·7	12·9	22·8	24·5	27·5	27·6	24·1
	·795	9·23	10·3	8·35	12·8	13·2	13·8	14·2	12·8
	·227	4·0	4·13	3·81	4·6	5·1	5·71	5·43	5·4
·02	5·60	8·93	...	...	...	10·2			
	·795	6·09	...	...	...	6·63			
	·227	2·97	...	...	...	3·15			
·01	5·60	6·02	6·97	6·77	6·99	7·06	6·33	6·00	6·1
	·795	4·27	4·90	4·69	4·78	4·84	4·56	4·30	4·05
	·227	2·17	2·23	2·31	1·97	2·47	2·30	2·20	1·86
·005	5·60	5·47	...	...	...	5·27			
·002	...	...	4·03	...	3·89	...	...	3·73	
			2·80	...	2·73	...	...	2·59	
			1·50	...	1·24	...	...	1·45	

Concentration of solution.	E.M.F.	NaF.	NaCl.	NaBr.	NaI.	$NaNO_3$.	$\frac{N}{2}Na_2SO_4$.	$\frac{N}{2}Na_2CO_3$.	NaOH.
·5 normal	5·60	8·98	7·66	9·24	12·5	...	12·6	11·4	12·0
	·795	4·78	5·37	5·38	5·83	...	6·64	6·60	6·20
	·227	2·00	2·07	2·11	2·21	...	2·49	2·45	2·61
·2	5·60	4·03	4·54	5·56	6·72	5·73	5·99	5·67	
	·795	2·65	3·14	3·32	3·36	3·77	3·75	3·64	
	·227	1·39	1·42	1·41	1·39	1·85	1·61	1·71	
·1	5·60	...	3·49	...	...	3·88	...	3·78	
	·795	...	2·45	...	...	2·67	...	2·65	
	·227	...	1·15	...	...	1·30	...	1·30	
·05	5·60	2·91	2·95	3·76	...	...	3·09	3·02	3·00
	·795	2·08	2·21	2·50	...	...	2·16	2·12	2·07
	·227	·98	1·05	·97	...	...	·96	·97	·98

Concentration of solution.	E.M.F.	LiCl.	$LiNO_3$.	RbCl.	$RbNO_3$.	CsCl.	$CsNO_3$.	HCl.
·5 normal	5·60	1·88	2·28	...	...	...	...	1·08
	·795	1·09	1·59					
	·227	·56	·89	...	...	...	...	·35
·1	5·60	1·29	1·47	41·4	213·0	123	303	
	·795	·87	·99	26·4	82·4	60·5	115	
	·227	·41	·53	11·3	25·9	22·2	36·6	
·02	5·60	...	...	14·8	19·4	17·6	20·1	
	·795	...	...	9·65	11·6	11·7	13·1	
	·227	...	...	4·71	5·14	5·9	6·2	
·004	5·60	...	...	6·46	5·44	7·98	7·86	
	·795	...	...	4·51	4·18	5·70	5·51	
	·227	...	...	2·41	2·26	3·02	2·97	

The foregoing tables show—

I. That at small concentrations equivalent solutions of all salts of the same metal impart the same conducting power to the flame.

II. That at higher concentrations the equality mentioned in I no longer holds good, the oxysalts showing a greater conducting power than the haloid salts. This difference increases with increasing concentration, and with increasing electromotive force.

A clearer conception of the relations which hold is obtained by expressing the conducting power in terms of molecular conductivities. It is impossible to give absolute molecular conductivities, as this would require an exact knowledge of the concentration of salt vapour between the electrodes and of the capacity of the electrode system. In the following table the numbers, which are proportional to the molecular conductivities, have been obtained by dividing the numbers of the preceding table by the concentration of the solution, taking $\frac{1}{10}$ normal solutions as of unit concentration.

The values all refer to an E.M.F. of ·227 volt. At this E.M.F. Ohm's law is obeyed with close approximation in our experiments, so that the conditions correspond in this respect with those of conduction in aqueous solution, from which the idea of molecular conductivity is drawn.

Concentration of solution.	KCl.	KBr.	$KClO_3$.	KI.	KNO_3.	$\frac{N}{2}K_2SO_4$.	$\frac{N}{2}K_2CO_3$.	KOH.
·2	3·67	3·66	3·31	6·4	10·8	...	...	13·3
·1	5·75	5·74	...	6·2	9·35	11·0	11·2	
·05	8·0	8·26	7·62	9·2	10·2	11·42	10·86	10·8
·02	14·8	...	...	...	15·7			
·01	21·7	22·3	23·1	19·7	24·7	23·0	22·0	18·6
·002	...	75·0	...	67·0	...	...	72·5	

Concentration of solution.	NaF.	$NaCl$.	$NaBr$.	NaI.	$NaNO_3$.	$\frac{N}{2}Na_2SO_4$.	$\frac{N}{2}Na_2CO_3$.	$NaOH$.
·5	·40	·41	·42	·44	...	·50	·49	·52
·2	·69	·71	·72	·70	·92	·80	·85	
·1	...	1·15	...	...	1·30	...	1·30	
·05	1·96	2·10	...	...	...	1·92	1·94	1·96

Concentration of solution.	$LiCl$.	$LiNO_4$.	$RbCl$.	$RbNO_3$.	$CsCl$.	$CsNO_3$.	HCl.
·5	·11	·18	...	...	...	...	·07
·1	·41	·53	11·3	25·9	22·2	36·6	
·02	...	...	23·5	25·7	29·5	31·0	
·004	...	...	60·2	56·3	75·5	74·2	

The following conclusions may be drawn from the above tables:—

I. In general, the molecular conductivity of a salt increases with increasing dilution.

II. The oxysalts of all alkali metals behave differently to the haloid salts.*

III. At all concentrations investigated, the conducting power of the oxysalts of any one metal is approximately the same.

With regard to the halogen salts, it appears that potassium iodide occupies an intermediate position, forming a transition member from the haloids to the oxysalts. Sodium bromide and iodide would appear to occupy a similar position among the sodium salts.

It appears also that with increasing concentration the molecular conductivity of the oxysalts attains a minimum value. This is very evident with the oxysalts of potassium, and recognisable in those of rubidium and cæsium, the minimum values being 11, 26, and 34 in the respective cases.

---

* Potassium chlorate, being converted by the flame into the chloride, is an exception to this statement.

No minimum is observable in the case of the sodium or lithium salts, nor in the case of the chlorides of cæsium, rubidium, and potassium, and the bromide of potassium. It is probable, however, that in the case of these salts a minimum value of the molecular conductivity would be found, if higher concentrations could be investigated, for this is distinctly observable in the case of potassium iodide. It will also be seen that with the oxysalts of potassium the minimum is only attained at higher concentrations than in the case of the oxysalts of rubidium and cæsium.

The intermediate character of iodide of potassium is once more evident, for the minimum value of the molecular conductivity is about 6, whilst that for the bromide and chloride must be considerably less, and that for the oxysalts is 11.

In the case of the haloid salts the variation of conducting power with concentration may be approximately expressed by the equation

$$c = k\sqrt{q},$$

when c is conductivity, q the concentration, and k a constant.

This is shown for NaCl and KCl in the following tables:—

KCl.					NaCl.				
Concentration of solution.	E.M.F. 5·6 volts.		E.M.F. ·227 volt.		Concentration of solution.	E.M.F. 5·6 volts.		E.M.F. ·227 volt.	
	Found.	Calculated.	Found.	Calculated.		Found.	Calculated.	Found.	Calculated.
·2	31·9	(31·9)	7·34	(7·34)	·5	7·66	(7·66)	2·07	(2·07)
·1	21·0	22·6	5·75	5·20	·2	4·54	4·84	1·42	1·31
·05	14·1	15·9	4·00	3·67	·1	3·49	3·42	1·15	·93
·02	8·93	10·1	2·97	2·32	·05	2·95	2·42	1·05	
·01	6·02	7·1	2·17	1·65					
·005	5·47	5·0							
·002	4·03	3·2							

The agreement which here is by no means complete cannot be recognised at all in the oxysalts except at low concentrations. In this respect our results differ from those of Arrhenius, who gives the relation $c = k\sqrt{q}$ as one of general applicability. The chief cause of the difference in our results lies doubtless in the fact already mentioned, that, with a solution of given concentration, much more salt was carried into the flame in our experiments than in those of Arrhenius.

It is of interest to determine the variation of conducting power from metal to metal of the series of salts investigated. This is most satisfactorily done by making use of concentrations at which the individualities of the several salts of one metal have disappeared. On account of the relatively bad conducting power of the lithium and sodium salts, experiments at great dilutions were not attempted with them. The greatest dilution at which all salts were examined was $\frac{1}{10}$ normal, and, although at this concentration the various salts of each metal do not conduct equally well, we may take the chloride as representing the haloid salts and the nitrate as representing the oxysalts. The following table gives the comparison :—

	Chlorides.			Nitrates.		
E.M.F.	5·60	·795	·227	5·60	·795	·227
Cæsium	123	60·5	22·2	303	115	36·6
Rubidium . . .	41·4	26·4	11·3	213	82·4	25·9
Potassium . . .	21·0	13·4	5·75	68·4	29·3	9·35
Sodium	3·49	2·45	1·15	3·88	2·67	1·32
Lithium	1·29	·87	·41	1·47	·99	·53
Hydrogen . . .	·75	...	·27			

It is evident that the conductivity increases with increasing atomic weight of the metal, and that the increase is more rapid in the case of the oxysalts than in that of the haloids.

Influence of Temperature on Conductivity.

In the form of apparatus used by us a column of hot gases ascends from the lower cone of combustion to the upper one, the temperature rapidly decreasing. By altering the relative position of the electrodes and the mouth of the flame tube the electrodes could be brought into regions of different temperature. We do not attempt in the present enquiry to deal fully with the relation between the temperature of salt vapours and their conductivity, but have contented ourselves with making a few experiments, so as to gain an idea of the general order of the relationship.

By means of two millimetre scales etched on opposite sides of the glass cylinder, G (fig. 3), the distance between the tip of the inner cone and the lower edge of the electrodes (which in these experiments were square plates of platinum) was adjusted to 5, 15, and 25 millims. in the respective cases.

To measure the temperature, a platinum platinum-rhodium thermo-couple was placed first 2 millims. above, then 2 millims. below the electrodes in the axis of the flame, and the mean of the galvanometer readings taken to represent the temperature of the vapour between the electrodes. The solution sprayed in all cases was $\frac{1}{4}$ normal sodium carbonate.

The following table contains the results; the observations were repeated in the order given, so as to control their accuracy :—

	Height of lower edge of electrodes above tip of inner cone.	Position of thermo-couple.	Temperature deflection.	Conductivity.
(1)	·5 millims.	2 millims. above	256	235
	·5 ,,	2 ,, below	375	215
(2)	25 ,,	2 ,, above	237	85
	25 ,,	2 ,, below	311·5	80
(3)	15 ,,	2 ,, above	250	135
	15 ,,	2 ,, below	334	125
(4)	25 ,,	2 ,, above	240	86
	25 ,,	2 ,, below	314	80
(5)	5 ,,	2 ,, above	259	235
	5 ,,	2 ,, below	376	215

It will be seen from the above table that the temperature registered by the thermo-couple, when below the electrodes, is much higher than when above. The numbers representing the conducting power are otherwise, a slightly lower reading being obtained when the thermo-couple was below the electrodes. This is, of course, due to the cooling effect of the thermo-couple, which in the lower position was immersed in the gases before they reached the electrodes.

It should be stated that the cylindrical column of interconal gases and salt vapour was not appreciably altered by changes in position of the flame tube, so that the quantity of salt vapour between the electrodes was sensibly the same in all cases.

The end result of the above experiments may be expressed as follows :—

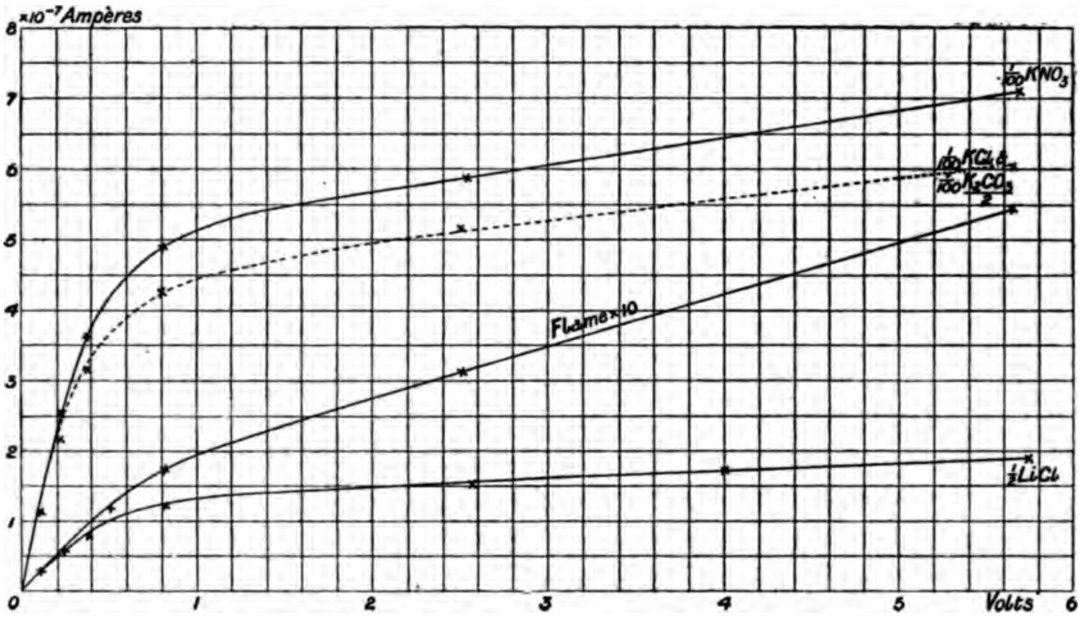
Temperature of vapour.	Conducting power.
316·5	225
292	130
275·6	82·7

These readings are on an arbitrary scale, but they show very clearly that the conducting power increases rapidly with increasing temperature, and that at temperatures not greatly below those which the vapour attains in flames the conductivity would become inappreciable. It may be stated that the temperature intervals in the above table correspond approximately to 140 and 95 Centigrade degrees.

Relation between Current Strength and Electromotive Force.

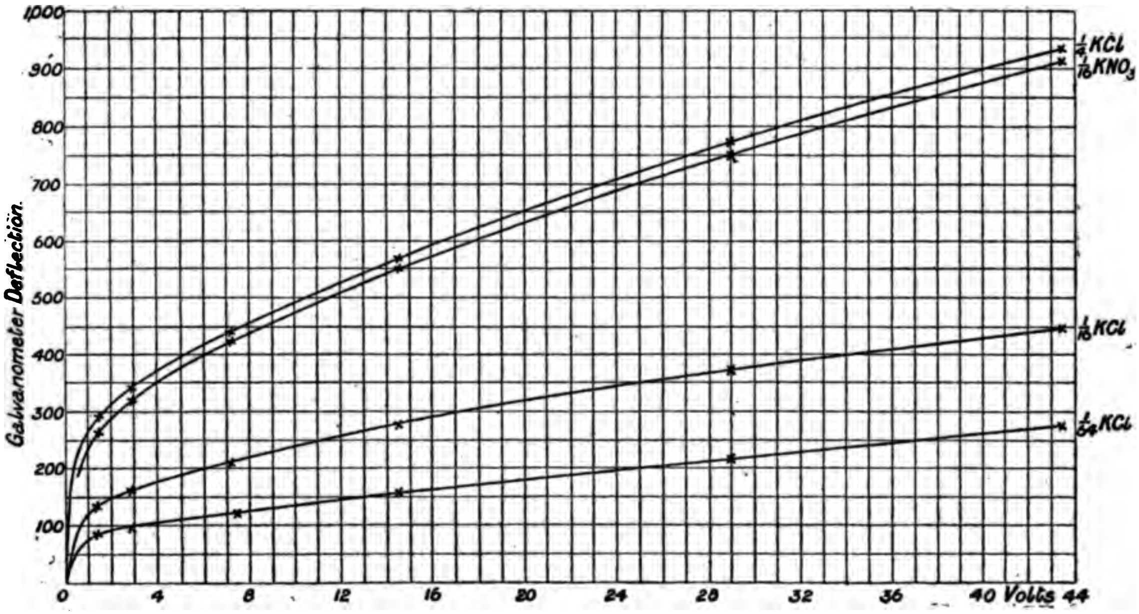
Experiments on the relation between current strength and electromotive force were carried out with a large number of salts, and with a difference of potential between the electrodes, which varied from ·01 volt to 45 volts. The results show that with small electromotive forces up to ·2 volt, Ohm's law is accurately obeyed. With greater electromotive forces the law is not obeyed, and the deviations increase with increasing electromotive force. The results are plotted on the Curves V to IX.

Curve V.



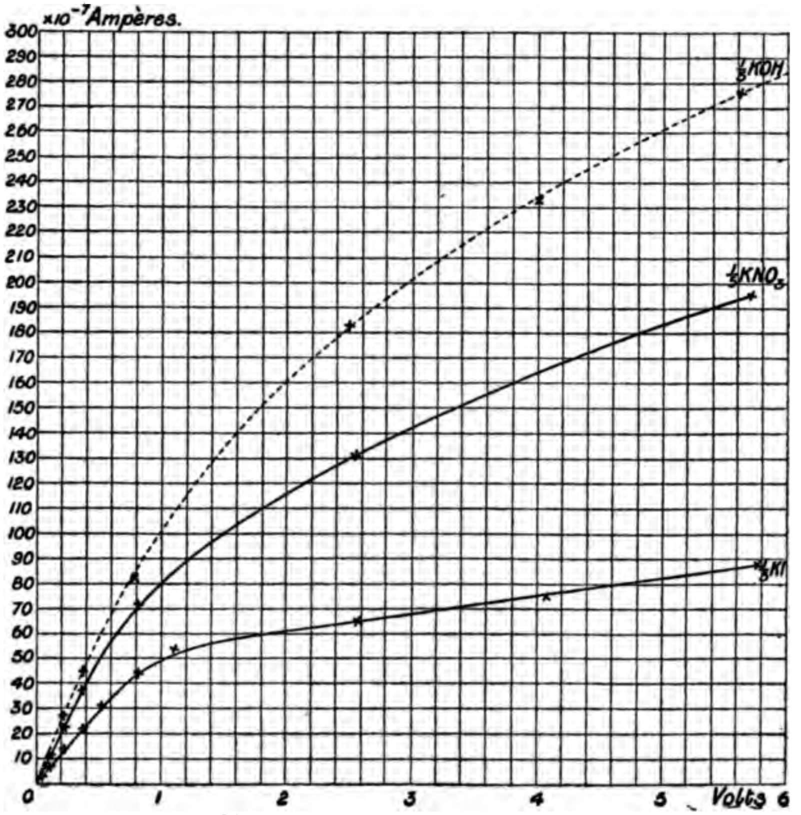
Variation of current with E.M.F.

Curve VI.



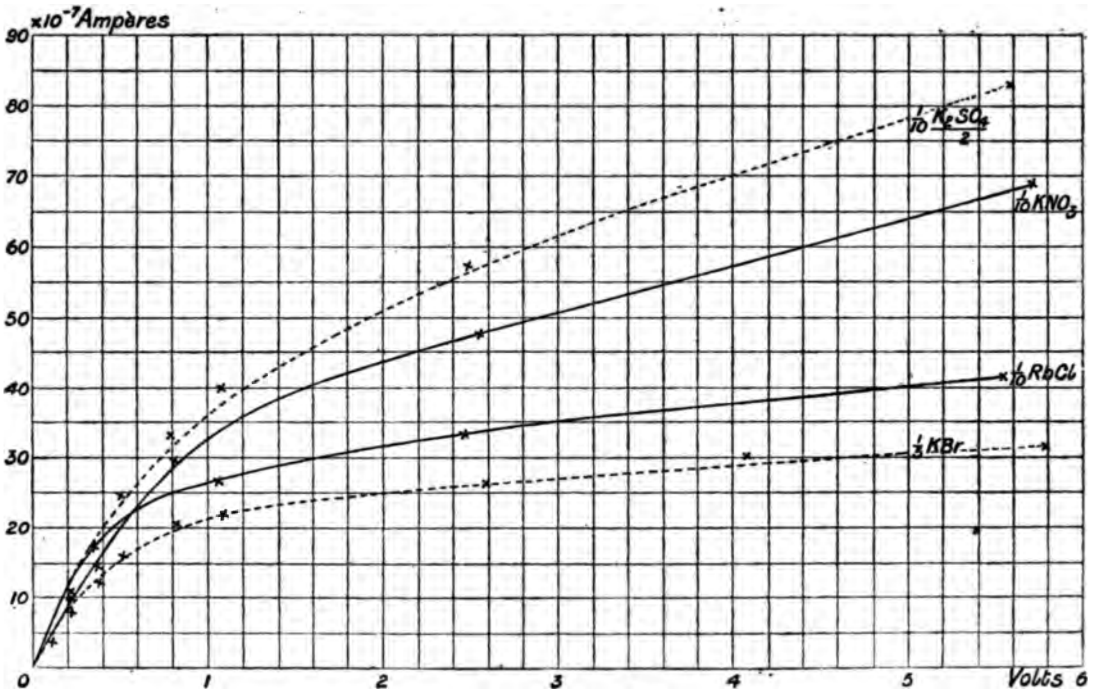
Variation of current with E.M.F.

Curve VII.



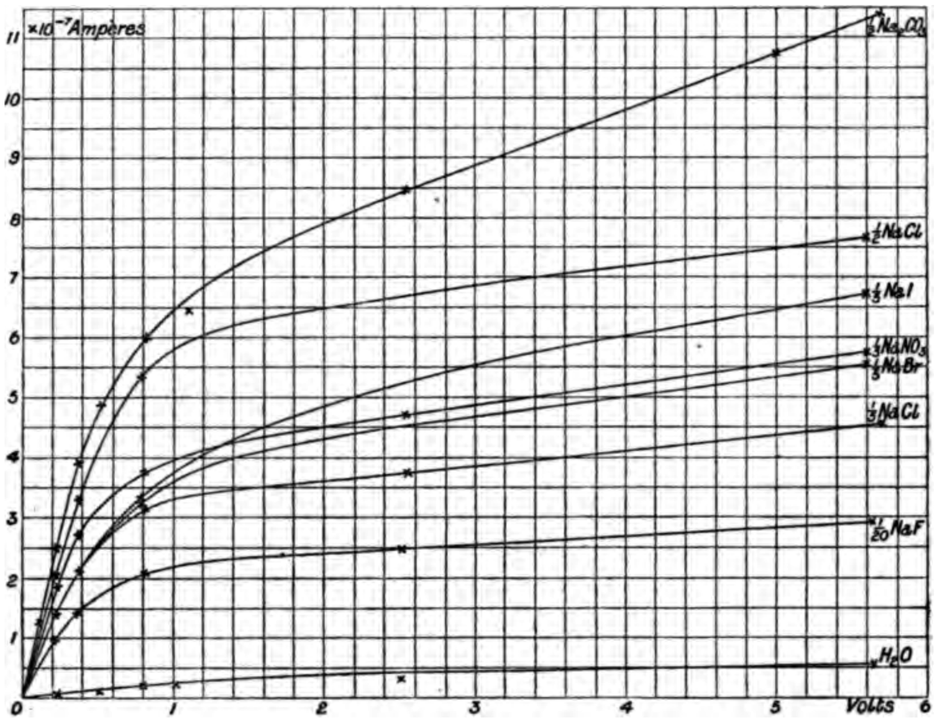
Variation of current with E.M.F.

Curve VIII.



Variation of current with E.M.F.

Curve IX.



Variation of current with E.M.F.

The following two tables sufficiently indicate the degree to which Ohm's law is obeyed by vaporised salts:—

E.M.F.	Ratio.	$\frac{N}{64}$ KBr.		$\frac{N}{64}$ K_2SO_4.		$\frac{N}{16}$ KCl.		$\frac{N}{16}$ KBr.	
		Deflection.	Ratio.	Deflection.	Ratio.	Deflection.	Ratio.	Deflection.	Ratio.
volt.									
·1	10	30·2	9·90	27·3	9·92	53·1	10·03	50·7	10·15
·08	8	23·9	7·84	21·0	8·00	42·1	7·94	40·6	8·12
·06	6	18·2	5·97	16·65	6·06	32·0	6·04	30·6	6·10
·04	4	12·0	3·93	10·95	3·98	21·05	3·96	20·25	4·05
·02	2	6·1	2·00	5·45	1·98	10·65	2·01	10·15	2·03
·01	1	3·05	1·00	2·75	1·00	5·3	1·00	5·0	1·00

E.M.F.	Ratio.	$\frac{N}{64}$ KI.		$\frac{N}{4}$ KCl.		$\frac{N}{16}$ KNO_3.	
		Deflection.	Ratio.	Deflection.	Ratio.	Deflection.	Ratio.
volts.							
44·5	3708	305	85·9	935	118	611	129
1·5	126	87·2	24·5	290	36·7	188	39·6
·12	10	24·7	9·77	78·5	9·94	47·5	10
·012	1	3·55	1	7·9	1	4·75	1

Our results are in harmony with those of **Arrhenius** in so far as they show that Ohm's law is only valid at low electromotive forces in the case of vaporised salts.

To express the general relationship between current strength and electromotive force, **Arrhenius** gives the equation $C = Af(E)$, where C = current strength, E = electromotive force, and A = a constant dependent on the solution sprayed. He found that for any electromotive force (E), $f(E)$ was the same for solutions of different salts and of different concentrations.

The validity of this equation is also confirmed by our observations up to a certain point, but with more concentrated solutions marked divergence is apparent. This will be evident from the following table.

The $\frac{1}{100}$ normal solutions of potassium salts which we investigated all gave approximately the same current for any one electromotive force. We have, therefore, used these to calculate some values of $f(E)$. Taking $f(E) = 1$ when $E = 5$ volts, we get from the numbers for $\frac{1}{100}$ normal KCl and $\frac{1}{100}$ normal K_2CO_3 solutions the following values:—

E.	f(E).
volts.	
2	·850
1	·760
·7	·709
·5	·638
·2	·332

From these values of f (E) we calculate the current which should be found in the case of a series of salts, and the numbers so obtained are inserted in a table side by side with those indicating the currents actually measured:—

E.M.F.	$\frac{N}{100}$ KNO_3.		$\frac{N}{5}$ KOH.		$\frac{N}{5}$ KBr.		$\frac{N}{2}$ Na_2CO_3.		$\frac{N}{5}$ NaCl.	
	Observed.	Calculated.	Observed.	Calculated.	Observed.	Calculated.	Observed.	Calculated.	Observed.	Calculated.
2	5·64	5·80	15·8	22·2	2·47	2·62	7·86	9·12	3·62	3·73
1	5·07	5·18	9·85	19·8	2·16	2·34	6·38	8·16	3·28	3·33
·7	4·70	4·84	7·60	18·5	1·90	2·18	5·65	7·61	3·00	3·11
·5	4·22	4·35	3·60	16·6	1·53	1·96	4·73	6·84	2·63	2·80
·2	2·25	2·26	2·31	8·65	·64	1·02	2·30	3·56	1·30	1·46

It is evident from the foregoing table that the formula $C = Af(E)$ does not express the relation existing between current strength and electromotive force over the series of observations made by us.

We are indebted to Professor J. J. **Thomson** for a suggestion which has led us to an equation capable of expressing the relationship in question in a remarkably complete way. In a paper published by Professor THOMSON and Mr. RUTHERFORD in the 'Phil. Mag.' ((V.) vol. 42, p. 392 (1896)), an account is given of the passage of electricity through gases exposed to Röntgen rays. According to the authors, the current through a gas exposed to Röntgen radiation between two parallel plates increases more and more slowly as the potential difference between the plates is increased, until finally a maximum current is obtained which remains constant under all further increase of potential difference. The electromotive force corresponding to the attainment of this state is called a saturating E.M.F. Assuming that the conductivity of Röntgenised gas is due to the presence of free ions, and that the rate of disappearance of these ions by recombination to form neutral molecules is proportional to the square of the concentration of the ions, THOMSON and RUTHERFORD give the following formula, $I - i = A\frac{i^2}{E^2}$, where I is the maximum current for a saturating E.M.F., i is the current strength for any electromotive force E, and A a constant.

Röntgenised gases and the gases of a flame exhibit a noteworthy similarity in their behaviour. Thus both rapidly lose their power of conducting electricity when they pass from the source where they have acquired this power. Both also lose their conductivity when passed between a pair of electrodes maintained at different potentials. (GIESE, 'Wied. Ann.,' vol. 17, p. 517, 1882.) That the conductivity in both cases may be due to ions has been suggested by previous investigators.

A glance at the curves in which THOMSON and RUTHERFORD plot the relation between current strength and electromotive force, will show that there is a general resemblance to the Curves I to V, by which our own results are plotted. The current through the flame, however, continues to increase even when a large E.M.F. is applied, whilst that through a Röntgenised gas reaches an almost constant value.

For potentials above one volt our curves are almost rectilinear, so that the relationship between current and E.M.F. may, for E.M.F.s above 1 volt, be expressed by the equation

$$c = I + k_1 E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

that is to say, the current strength may be represented as composed of two parts, one being a constant quantity and the other a quantity proportional to the E.M.F. This equation, however, does not give us the value of the current for small E.M.F.s.

If we represent the relation between the current and E.M.F. at all E.M.F.s by the equation

$$c = i + k_1 E,$$

then i is a variable which at high E.M.F. attains a constant value I exactly as the current through a Röntgenised gas attains a constant value.

From the analogy of the flame conductivity to that of Röntgenised gases, we assume that the equation used by THOMSON and RUTHERFORD to represent the relation between the current and the E.M.F. in their experiments should also represent the relation between i and the E.M.F. in our experiments.

Accordingly, then, the equations

$$c = i + k_1 \mathrm{E} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (2)$$

$$\mathrm{I} - i = k_2 \frac{i^2}{\mathrm{E}^2} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (3)$$

should represent the value of the current strength at all E.M.F.s in our experiments. In the case of Röntgenised gases $k_1 = 0$, or is very small, so that $c = i$.

In the following tables the observed currents for various salts are compared with those calculated by means of equations (2) and (3), and curves calculated by means of these equations, using in each case the values given below for k_1, k_2 and I, are dotted in the curve diagrams V, VII, and VIII. The points marked near them show the experimentally determined currents.

$\frac{N}{100}$ KCl and $\frac{N}{2}$ K_2CO_3.

$\mathrm{I} = 4{\cdot}50 \times 10^{-7}$.
$k_1 = {\cdot}278 \times 10^{-7}$.
$k_2 = {\cdot}020 \times 10^{+7}$.

E.M.F.	Current (calculated).	Current (from curve in Diagram IV).
volts.	amperes.	amperes.
·076	1·02 × 10^{-7}	·80 × 10^{-7}
·179	2·05 „	1·83 „
·346	3·10 „	3·10 „
·80	4·22 „	4·30 „
1·09	4·50 „	4·53 „
2·00	4·96 „	5·00 „
5·00	5·87 „	5·90 „

$\frac{N}{10} \frac{K_2SO_4}{2}$.

$\mathrm{I} = 4{\cdot}23 \times 10^{-6}$.
$k_1 = {\cdot}74 \times 10^{-6}$.
$k_2 = {\cdot}148 \times 10^{+6}$.

E.M.F.	Current (calculated).	Current (from experimental curve).
volts.	amperes.	amperes.
·10	·574 × 10^{-6}	·55 × 10^{-6}
·214	1·16 „	1·12 „
·516	2·38 „	2·50 „
1·04	3·77 „	3·90 „
3·20	6·36 „	6·40 „
5·00	7·83 „	7·87 „

$\frac{N}{5}$ KOH.

$I = 16{\cdot}9 \times 10^{-5}$.
$k_1 = 2{\cdot}10 \times 10^{-6}$.
$k_2 = {\cdot}141 \times 10^{+6}$.

E.M.F.	Current (calculated).	Current (from experimental curve).
volts.	amperes.	amperes.
·085	1·07 × 10⁻⁶	·96 × 10⁻⁶
·184	2·28 "	2·13 "
·407	4·75 "	4·80 "
·666	7·30 "	7·30 "
1·18	11·3 "	11·1 "
2·00	16·1 "	15·8 "
4·08	23·6 "	23·6 "
5·97	28·4 "	28·2 "

$\frac{N}{5}$ KBr.

$I = 2{\cdot}41 \times 10^{-6}$.
$k_1 = {\cdot}135 \times 10^{-6}$.
$k_2 = {\cdot}158 \times 10^{+5}$.

E.M.F.	Current (calculated).	Current (from experimental curve).
volts.	amperes.	amperes.
·213	·73 × 10⁻⁶	·70 × 10⁻⁶
·335	1·05 "	1·07 "
·625	1·59 "	1·73 "
·802	1·81 "	2·00 "
1·25	2·17 "	2·23 "
1·91	2·46 "	2·45 "
2·76	2·67 "	2·69 "
3·50	2·81 "	2·85 "
4·50	3·00 "	3·02 "
6·00	3·21 "	3·20 "

$\frac{N}{5}$ NaCl.

$I = 3{\cdot}10 \times 10^{-7}$.
$k_1 = {\cdot}258 \times 10^{-7}$.
$k_2 = {\cdot}030 \times 10^{+7}$.

E.M.F.	Current (calculated).	Current (from experimental curve).
volts.	amperes.	amperes.
·12	1·03 × 10⁻⁷	·80 × 10⁻⁷
·33	2·08 "	2·00 "
·40	2·30 "	2·30 "
·74	2·89 "	3·05 "
1·12	3·19 "	3·30 "
2·00	3·55 "	3·60 "
3·00	3·85 "	3·88 "
5·00	4·39 "	4·40 "

The agreement of the calculated and observed values in the above tables shows that the relation between the current and E.M.F. in our experiments can be represented with considerable accuracy by the formula in question.

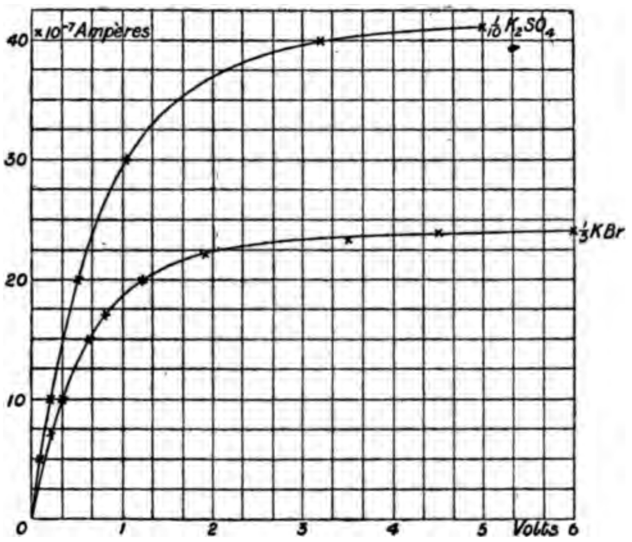
In order to show that *i* gradually attains a maximum value in our experiments as the E.M.F. is increased, the calculated values of *i* are inserted in a separate column of the foregoing tables.

In the case of $\frac{1}{100}$ normal KCl, $\frac{1}{100}$ normal K_2CO_3, $\frac{1}{10}$ normal K_2SO_4, and $\frac{1}{5}$ NaCl, it will be seen that *i* has almost reached its maximum value, that is, it has become nearly equal to I at an E.M.F. of 5 volts.

In the case of $\frac{1}{5}$ KBr, this is the case at 6 volts, whilst in the case of $\frac{1}{5}$ KOH the maximum value is not quite reached at 6 volts.

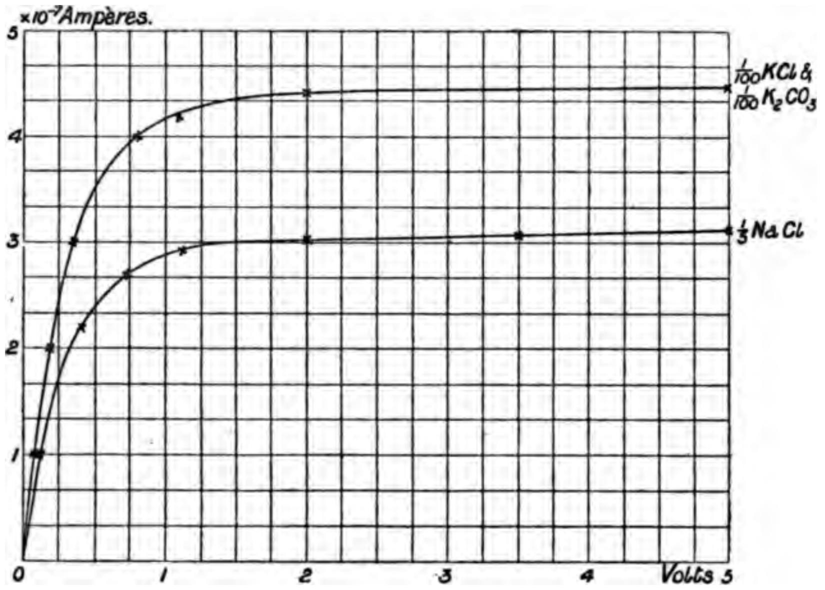
The values of *i* are plotted in Curves X and XI. As was to be expected, the form of these curves is perfectly similar to those given by Thomson and Rutherford.

Curve X.



Variation of *i* with E.M.F.

Curve XI.



Variation of *i* with E.M.F.

We will conclude this section by showing what relation must hold between the constants I, k_1, and k_2, in order that the equation used by ARRHENIUS (see *ante*, p. 113), $C = Af(E)$, where $f(E)$ has the same value for all solutions, may hold good.

Solving $I - i = k_2 \frac{i^2}{E^2}$ for i, we get

$$i = -\frac{E^2}{2k_2} \pm \frac{E}{2}\sqrt{\frac{E^2}{2k_2^2} + \frac{4I}{k_2}}.$$

Now $C = i + k_i E$.

Substituting the above value for i, we have

$$C = E\left\{k_1 - \frac{(E - \sqrt{4Ik_2 + E^2})}{2k_2}\right\}$$

$$= IE\left\{\frac{k_1}{I} - \frac{(E - \sqrt{4Ik_2 + E^2})}{2Ik_2}\right\}.$$

From this it follows that if Ik_2 and $\frac{k_1}{I}$ have the same values for all solutions, then $f(E)$ in the equation of ARRHENIUS, $C = Af(E)$, will be of the same form for all solutions.

According to THOMSON and RUTHERFORD (*loc. cit.*), k_2 should be a constant for any one substance, whatever the concentration.

The following table contains some values of the quantities I, k_1, k_2, $\frac{k_1}{\mathrm{I}}$, and $\mathrm{I}k_2$ for various solutions :—

Solution.*	$\mathrm{I} \times 10^{+7}$.	$k_1 \times 10^{+7}$.	$k_2 \times 10^{-7}$.	$\frac{k_1}{\mathrm{I}}$.	$\mathrm{I}k_2$.
$\frac{1}{5}$ KOH	169	21	·0141	·124	2·38
$\frac{1}{5}$ KI	46·0	7·1	·0140	·154	·645
$\frac{1}{10}$ K_2SO_4	42·3	7·4	·0148	·175	·626
$\frac{1}{3}$ KBr	24·1	1·35	·0158	·056	·381
$\frac{1}{20}$ K_2CO_3	14·5	2·4	·0143	·166	·207
$\frac{1}{50}$ KNO_3	6·66	·713	·0144	·107	·096
$\frac{1}{50}$ KCl	6·38	·55	·0144	·086	·092
$\frac{1}{100}$ KNO_3	4·83	·395	·0154	·082	·075
$\frac{1}{100}$ KCl	4·50	·278	·0200	·062	·090
$\frac{1}{2}$ Na_2CO_3	6·11	·927	·0301	·152	·184
$\frac{1}{2}$ NaCl	5·5	·383	·0296	·0696	·163
$\frac{1}{5}$ NaCl	3·1	·258	·0300	·083	·093
$\frac{1}{20}$ NaF	2·12	·147	·044	·0694	·093
$\frac{1}{10}$ $RbNO_3$	120	18·6	·00910	·155	1·09
$\frac{1}{10}$ RbCl	25·2	2·56	·00915	·102	·23
$\frac{1}{50}$ RbCl	10·8	1·04	·0124	·104	·134
$\frac{1}{250}$ RbCl	4·70	·412	·0228	·088	·107
$\frac{1}{10}$ $CsNO_3$	148	27·8	·0066	·188	·976
$\frac{1}{10}$ CsCl	53·5	6·43	·00682	·120	·365
$\frac{1}{50}$ CsCl	12·0	1·09	·0072	·091	·087
$\frac{1}{250}$ CsCl	5·91	·465	·0122	·0786	·072
$\frac{1}{2}$ LiCl	1·24	·114	·083	·092	·102
$\frac{1}{10}$ LiCl	·83	·0825	·128	·099	·106

It is clear from the above numbers that $\frac{k_1}{\mathrm{I}}$ varies considerably, according to the solution sprayed. Nevertheless, the values for the more dilute solutions do not differ much from ·08.

k_2 has nearly equal values for the salts of each metal, except for values of I less than about 5×10^{-7} ampere. Below this, k_2 increases as I diminishes, so that $\mathrm{I}k_2$ is nearly constant.

We have thus the following values of $k_2 \times 10^{-7}$ when I is greater than 5×10^{-7}.

---

* The concentration is given in terms of a normal solution.

Cæsium Salts . . .	·0066
Rubidium ,, . . .	·0091
Potassium ,, . . .	·0144
Sodium ,, . . .	·0300
Lithium ,, . . .	·08

When I is below 5×10^{-7}, then Ik_2 has nearly the same value for all solutions, viz., $Ik_2 = {}$·09.

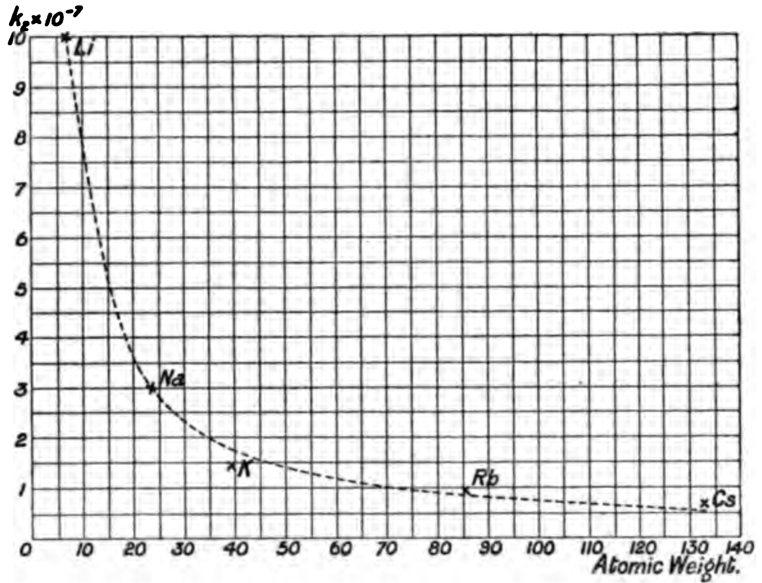
It thus appears that both $\frac{k_1}{I}$ and Ik_2 are nearly of the same value for all solutions, provided I is less than 5×10^{-7}, so that for dilute solutions ARRHENIUS'S equation $C = Af(E)$ holds good. When, however, I is greater than this, k_2 becomes constant, so that Ik_2 increases proportionally to I, and $f(E)$ changes in form.

This increase in Ik_2 corresponds to a decrease in the slope of the beginning of the curves, so that as the concentration of the solution is increased, the curves bend over more and more gradually towards the axis of E.M.F.

In Curve XII the variation of k_2 with the atomic weight of the metal is shown.

The points fall nearly on the curve, $Mk_2 = {}$·7 (M = Atomic Weight) which is drawn.

Curve XII.



Variation of k_2 with Atomic Weight.

Conductivity of Flames containing Acids.

Since the majority of acids are decomposed at the temperatures employed in our experiments, our investigation of the effect of acids on the conductivity of the flame was limited to hydrochloric and sulphuric acids.

It has already been shown that the free flame has a measurable though small conductivity, and, for the sake of comparison, the numbers obtained on p. 98 are appended to those obtained by spraying half normal solutions of HCl and H_2SO_4.

E.M.F.	Distilled water.	$\frac{N}{2}$ HCl.	$\frac{N}{2}$ H_2SO_4.
5·60 ·227	·54 ·055	1·46 ·396	·510 ·105

The conductivity of the acids is thus very small in comparison with that of the alkali salts.[*]

We also investigated the conductivity of ammonium chloride, and found it to be almost the same as that of an equivalent solution of hydrochloric acid. This is readily explained by the dissociation of the salt into HCl and NH_3 at the temperature of the experiment.

The following values, corrected for the conductivity of the free flame, were obtained :—

E.M.F.	$\frac{N}{2}$ NH_4Cl.	$\frac{N}{2}$ HCl.	$\frac{N}{2}$ H_2SO_4.
volts. 5·6 ·227	1·03 ·321	·92 ·341	– ·03 ·05

In the flame the H_2SO_4 is no doubt largely decomposed, yielding sulphur dioxide, whilst the hydrochloric acid is more stable. This accounts for the greater conductivity of the hydrochloric acid.

It is possible that in these cases, where the conductivity is small, dust particles entering with the air may have an appreciable effect. It is impossible to estimate this effect; we sought to reduce it as far as possible by filtering the air through cotton wool. If we consider it to have been insignificant, it is possible to make a comparison of the conducting power of water and hydrochloric acid in the flame. The quantity of water vapour in the flame with a spray of distilled water was many hundred times, probably many thousand times, the quantity of hydrochloric acid in the flame when a half normal solution was sprayed. The numbers given above show that the conductivity of the hydrochloric acid is at least two or three times that due to the water vapour alone.

Experiments with Decolorized Flames containing Salt Vapours.

The introduction of chloroform vapour into a flame coloured by the vapour of a lithium salt completely destroys the coloration (SMITHELLS, 'Phil. Mag.,' (V), 39, 122, 1895). The chloroform affords a convenient means of supplying an abundance of hydrochloric acid along with combustible carbon, and the first action in the flame may be expressed by the equation:

$$CHCl_3 + H_2O = CO + 3HCl.$$

Experiments were made with flames containing chloroform vapour together with salt spray, in order to discover if any relation existed between the conditions requisite for coloration and those which determine conductivity.

In order to introduce the chloroform at any moment into the flame without disturbance of the experimental conditions, the apparatus was so arranged that by means of a three-way tap the gas supply could be passed through one or other of two similar U-tubes placed "in parallel," one containing a little water and the other an equal volume of chloroform. The flame remained steady when the change was made from one course to the other, and by use of a thermo-couple we found that the temperature in the neighbourhood of the electrodes likewise was not sensibly affected.

The following table gives a comparison of the conductivity of the flame containing lithium salts in the coloured and colourless states:—

	$\frac{N}{2}$ LiCl.		$\frac{N}{10}$ $LiNO_3$.	
E.M.F.	5·60	·227	5·60	·227
Current strength:				
Coloured flame	2·14	·547	2·01	5·87
Colourless „	2·44	·251	4·50	5·39

The above figures show that the conducting power is not destroyed when the flame is decolorized. It appears that the influence of the introduction of chloroform on the conductivity, such as it is, varies with the E.M.F. employed. At 5·6 the conductivity is uniformly increased when chloroform is added, whilst at ·227 volt a diminution is observed in both cases.

If the conductivity of the flame alone (that is, without salt) and of the flame with chloroform alone be subtracted from the above numbers, we obtain the following values for the salt alone:—

	5·6 volts.	·227 volt.
$\frac{N}{2}$ LCl:		
Coloured flame . . .	1·7	·50
Decolorized flame . .	1·5	·17
$\frac{N}{10}$ $LiNO_3$:		
Coloured flame . . .	1·5	·52
Decolorized flame . .	3·0	·41

As the lithium salts have but a small conductivity in the flame, it was thought possible that the alteration of conductivity above tabulated might depend on the large excess of hydrochloric acid in the flame. The experiments were therefore extended to the salts of cæsium and potassium, which have so high a conductivity as to preclude the possibility just suggested.

In the following table the results are given in scale deflections :—

E.M.F. in volts . . .	5·60	1·08	·227	·028
CsCl solution :				
Coloured flame	850	740	264	33
Decolorized flame . . .	1410	700	212	26
KCl solution :				
Coloured flame	55	...	105	
Decolorized flame . . .	78	...	93	

The above figures show that just as in the case of lithium salts, the conductivity of flames containing the salts of cæsium and potassium does not disappear on removal of the colour by means of chloroform, and the influence of the E.M.F. on the changes of conductivity that are noticed is the same in form as in the case of the lithium salts. It appears that at a certain E.M.F. the addition of chloroform would produce no effect in the conductivity. For the cæsium chloride solution used it follows from the table that this E.M.F would be about 1·1 volt.

Conductivity of Salts vaporised in the Flame of Cyanogen.

It has already been stated in the introduction that the equal conductivity of various salts of the same metal was attributed by ARRHENIUS to the conversion of each salt into hydrate by the large excess of water vapour present.

It appeared of interest to investigate the behaviour of salts in a gaseous medium containing only a small percentage of water vapour, and for this purpose we chose the flame of cyanogen. To avoid the presence of water altogether is impracticable, if the salt has to be sprayed; but if we suppose that in a coal-gas flame any considerable proportion of the hydrogen has been burned before it reached the neighbourhood of the electrodes, the quantity of water vapour in such a flame will be very great compared with that of a cyanogen flame, in which the only water is that introduced by the sprayer (partly as drops and partly as water vapour). As we estimated this difference in the amount of water to be something like 10 to 1, we thought it probable that, if the hypothesis of ARRHENIUS were correct, marked differences of conductivity should be observable when salts in suitably concentrated solution were sprayed in the flames of coal-gas and cyanogen. For this purpose $\frac{1}{50}$ normal solutions of potassium salts were employed. In a coal-gas flame solutions of this concentration conduct equally well, but in a cyanogen flame, where no water is produced by combustion, such solutions should act like the more concentrated solutions in a coal-gas flame, that is to say, their individual conductivities should become apparent.

In carrying out the experiments the limited amount of cyanogen available* at one time necessitated some alterations in the apparatus. The vessels for the collection of coarse spray were removed, and the flame tube connected directly to the tubulated cylinder, into which the sprayer projected. The flame was not separated into its two cones, and just enough air used to keep the sprayer steadily in action. The amount of salt entering the flame was not greatly different from what it was with the apparatus in its ordinary form.

The following results were obtained with an E.M.F. of 5·55 volts :—

Experiment No. 1.		Experiment No. 2.	
Solution.	Conductivity.	Solution.	Conductivity.
$\frac{N}{50}$ KCl	167·0	$\frac{N}{50}$ KBr	199·0
$\frac{N}{50}$ K_2SO_4	177·0	$\frac{N}{50}$ KNO_3	205·0
Distilled water	6·4	Distilled water	5·7

After each of the above experiments a measurement was carried out with a coal-gas flame under exactly the same conditions of pressure. This gave the following numbers :—

Experiment No. 1.		Experiment No. 2.	
Solution.	Conductivity.	Solution.	Conductivity.
$\frac{N}{50}$ K_2SO_4	19·4	$\frac{N}{50}$ KBr	21·8

Subtracting the conductivity of the flame without salt, and correcting for a slight variation in the condition between the two cyanogen experiments, we have the following values :—

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* One cubic foot stored over mercury in an iron gas-holder.

$\frac{N}{50}$ KCl.	161,
„ KBr	163,
„ KNO_3	169,
„ K_2SO_4	171.

The conductivities of the four salts are almost equal at the concentration used; the two haloid salts have somewhat less values than the two oxysalts.

It appears, therefore, that the smaller quantity of water in the cyanogen flame has not had a marked effect in bringing out the individual conductivities of the salts. It would, however, be unsafe to draw any positive conclusion as to the cause of conductivity from these experiments. We were deterred from prosecuting the enquiry further, because of the non-comparable character of the two flames in respect of temperature, a factor which has so great an influence on conductivity.

It will be noticed that the conductivity of salts in a cyanogen flame is about ten times that which they have in a coal-gas flame. The cyanogen flame without salt also conducted about ten times as well as the coal-gas flame without salt. We found that a cyanogen flame, into which a bead of salt was introduced by means of a platinum wire, showed a very high degree of conductivity. This was largely due to the very rapid rate at which the bead was vaporised; it shows at the same time that high conductivity may occur in the absence of hydroxides.

Consideration of Results.

We have not given, and do not think it necessary to give, an account of all previous investigations on the electrical conductivity of flame gases.* In recent times the conductivity of salt vapours has been investigated by WIEDEMANN and EBERT ('Wied. Ann.,' 35, 209, 1888), J. J. THOMSON ('Phil. Mag.,' (V), 29, 356 and 441), and by ARRHENIUS (*loc. cit.*), with a view to determining its character, whether electrolytic or otherwise. WIEDEMANN and EBERT, working with high electromotive force and comparatively cool electrodes, came to the conclusion that the discharge through flames was of a disruptive character, and facilitated in different degrees by the vapours of different salts. THOMSON, using a highly heated porcelain tube, provided with platinum electrodes, considered his results to indicate an electrolytic conduction. The conclusions of ARRHENIUS, as has already been stated, are entirely in favour of the view that the conduction of salt vapours is electrolytic in character.

Our own results do not seem to admit of any other explanation than that the conduction of salt vapours is electrolytic in character. At the same time the features presented by the conduction in the case of salt vapours do not correspond in every particular to those of the conduction of salts when dissolved in liquid solvents. In the case of salt vapours, the high temperatures at which alone the conductivity can be examined, the correspondingly greater mobility of the molecules or ions, as well as the enormous reduction in the density and viscosity of the medium, and in the concentration of the salt, give ample ground for expecting characteristics in the phenomena of conduction very different from those which occur with liquid electrolytes at ordinary temperatures, although in both cases the conduction may be of a truly electrolytic character.

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* A good summary is given by HEMPTINNE ('Zeitschf. f. Phys. Chem.,' 12, 244, 1893).

To take first of all the relation between current strength and electromotive force, we have in the case of liquid electrolytes, provided polarisation of the electrodes is avoided, a strict applicability of Ohm's law. In the case of salt vapours this law only applies for low electromotive forces. This was found both by ARRHENIUS and ourselves. ARRHENIUS, from theoretical considerations, believes that Ohm's law should hold also for higher electromotive forces, and he concludes that the divergence from it must be regarded as only apparent. This, however, leaves the divergence entirely unexplained.

For an expression capable of representing the relationship between current strength and electromotive force we were led, as already stated, to a formula derived by Professor J. J. THOMSON and Mr. RUTHERFORD from their study of the conductivity of gases subjected to Röntgen rays. In a gas exposed to Röntgen rays a steady supply of ions is supposed to be generated and the resulting concentration of ions is then determined by the fact that the rate at which they combine is proportional to the square of the number present, assuming equal numbers of positive and negative ions to be distributed throughout the gas. In a flame containing salt vapour it may be supposed that a steady supply of ionising salt is carried up between the electrodes, so that the conditions would be, to this extent, analogous in the two cases.

In applying the formula to our results we have had to recognise a feature distinguishing the behaviour of salt vapours from that of Röntgenised gases, namely, the fact that the current strength continues to rise slowly even at high E.M.F.s. The explanation of this difference does not seem to be difficult.* It would be accounted for by the increased electrostatic field either bringing in ions from the neighbourhood of the electrodes or increasing the rate of ionisation between them.

We have now to consider our results in reference to the question of the state in which the salts exist in the flame, and give rise to the conductivity. It could hardly be expected, *prima facie*, that the salts would all be vaporised without change, for even those among them that are most stable under ordinary conditions are likely to undergo some change of composition at the very high temperature reigning in the flame, and in the presence of various flame gases. The liability to change is indeed so great, and, at the same time, the precise character and extent of the change so little known from independent evidence, that great caution is necessary before declaring that the results of such experiments as we have undertaken establish any one possible view.

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* A kind of convective conduction, proportional to the E.M.F. which ARRHENIUS recognised in his experiments on alkaline earth metals, cannot here be in question. We have referred to this subject in another connection on p. 98.

The most important generalisations derivable from our study of the relation between current strength and concentration of different sets of salts of different metals are first, that the conductivities differ always according to the metal; secondly, that among salts of the same metal differences of conductivities evident at high concentrations disappear when the dilution is greater; and thirdly, that the conductivity of haloid salts is different from the conductivity of oxysalts.

We will give the explanation of these general facts, which appears to be most in conformity with our results, and most compatible with chemical evidence.

The fact that the conductivity of haloid salts at higher concentrations is approximately proportional to the square root of the concentration is consistent with the presence of a binary electrolyte, and as we have found that the conductivity of chlorides is maintained when the presence of a large quantity of chloroform in the flame forbids us to suppose that the chlorides are chemically altered, we conclude that the binary electrolyte in question is the haloid salt itself.

Again, the approximately equal conductivity of the oxysalts of any one metal which approaches that of the hydrates, indicates that in the flame they are changed into the same electrolysable substance, which we conclude is the hydroxide or oxide.

At the same time, whilst we recognise the haloid salts as being present to a considerable extent undecomposed in the flame, and acting as electrolytes, the fact that at high dilution the haloid salts and oxysalts alike of any one metal have the same conducting power, makes it probable that under these circumstances the haloid salts have also been converted into hydroxides, thus giving a common dissociating body.

The fact that potassium iodide at higher concentrations has a greater conductivity than potassium chloride, or bromide, is compatible with the greater readiness with which this salt is acted upon by oxidising agents. Whilst the chloride and bromide preserve their individuality, the iodide is largely converted into the oxide, which has a higher conductivity.

Coming lastly to the question whether the luminosity in flames coloured by salt vapours is connected with their electrical conductivity, we think our observations on flames containing chloroform give a definite decision in the negative.

The addition of chloroform to a flame produces hydrochloric acid. Now the conductivity of a flame containing either chloroform or hydrochloric acid, but no salt, is shown by our experiments (see p. 121) to be extremely small. Since in a flame containing an alkaline chloride the conductivity depends on the ionisation of this salt, the increased concentration of the chlorine ions due to the introduction of chloroform is so small that the degree of ionisation is not materially reduced and the conductivity therefore is not greatly affected.

The coloration of the flame is, however, entirely destroyed by the addition of chloroform.

We can say, therefore, that hydrochloric acid prevents the occurrence in the flame of the substance on which the emission of light depends, and that this substance is not merely the metal existing as an ion.

Thus the question which originally impelled us to our experiments is answered. Since it is answered in the negative, some other explanation of the luminosity of salt vapours must be sought, and we fall back upon the alternatives which have already been discussed by one of us on a previous occasion ('Phil. Mag.,' (V), 37, 245 (1894)).

It seems clear that the coloration of a flame containing vaporised sodium chloride is dependent upon the presence of the vapour of the metal in the non-ionic state, and the explanation of this, most conformable to our experiments, is that a very small proportion of the chloride is first converted into oxide by oxidising gases in the flame and the oxide then reduced by reducing gases in the flame. The presence of hydrochloric acid prevents the formation of oxide and hence prevents also the liberation of the metal. Whether the metal vapour glows solely in consequence of its high temperature, or because of vibrations imparted to its atoms during chemical change, is a question which our experiments were not designed to answer.

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Röntgen rays. Let us consider what takes place when the gas has been removed from the action of the rays, disregarding for the present the effect of recombination. The ions may be considered as constituting a separate gas, the molecules of which may be either bigger or smaller than the molecules of the gas in which they are immersed. When an ion comes into contact with the surface of the sphere, it loses its charge, so that the metal may be regarded as a body which completely absorbs the ions. The reduction in the conductivity by the diffusion of the ions to the sides is exactly analogous to the removal of moisture from a gas by bubbling it through sulphuric acid. The more rapidly the water vapour diffuses through the gas, the greater will be the number of water molecules which come into contact with the acid round the bubble. If the quantity of moisture which is removed be found experimentally, the coefficient of diffusion of water vapour into the gas can be deduced.* It would be impracticable to use this method to find the coefficient of diffusion of ions into a gas contained in a large vessel, as the loss of conductivity due to recombination would be large compared with the loss due to the sides.

The method which was employed was to pass a uniform stream of gas through fine metal tubing, and to allow the rays to fall on the gas immediately before entering the tubing. The bore of the tubing can be so adjusted that the number of ions which come into contact with the sides will be large compared with the number which recombine.

It is convenient to use tubing ot such a length that the conductivity will be reduced to about one-half its initial value.

In order to obtain the coefficient of diffusion, when the reduction in the conductivity is known, the following problem presents itself.

If a small quantity of a gas, A, is mixed with another gas, B, and the mixture passed along a tube, the sides of which completely absorb A, to find what quantity of A emerges from the tube with B.

It will be immediately seen that if the gases diffuse rapidly into each other, a large proportion of the molecules of the gas A will come into contact with the surface of the tube, and will there be absorbed. If on the other hand the rate of interdiffusion is very small, the molecules of A will travel down the tube in straight lines parallel to the axis of the tube, and practically none of them will come into contact with the surface.

The complete solution of the above problem, taking into account the variation of the velocity at points along a radius of the tube, is given in Section I. The results of the experiments and the conclusions to which they lead are contained in Section II.

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* JOHN S. TOWNSEND, 'Phil. Mag.,' June, 1898.

SECTION I.

Mathematical Investigation.

1. In a conducting gas we have two distinct sets of bodies to deal with : the ions, which are charged, and whose motion under an electromotive force constitutes conduction, and the uncharged molecules, the number of which is very much greater than the number of ions [the latter number multiplied by 10^{12} gives about the order of the number of molecules]. The ions, which for the present we will suppose to consist of an equal number of positively and negatively charged carriers, may be considered as a distinct gas, A, and the rest of the molecules through which they move as another gas, B, the two together constituting a conducting gas. When the carriers come into contact with a metal surface, they either give up their charge to the metal or remain in contact with the surface, so that the metal behaves like a perfect absorber of the ions.

In a paper On the Dynamical Theory of Gases,* Maxwell has given the general equations of motion of two gases diffusing into each other.

The equations are of the form :

$$\rho_1 \frac{du_1}{dt} + \frac{dp_1}{dx} = k\mathrm{A}\rho_1\rho_2 (u_2 - u_1) + \mathrm{X}\rho_1,$$

where ρ_1 and ρ_2 are the densities of the gases; p_1 and p_2 their partial pressures; u_1 and u_2 their mean velocities in the x direction; $k\mathrm{A}$ a constant for the two gases which depends upon the temperature ; and X the force acting on unit mass.

The first and last terms in the above equation may be omitted, as they are small compared with the other two, but in dealing with a gas which is made up of small charged bodies a new term must be introduced when electric forces are acting.

Thus the term $\rho_1\mathrm{X}$ arising from the force of gravity, for example, is $981 \times m_1 n_1$; where n_1 is the number of molecules of the first gas per cub. centim. and m_1 the mass of each (m_1 expressed in grammes is of the order 10^{-23}).

In order to estimate dp_1/dx roughly, we will suppose that the gases are contained in a tube of ·15 centim. radius, and that $p_1 = 0$ at the surface. In this case $\frac{dp_1}{dx}$ will be of the order $\frac{p'_1}{\cdot 15}$, where p'_1 is the value of p_1 at the centre. Letting ξ_1 η_1 and ζ_1 denote the mean velocities of agitation in the directions x, y, and z, $p'_1 = m_1 n'_1 \xi_1^2$, and we obtain $\frac{p'_1}{\cdot 15} = \frac{m_1 n'_1 \xi_1^2}{\cdot 15}$, which is large compared with $981 \times m_1 n'_1$, since ξ_1 is of the order 10^4.

The first term $m_1 n_1 du_1/dt$ is small compared with dp_1/dx, since the resistance to the motion is so great; the acceleration in the cases with which we are concerned is less than the acceleration of a body falling under gravity, and consequently is of a much smaller order than dp_1/dx.

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* J. C. Maxwell, 'Phil. Trans.,' vol. 157, 1866.

When each of the n_1 molecules of the first gas carries an atomic charge (6×10^{-10} electrostatic unit), forces come into play which may be of any order compared with $\frac{dp_1}{dx}$. In an electric field having a potential gradient of 1 volt per centim., the force on n'_1 ions would be $\frac{1}{300} \times 6 \times 10^{-10} \times n'_1$, which is large compared with the above value of $\frac{dp}{dx}\left[\frac{10^{-15}n'_1}{\cdot 15}\right]$.

In general six equations of the form given by MAXWELL are required, but when, as in the present case, one of the gases is present in very small quantities, the system of equations reduces to three, and the process of diffusion of the ions may be considered as having no effect on the mean velocities of the gas through which they diffuse. The second gas, B, has practically no motion in passing along a tube, except along the axis, which we take as coinciding with the axis of coordinates z. The notation can therefore be simplified, and in what follows we shall let n be the number of ions per cub. centim.; p, their partial pressure; e, the charge on each ion; X, Y, and Z, the electric forces at any point; u, v, and w, the velocities of the ions; W, the velocity of the gas, B, through the tube; (a) the radius of the tube; and K, the coefficient of diffusion of A into B.

The differential equations giving the motion are :—

$$\frac{1}{K}\,pu = -\frac{dp}{dx} + nXe,$$

$$\frac{1}{K}\,pv = -\frac{dp}{dy} + nYe,$$

$$\frac{1}{K}\,pw = -\frac{dp}{dz} + nZe + \frac{1}{K}\,pW.$$

When the steady state is reached, p is constant at any point in the tube with respect to the time, and the equation of continuity becomes

$$\frac{d}{dx}(pu) + \frac{d}{dy}(pv) + \frac{d}{dz}(pw) = 0.$$

dp/dz can be omitted from the third equation, as it is small compared with the other terms, thus, in practice, $\frac{1}{p}\frac{dp}{dz}$ is of the order $\frac{1}{20}$, W = 100, and K = $\cdot 03$; so that dp/dz is only about one ten-thousandth of $\frac{1}{K}\,p$W.

In the case with which we are dealing, $W = \frac{2V}{a^2}(a^2 - r^2)$, where V is the mean velocity defined by the condition, $\pi a^2 Vt$ = total volume of the gas B, crossing any section in a time t. Confining the investigation to the case where the numbers of positive and negative carriers are equal, the forces X, Y, and Z vanish, and the equation for p becomes:—

$$K\left(\frac{d^2p}{dx^2}+\frac{d^2p}{dy^2}\right)-\frac{2V}{a^2}(a^2-r^2)\frac{dp}{dz}=0\,;$$

which, expressed in cylindrical coordinates, becomes

$$r^2\frac{d^2p}{dr^2}+r\frac{dp}{dr}-\frac{2Vr^2}{a^2K}(a^2-r^2)\frac{dp}{dz}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

We have to find a solution of this equation which will satisfy the conditions:

$p = p_0$ a constant when $z = 0$ for all values of r, since A is distributed evenly throughout B on entering the tube.

$p = 0$ when $r = a$ for all values of z, since A gets absorbed by coming into contact with the tube.

Let $p = \phi e^{-\frac{\theta^2 a^2 K}{2V}z}$, where ϕ is a function of r, and θ^2 a constant to be determined afterwards.

Substituting this value of p in Equation (1), we obtain

$$r^2\frac{d^2\phi}{dr^2}+r\frac{d\phi}{dr}+\theta^2r^2(a^2-r^2)\phi=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

One solution, M, of this equation can be found in the form of a series.

Let

$$A_m r^m + A_{m+2}r^{m+2} + A_{m+4}r^{m+4}$$

be three consecutive terms in the expansion of M in powers of r.

Substituting in (2) we find, by equating to zero the coefficient of the $(m + 4)^{\text{th}}$ power of r, that

$$(m+4)^2 A_{m+4} + \theta^2a^2A_{m+2} - \theta^2A_m = 0.$$

If $A_{m+4}r^{m+4}$ is the first term in M, $(m + 4)^2$ must vanish. Hence, the first term must be a constant, which we will take as unity. Thus

$$M = 1 + B_1r^2 + B_2r^4 + B_3r^6 + \&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where B_1, B_2, B_3, &c., are found from the equations

$$4B_1 + \theta^2a^2 = 0,$$
$$16B_2 + \theta^2a^2B_1 - \theta^2 = 0,$$
$$36B_3 + \theta^2a^2B_2 - \theta^2B_1 = 0,\ \&c.$$

2. If Equation (3) be written in the form

$$M = 1 - \beta_1 \left(\frac{r}{a}\right)^2 + \beta_2 \left(\frac{r}{a}\right)^4 - \beta_3 \left(\frac{r}{a}\right)^6 + \&c. \ldots,$$

the relation connecting any three consecutive coefficients becomes

$$4n^2 \beta_n = \theta^2 a^4 (\beta_{n-1} + \beta_{n-2}),$$

from which it is easy to see that the series we have found for M is convergent.

Let $2n$ be greater than $\theta^2 a^4$, and let $\beta_{n-1} + \beta_{n-2} = S$. Then

$$\beta_n < \frac{S}{2n}, \text{ and } \beta_{n+1} < \frac{S}{2n}.$$

Similarly,

$$\beta_{n+2} < \frac{S}{2n(n+2)}, \text{ and } \beta_{n+3} < \frac{S}{2n(n+2)}.$$

Proceeding in this way, we see that

$$\beta_{n+2m} < \frac{S}{2n(n+2)(n+4)\ldots(n+2m)},$$

from which it follows that the series (3) is convergent.

3. The Equation (2) has a second independent solution, N, which can be found by using the solution $\phi = M$, the complete solution being $\alpha M + \beta N$. It will be seen from what follows that, when $r = 0$, N becomes infinite; so that it must be neglected when the gas A, as in the present case, extends to the centre of the tube.

Substituting for ϕ in Equation (2) $N = Mu$, we obtain

$$Mr^2 \frac{d^2u}{dr^2} + 2r^2 \frac{dM}{dr}\frac{du}{dr} + rM\frac{du}{dr} = 0,$$

or

$$\frac{1}{du/dr}\frac{d^2u}{dr^2} + \frac{2}{M}\frac{dM}{dr} + \frac{1}{r} = 0,$$

which, on integration, gives $rM^2 \, du/dr = c$. Hence

$$u = c\int\frac{dr}{rM^2}.$$

Expanding $\frac{1}{rM^2}$ in partial fractions, and integrating, we see that u has a term $c \log r$, so that N becomes infinite when $r = 0$.

Thus p cannot contain N in its expansion, and we get

$$p = \Sigma c_\theta M_\theta e^{-\frac{\theta^2 a^2 K}{2V} z} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4).$$

The boundary condition, $p = 0$ when $r = a$, requires that such values of θ^2 be chosen as will make $M_\theta = 0$ when $r = a$.

Substituting (a) for (r) in the function M, we obtain a function of $\theta^2 a^4$ with numerical coefficients. Let x_1, x_2, x_3, &c., be the values of $\theta^2 a^4$ which satisfy the equation $M_{r=a} = 0$, and let θ_1, θ_2, θ_3, &c., be the corresponding values of θ; and equation (4) becomes

$$p = c_1 M_1 e^{-\frac{\theta_1^2 a^2 K}{2V} z} + c_1 M_2 e^{-\frac{\theta_1^2 a^2 K}{2V} z} +, \text{\&c.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

4. Before proceeding to determine the coefficients c_1, c_2, &c., it is necessary to prove some general properties of the solution of the equation $\nabla^2\phi + \theta_n^2 f(x, y, z)\,\phi = 0$; $f(x, y, z)$ being any function of x, y, z. Let ϕ_n and $\phi_{n'}$ be solutions corresponding to values θ_n and $\theta_{n'}$ of the parameter θ.

By GREEN'S theorem, we have

$$\iiint [\phi_n \nabla^2 \phi_{n'} - \phi_{n'} \nabla^2 \phi_{n'}]\, dx\, dy\, dz = \iint \left(\phi_n \frac{d\phi'}{d\nu} - \phi' \frac{d\phi}{d\nu}\right) dS.$$

Let θ_n and $\theta_{n'}$ be such values of θ as will make ϕ_n and $\phi_{n'}$ vanish at the surface S ot the region throughout which the above volume integral is taken. The surface integral vanishes under these conditions, and we get, on substituting for $\nabla^2\phi_n$ and $\nabla^2\phi_{n'}$ their values,

$$(\theta_n^2 - \theta_{n'}^2) \iiint \phi_n \phi_{n'} f(x, y, z)\, dx\, dy\, dz = 0 \quad . \quad . \quad . \quad . \quad . \quad (6a),$$

which shows that the triple integral vanishes when θ_n and $\theta_{n'}$ do not coincide.

Let us suppose $\phi_{n'}$ to be got from ϕ_n by changing θ_n^2 into $\theta_n^2 + d\theta^2$, and GREEN'S theorem gives

$$- d\theta^2 \iiint \phi_n^2 f(x, y, z)\, dx\, dy\, dz = d\theta^2 \iint \left[\phi_n \frac{d^2\phi_n}{d\theta_n^2 d\nu} - \frac{d\phi_n}{d\theta_n^2} \frac{d\phi_n}{d\nu}\right] dS,$$

so that

$$\iiint \phi_n^2 f(x, y, z)\, dx\, dy\, dz = \iint \frac{d\phi_n}{d\theta^2} \cdot \frac{d\phi_n}{d\nu}\, dS \quad . \quad . \quad . \quad . \quad . \quad (6b).$$

We also have

$$\iiint \nabla^2 \phi_n\, dx\, dy\, dz = \iint \frac{d\phi_n}{d\nu}\, dS,$$

from which we derive

$$\theta_n^2 \iiint \phi_n f(x, y, z)\, dx\, dy\, dz = - \iint \frac{d\phi_n}{d\nu}\, dS \quad . \quad . \quad . \quad . \quad . \quad (6c).$$

5. Let $f(x, y, z)$ be $(a^2 - r^2)$ and ϕ a function of the cylindrical coordinate r. The equation $\nabla^2\phi + \theta^2 f(x, y, z)\,\phi = 0$ then reduces to $\frac{1}{r}\frac{d}{dr}\left(r \frac{d\phi}{dr}\right) + \theta^2 (a^2 - r^2)\,\phi = 0$, so that we can substitute M for ϕ in the three equations (6a), (6b), and (6c). If the surface integrals be taken over the surface of the cylinder of radius a, we obtain

$$\int_0^a \mathrm{M}_n \mathrm{M}_{n'} (a^2 - r^2)\, r\, dr = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7a).$$

$$\int_0^a \mathrm{M}_n^2 (a^2 - r^2)\, r\, dr = a \left[\frac{d\mathrm{M}_n}{d\theta_n^2} \cdot \frac{d\mathrm{M}_n}{dr}\right]^{r=a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7b).$$

$$\int_0^a \mathrm{M}_n (a^2 - r^2)\, r\, dr = -\frac{a}{\theta_n^2} \left[\frac{d\mathrm{M}_n}{dr}\right]^{r=a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7c).$$

From these three equations the coefficients c_1, c_2, &c., can be determined.

Since $p = p_0$, when $z = 0$, we have

$$p_0 \equiv c_1 \mathrm{M}_1 + c_2 \mathrm{M}_2 +, \text{ \&c.}$$

Multiply this identity by $\mathrm{M}_n (a^2 - r^2)\, r\, dr$, and integrating from $r = 0$ to $r = a$, we obtain

$$-\frac{a p_0}{\theta_n^2} \left[\frac{d\mathrm{M}}{dr}\right]^{r=a} = \mathrm{C}_n \, . \, a \left[\frac{d\mathrm{M}}{dr} \frac{d\mathrm{M}}{d\theta^2}\right]^{r=a}.$$

Hence

$$\mathrm{C}_n = \frac{-p_0}{\theta_n^2 \left[\dfrac{d\mathrm{M}_n}{d\theta_n^2}\right]^{r=a}},$$

and

$$p = -p_0 \left[\frac{\mathrm{M}_1}{\theta_1^2 \left[\dfrac{d\mathrm{M}_1}{d\theta_1^2}\right]^{r=a}} \cdot e^{-\frac{\theta_1^2 a^2 \mathrm{K} z}{2\mathrm{V}}} + \frac{\mathrm{M}_2}{\theta_2^2 \left[\dfrac{d\mathrm{M}_2}{d\theta_2^2}\right]^{r=a}} \cdot e^{-\frac{\theta_2^2 a^2 \mathrm{K} z}{2\mathrm{V}}} + \text{\&c.}\right] \quad . \quad . \quad (8).$$

On entering the tube the quantity of the gas A, per cub. centim., is proportional to p_0, so that $p_0 \pi a^2 \mathrm{V}$ is proportional to the quantity of A entering the tube per second (which is found by the conductivity when A consists of ions). The quantity of A which crosses a section at a distance z from the origin per second is proportional to $\int_0^a p \times \frac{2\mathrm{V}}{a^2} (a^2 - r^2)\, 2\pi r\, dr$, where p has the value given in Equation (8). The ratio R of the quantity of A which passes a section at a distance z to that which enters the tube is

$$\frac{4}{p_0 a^4} \int_0^a p\, (a^2 - r^2)\, r\, dr.$$

Substituting for p its value and using Equation 7 (c), we get

$$\mathrm{R} = \frac{4}{a^3} \left[\frac{1}{\theta_1^4} \left\{\frac{d\mathrm{M}_1/dr}{d\mathrm{M}_1/d\theta_1^2}\right\}^{r=a} \cdot e^{-\frac{\theta_1^2 a^2 \mathrm{K} z}{2\mathrm{V}}} + \frac{1}{\theta_2^4} \left\{\frac{d\mathrm{M}_2/dr}{d\mathrm{M}_2/d\theta_2^2}\right\}^{r=a} \cdot e^{-\frac{\theta_2^2 a^2 \mathrm{K} z}{2\mathrm{V}}} + \text{\&c.}\right] \quad . \quad (9).$$

The values of θ which are admissible are roots of the equation $\mathrm{M}_{r=a} = 0$ regarded as an equation in θ.

[We may here point out that, if the gas, A, on entering the tube was distributed across the section according to the law $p = \chi(v)$, where χ denotes any function, the coefficients, in the expansion of p in Equation (5), would be determinable by means of the identity

$$\chi(v) \equiv C_1'M_1 + C_2'M_2 + \&c.,$$

and by using 7 (a) and 7 (b) we see that

$$C_n' = \frac{\int_0^a \chi(r) M_n (a^2 - r^2)\, r dr}{a\, (dM_n/dr \,.\, dM_n/d\theta^2)^{r=a}}\,.$$

Hence any function χ can be expanded in a series of the "M" functions.

It can further be seen from Equations 6 (a) and 6 (b) that $\chi(r)$ can be expanded in a series of functions ϕ, where ϕ is a solution of $\frac{1}{r}\frac{d}{dr} r \frac{d\phi}{dr} + \theta^2 f(r)\, \phi = 0$; those values of θ being selected which make ϕ vanish at the boundary of the cylinder to which χ applies.]

6. Before determining the roots of the equation $M_{r=a} = 0$, it will be found useful to establish the two following propositions:—

1. All the coefficients $\frac{1}{a^3\theta^4}\left[\frac{dM/dr}{dM/d\theta^2}\right]^{r=a}$ in the expansion of R in Equation (9), are positive, and their sum is $\frac{1}{4}$.
2. All the roots of the Equation $M_{r=a} = 0$ are positive.

When $z = 0$, R must be unity, so that

$$\sum_{n=1}^{n=\infty} \frac{1}{a^2\theta_n^2}\left[\frac{dM_n/dr}{dM_n/d\theta_n^2}\right]^{r=a} = \tfrac{1}{4}.$$

Also

$$\frac{1}{a^3\theta_1^4}\left[\frac{dM_1/dr}{dM_1/d\theta^2}\right]^{r=a} = \frac{1}{a^3\theta_1^4}\left[\frac{(dM_1/dr)^2}{dM_1/d\theta^2 .\, dM_1/dr}\right]^{r=a} = \frac{1}{a^2\theta_1^4}\,\frac{(dM_1/dr)^2}{\int_0^a M_1^2 (a^2 - r^2)\, r dr} \text{ by 7 } (b).$$

This last expression is essentially positive, since r is less than a, hence none of the coefficients in the series (9) can exceed ($\frac{1}{4}$ — sum of preceding coefficients).

The second proposition is easily proved by a geometrical method, which shows that when θ^2 is negative $M_{r=a}$ is a positive quantity greater than unity.

The first few terms in M are

$$M = 1 - \frac{\theta^2 a^2}{4} r^2 + \tfrac{1}{16}\left(\theta^2 + \frac{\theta^4 a^4}{4}\right) r^4 + \&c.$$

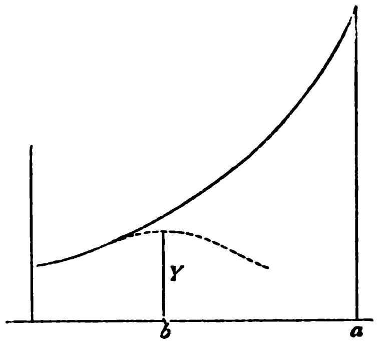
Let us suppose that θ^2 is negative, and let a curve be drawn, the x axis of which is r, and the y axis M.

When $x = 0 : y = 1$, $dy/dx = 0$, and d^2y/dx^2 is positive.

Hence the curve cuts the axis of y at unit distance from the origin, the tangent to the curve is here parallel to the axis of x, and as x increases the tangent begins to slope at a positive angle to the axis of x.

From the differential equation of the curve $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-\theta^2(a^2-x^2)x^2y$, we can easily trace qualitatively the form that the curve takes as r is increased from 0 to a.

We have seen that initially, when x is small, y, dy/dx, and d^2y/dx^2 are positive quantities. Let us suppose that it is possible for dy/dx to be negative for any value of r less than a, the curve taking the form of the dotted line in the figure.



Before dy/dx, which starts with being positive, can become negative, it must pass through a zero value at $x = b$.

The differential equation then gives

$$b^2\frac{d^2y}{dx^2}=-\theta^2(a^2-b^2)b^2\mathrm{Y}.$$

Hence d^2y/dx^2 is positive, therefore as we go along the axis of x in the positive direction from b, the tangent to the curve again begins to make a positive angle with the axis of x, so that y begins to increase. This shows that dy/dx cannot be negative at any point between $x = 0$ and $x = a$. Hence the curve must have a form somewhat similar to the continuous line in the figure, the value of y when $x = a$ being greater than the value of y at the origin. Hence the function $\mathrm{M}_{r=a}$ cannot vanish for any negative value of θ^2.

7. When r is made equal to a in M, the expression becomes a function of θ^2a^4, with numerical coefficients. The two smallest roots of the equation $\mathrm{M}_{r=a}=0$ are $\theta_1^2a^4 = 7{\cdot}313$, and $\theta_2^2a^4 = 44{\cdot}56$, which were found by expanding the function $\mathrm{M}_{r=a}$ in ascending powers of θ^2a^4. For the determination of these roots, eight terms in the expansion were found; the larger roots cannot conveniently be found by this method, but for the purposes of this investigation their determination is not necessary, as the terms which they introduce into R are smaller than the experimental errors.

The other numbers which are required are

$$\frac{1}{\theta_1^2a^3}\left[\frac{d\mathrm{M}_1}{dr}\right]^{r=a}=\cdot1321,\qquad \frac{1}{\theta_2^2a^3}\left[\frac{d\mathrm{M}_2}{dr}\right]^{r=a}=\cdot0302,$$

$$\frac{1}{a^4}\left[\frac{d\mathrm{M}_1}{d\theta_1^2}\right]^{r=a}=\cdot0926,\qquad \frac{1}{a^4}\left[\frac{d\mathrm{M}_2}{d\theta_2^2}\right]^{r=a}=\cdot0279.$$

Hence,

$$R = 4\left[\frac{\cdot 1321}{7\cdot 313 \times \cdot 0926}\, e^{-\frac{7\cdot 313 Kz}{2a^2V}} + \frac{\cdot 0302}{44\cdot 56 \times \cdot 0279}\, e^{-\frac{44\cdot 56 Kz}{2a^2V}} + \&c.\right] . \quad . \quad . \quad (10).$$

This formula holds for gases in general, when the gas which is being absorbed is present in small quantities. This restriction is necessary, since the effect of gravity would disturb the distribution of pressure given in equation (8), especially when the gases A and B differ much in density.

We therefore conclude that, when two gases, A and B, are mixed together and passed along a tube, the surface of which absorbs A, the ratio of the quantity of A which emerges to that which enters is

$$4\left[\cdot 1952\, e^{-\frac{7\cdot 313 Kz}{2a^2V}} + \cdot 0243\, e^{-\frac{44\cdot 56 Kz}{2a^2V}} + \&c.\right],$$

where a is the radius of the tube, z its length, K the coefficient of interdiffusion of the gases A and B, and V the mean velocity of the gases in the tube.

The effect of the velocity being greater at the centre than at the surface of the tube, is to increase the quantity of A that comes through the tube with B. This can be seen* by comparing the formula (10) with the function

$$4\left[\frac{e^{-\frac{(2\cdot 404)^2 Kz}{a^2V}}}{(2\cdot 404)^2} + \frac{e^{-\frac{(5\cdot 52)^2 Kz}{a^2V}}}{(5\cdot 52)^2} + \&c.\right],$$

which is the ratio of the quantity of A coming through the tube to the quantity entering, calculated on the supposition that the velocity of the gases is the same at every point.

If a gas, with ions uniformly distributed in it, has a conductivity c_1 after passing through a tube of length l_1, and a conductivity c_2 after passing through a tube of length l_2, we see from Equation (10) that

$$\frac{c_1}{c} = \frac{\cdot 195 e^{-\frac{7\cdot 31 K l_1}{2a^2V}} + \cdot 024 e^{-\frac{44\cdot 5 K l_1}{2a^2V}}}{\cdot 195 e^{-\frac{7\cdot 31 K l_2}{2a^2V}} + \cdot 024 e^{-\frac{44\cdot 5 K l_2}{2a^2V}}} \quad . \quad . \quad . \quad . \quad (11).$$

8. When the ionization is produced by Röntgen rays the ratio c_1/c_2 can be easily determined for most gases when $l_1 = 10$ centims., $l_2 = 1$ centim., $a = 1\cdot 5$ millim., and V about 100 centims. per second. Letting $c_1/c_2 = y$, and $\frac{7\cdot 31 K l_1}{2a^2V} = x$, the values of y corresponding to a series of values of x were found, and a curve representing the connection between x and y was drawn. The part of this curve which includes the values of y, which were obtained experimentally, is given in the first diagram, and from it the values of $\frac{7\cdot 31 K l_1}{2a^2V}$ can be immediately found.

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* JOHN S. TOWNSEND, 'Phil. Mag.,' June, 1898.

When hydrogen was passed through the tubing 10 centims. long it was found that its conductivity was so much reduced that it could not be accurately determined. It was therefore necessary to use another apparatus in which $l_1 = 4$ and $l_2 = 1$. The curve showing the connection between x and y for the case where $l_1/l_2 = 4$ is given in the second diagram.

Diagram 1.

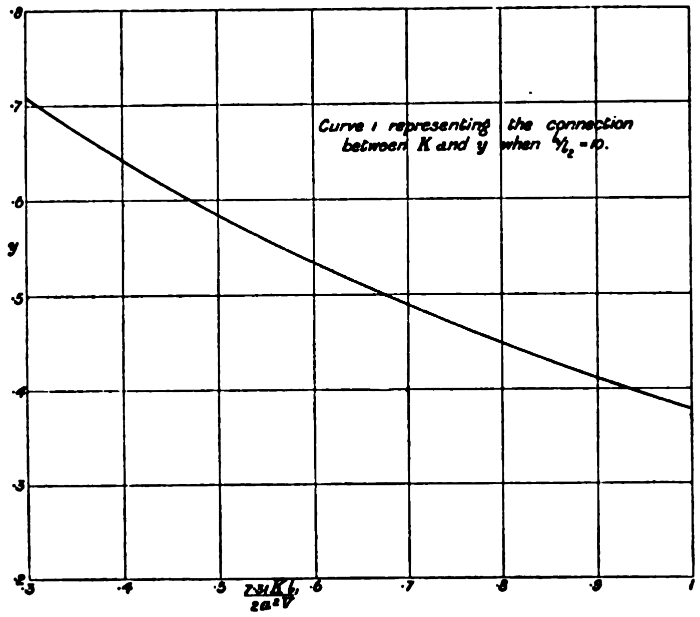
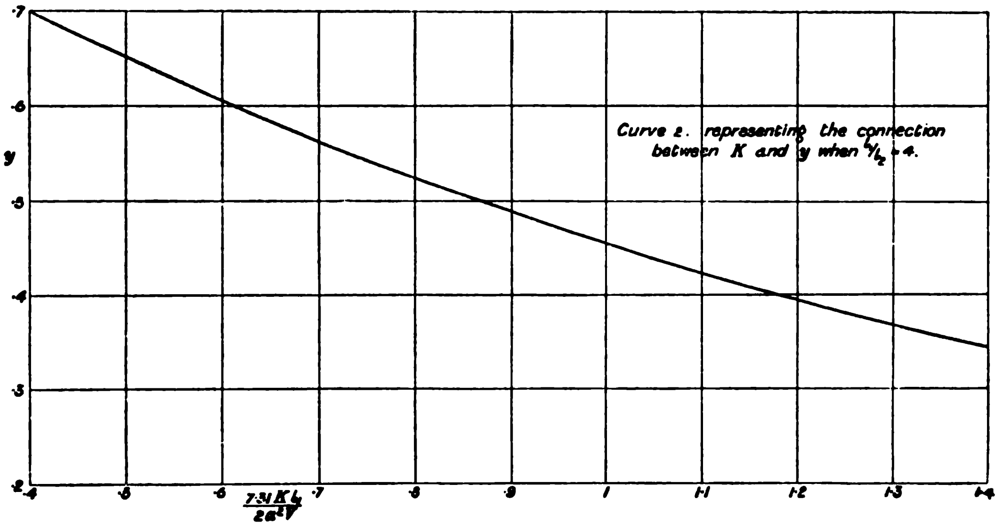


Diagram 2.



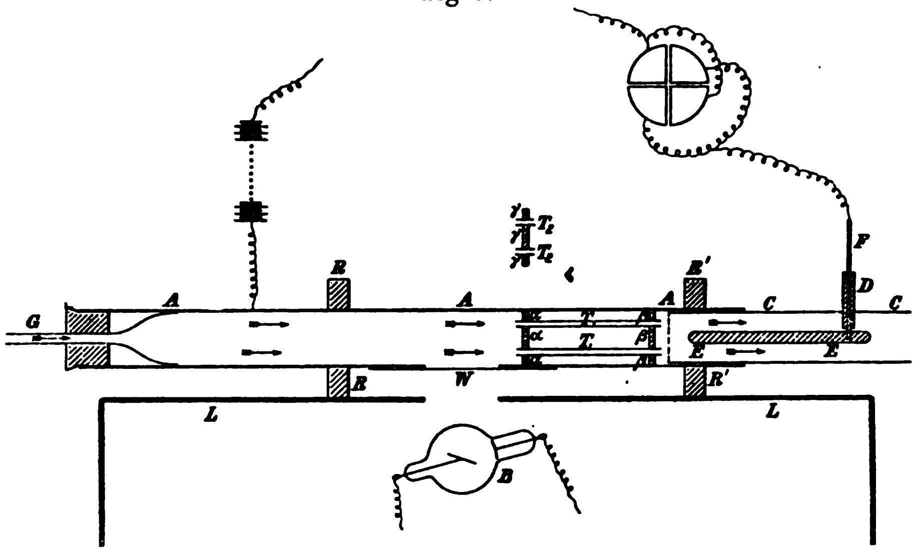
SECTION II.

DESCRIPTION OF APPARATUS.

The apparatus which was used for experiments with air, is shown in fig. 1. It consisted of a brass tube, A, 50 centims. long and 3·2 centims. in diameter, with a window, W, through which the rays from the Crookes tube, B, could pass. A second brass tube, C, 17 centims. long, fitted tightly into A, and could be moved into any desired position. The rod, F, to which the electrode, E, was fixed, passed through the ebonite plug, D, which insulated it from the tube C. The electrode, having no other support except D, could thus be put into any position in the tube A by moving C. A series of very fine wires (·1 millim. in diameter) were soldered parallel to one another, at distances 2 millims. apart, across the end of the tube C; the purpose served by this grating will be explained when we come to deal with recombination.

The gas entered the apparatus through the glass tube G, and, before reaching the electrode, passed through the tubes T_1. These tubes were soldered into holes bored in two brass discs, α and β, which fitted exactly into the tube A, so that no gas could pass between the discs and the tube. The holes in the discs were equidistant from one another, and lay on a circle whose centre was the centre of the disc. Twelve tubes, 10 centims. long and ·3 centim. in diameter, were thus arranged parallel to one another, two of which are shown in the figure. The symmetry of this arrangement ensured that the velocity along each of the small tubes would be the same. Another twelve tubes, 1 centim. long and ·3 centim. in diameter, were soldered into the disc γ.

Fig. 1.

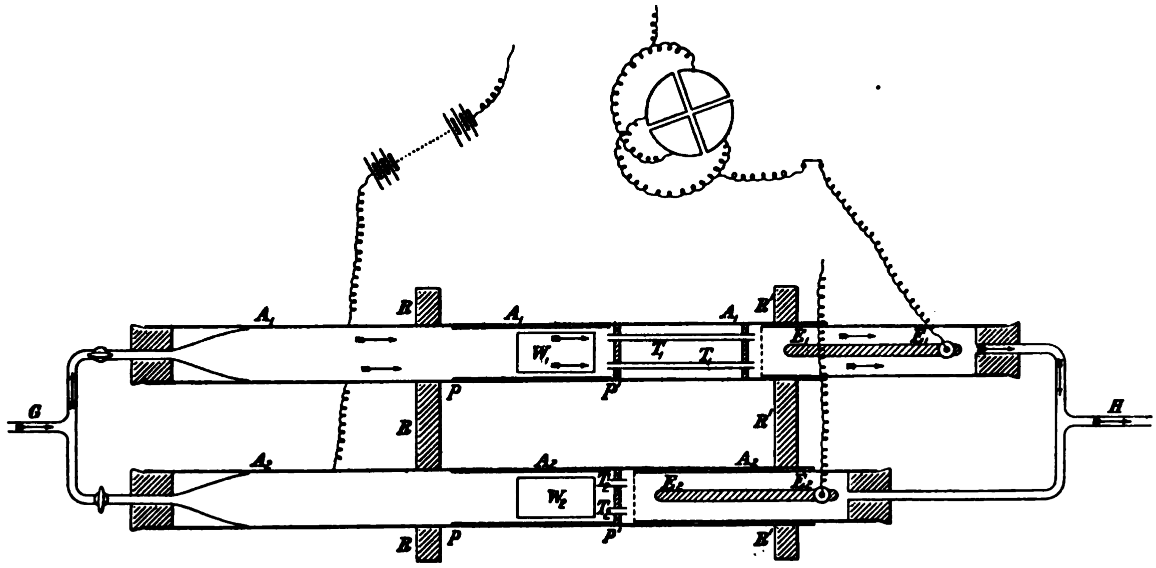


The bulb B, and the Ruhmkorff coil with which it was worked, were contained inside a box covered with lead, L. A rectangular hole was cut in the box and the lead, through which the rays from the bulb could pass. The lead covering prevented the rays from falling on any other part of the apparatus except the aluminium window W, and also screened the wire connecting F to the electrometer from electrostatic influence.

The tube A was supported by two ebonite rings, R and R′, which rested on the lead, L, and insulated the tube. The potential of the tube was raised to 80 volts by connecting it to a terminal of a battery of 40 lead cells, the other terminal of which was connected to earth. The electrode, E, was joined to a pair of quadrants of an electrometer, the other quadrants and the case being connected to earth.

In order to obtain a uniform stream of air, the tube, G, was connected to a gasometer, and the velocity of the gas along the tubes, T, could be calculated by observing the rate at which the cylinder of the gasometer fell. When the bulb is giving out rays, the gas, as it passes the aluminium window, becomes a conductor, and the ions are carried with the stream into the tubes T_1. In passing through these tubes some of the ions are discharged by the sides, and the rest on coming into the field of force (caused by the difference of potential of 80 volts between the electrode and the tube C) are removed from the gas. It will be seen that no external force acts on the ions until they escape from the tubes T_1, since all parts of the apparatus, except E, are in metallic connection with the large tube A.

Fig. 2.



When the potential of A is positive, the positively charged ions are collected on the electrode, and the deflection of the electrometer needle is proportional to the number of these ions, which come through the small tubes. The negative ions are collected on the electrode by making the potential of A negative.

If the motion of the gas past the electrode were steady, it would only require a difference of potential of a few volts between the electrode and the tube in order to remove all the ions from the gas. This, however, is not the case, since the motion of the gas as it escapes from the tubes, T, is turbulent, so that it is necessary to use a large force in order to get the maximum deflection on the electrometer scale. It was found that when the potential difference was changed from 80 to 40 volts, that the deflections were not appreciably altered; any voltage, therefore, between 40 and 80 would suffice to remove all the ions.

When it was required to find the conductivity of air after passing through short tubes, the tubes T_1 were removed and the disc γ was placed in the position occupied by α, then the electrode was moved up near the disc so that the electric force should act on the air immediately after leaving the tubes, T_2.

Experiments with oxygen, hydrogen, and carbonic acid were made with the apparatus, the horizontal section of which is shown in fig. 2. It consisted of two long tubes, A_1 and A_2, each exactly similar to A in fig. 1. In one of them the long tubes, T_1, were set up, and in the other the tubes T_2. The tubes, G and H, were connected to two gasometers, so that the gas could be passed from one to the other, through either of the tubes, A_1 or A_2. The two tubes were fixed tightly into two rectangular pieces of ebonite, R and R′, which rested on the top of the box containing the bulb. Two wooden rails were screwed to the box at such a distance that the ebonite supports fitted exactly between them, so that, by sliding the apparatus from side to side, the window in either of the tubes could be brought exactly over the bulb.

It was found necessary to put a cylinder of aluminium inside each of the tubes, extending from p to p', to prevent the rays from falling on the inner surfaces of the tubes, which were of brass. Before these cylinders were put in experiments were made to see whether the ionization produced in a stream of air passing along A_1 was equal to that produced in an equal stream through A_2, and it was found that there was a considerable difference between the conductivities in the two cases. The inequality was not due to any difference in the thickness of the aluminium covering the two windows, but was traced to differences in the state of the surfaces of the brass tubes opposite the windows.

It has been shown by PERRIN* that the ionization produced by Röntgen rays in a gas in contact with a metal is considerably increased by allowing the rays to fall normally on the metal surface. This effect upon the ionization is different for different metals, and depends also upon the state of the surface. According to PERRIN, only a very small increase in conductivity is produced when the rays fall upon an aluminium surface. It was found that, when the two aluminium cylinders were put inside A_1 and A_2, the difference in conductivity which was first observed disappeared entirely.

METHOD OF CONDUCTING THE EXPERIMENTS.

When working with the first form of apparatus the experiments were conducted in the following manner:—The tube A is raised to a potential of 80 volts positive, and the quadrants to which the electrode is joined are insulated. The stream of air from the gasometer is thus allowed to pass through the apparatus, and, when the velocity is steady, the coil working the bulb is turned on for a fixed time (20 seconds generally) and a deflection of n_1 divisions is obtained on the electrometer scale.

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* 'Comptes Rendus,' vol. 124, p. 455.

The potential of A is then changed to 80 volts negative, and the same experiment is repeated and a negative deflection, n'_1, is obtained.

The tubes, T_1, are then removed, and the short tubes, T_2, are put in their place as already described. Two similar experiments are then made, and larger deflections, n_2 and n'_2, are obtained when the rays are turned on the same stream of air for the same time.

These four experiments are then repeated several times, and the mean value of the observations is taken in order to eliminate errors arising from variations in the strength of the rays. It was found that the constancy of the bulb was improved by allowing a fixed time (3 minutes) to elapse between each experiment. When this precaution was taken, it was possible to get rays which remained constant within 5 per cent. for the space of an hour.

When working with the second apparatus the numbers n_1 and n'_1 are obtained by sending the gas along the tube A_1, and finding the deflections when the rays fall on the window, W_1, for 20 seconds. In order to obtain n_2 and n'_2 it is only necessary to move the apparatus along the top of the box till the window in A_2 comes over the bulb, and to make similar observations with the electrode E_2 joined to the quadrants of the electrometer and the stream of gas passing along A_2.

Correction for Recombination.

Before the coefficients of diffusion can be calculated from the above observations it is necessary to make a correction for the loss of conductivity due to recombination. Let us denote by cn the number of ions which, when collected on the electrode, give a deflection of n divisions on the electrometer scale. The above experiments show that there are cn_2 positive ions which pass the section of the tubes, T_1, at a distance of 1 centim. from the end near the window, W_1. Of these cn_2 ions $c(n_2 - n_1)$ are lost in the remaining 9 centims. of the tubes. The loss is principally due to the ions coming into contact with the sides of the tubes; but the loss is also to a small extent due to collisions between positive and negative ions; it is necessary to find how much the observed value of n_1 must be increased in order to compensate for the loss of ions arising from recombination. If cM is the number of positive ions which encounter negative ions and do not come into contact with the sides, then the ratio $\frac{n_1 + M}{n_2} = y$ is the number which is required in order to calculate the coefficient of diffusion from the curve given in Section I. Recombination also takes place in the short tubes T_2, but this effect is too small to take into account.

In order to find M it is necessary to find the rate at which the gas loses conductivity due to recombination, and this can be easily done by making a change in the arrangement of the apparatus. The tubes T_1 and T_2 were removed from A_1 and A_2, and the electrodes were placed in the positions shown in fig. 2, the wire grating in front of the electrode E_1 being 12 centims. from the window in A_1, and the grating in front of E_2 3 centims. from the window in A_2. The gratings in front of the electrodes, being in metallic connection with the tubes A_1 and A_2, prevent the fields of force from extending up the tubes, so that when ions are produced in a stream of gas they are not acted upon by any force till they cross the grating.

A stream of gas is passed through the tube A_1 and the rays allowed to fall upon it for 20 seconds. The positive ions are collected on the electrode as before, and a deflection N_1 is obtained on the electrometer scale (it is not necessary in this case to make a similar experiment with the negative ions). The apparatus is then moved so as to bring the window W_2 over the bulb, and the stream of gas is sent through the tube A_2, and the deflection N_2, which is greater than N_1, is obtained when the same experiment is made with the electrode E_2 joined to the electrometer. If d_1 and d_2 are the distances of the gratings from the windows W_1 and W_2, we see that the conductivity falls from N_2 to N_1, while the gas travels the distance $(d_1 - d_2)$. This reduction in conductivity is nearly entirely due to recombination, since the tubing is so wide that the loss due to diffusion to the sides is inappreciable. The mean time T that the gas takes to traverse the distance $d_1 - d_2$ can be found from the rate at which the gas escapes from the gasometer. It is important that this rate of escape should be the same as the rate of escape in the experiments in which n_1 and n_2 were determined, so that the ions should be distributed throughout the same volume of gas.

If cN is the number of ions in a gas in which no new ions are being produced, then the rate at which N varies with the time is given by the formula :

$$dN/dt = -\alpha N^2,\text{*}$$

when no other influences except recombination contribute to the reduction in N.

Hence, by integration,

$$\frac{1}{N_1} - \frac{1}{N_2} = \alpha T,$$

where T is the time in which the conductivity falls from N_2 to N_1 due to recombination. The value of α can therefore be determined by substituting the observed values of N_1, N_2, and T in this equation.

Returning to the case where the gas passes along the fine tubes, the conductivity falls from n_2 to n_1 while the gas passes along the last nine centims. of T_1. Let θ be the average time that any portion of the gas takes to traverse these nine centims. The amount of ionization per cub. centim. of gas can easily be reduced so that the loss of conductivity due to recombination is only $\frac{1}{20}$th of that due to diffusion to the sides.

From formula 10, Section I., we see that the conductivity n at any section of the tubes T_1 is given approximately by the formula :

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* J. J. THOMSON and RUTHERFORD, 'Phil. Mag.,' November, 1896.

$$n = n_2 e^{-\beta t},$$

where t is the time occupied by the gas in travelling from the section where its conductivity is n_2 to the section where its conductivity is n.

From observation we have:

$$n_1 = n_2 e^{-\beta\theta},$$

so that

$$\beta = \frac{1}{\theta} \log \frac{n_2}{n_1}.$$

Let cp be the number of positive ions which recombine with negative ions in the tubes T_1. Then

$$dp/dt = an^2 = an_2^2 e^{-2\beta t},$$

and $p = 0$ when $t = 0$.

Let P be the value of p at the end of the tubes, the value of t being θ at that point.

By integration we obtain:

$$P = a \frac{n_2^2 - n_1^2}{2\beta} = a\theta \frac{n_2^2 - n_1^2}{2 \log \frac{n_2}{n_1}}.$$

Substituting for a its value this equation becomes

$$P = \frac{\theta}{T} \cdot \frac{n_2^2 - n_1^2}{2 \log \frac{n_2}{n_1}} \cdot \frac{N_2 - N_1}{N_1 N_2}.$$

The effect of recombination would obviously be over-corrected for if M were taken equal to P, for although cP ions are lost by encounters with others of opposite sign, still it must be remembered that, had no recombination taken place, some of these cP ions would have lost their charge to the sides of the tube. The number cM should only include those ions which encounter others of opposite sign, and would not subsequently come into contact with the sides if their rate of diffusion were unaltered by the collisions.

It is easy to see that M is less than P and greater than $\frac{n_1}{n_2}$ P; a fairly accurate estimation of its value can be arrived at by the following method:—

At a section z of the tubes, where the conductivity is n, the number that recombine in a time dt is cdp, where $dp = an^2dt$.

From the formula 10, Section I., it can be seen that of these $c\delta p$ ions, $c\delta p \frac{n_1}{n}$ would not come into contact with the sides as they pass from z to the end of the tubes T_1, if their rate of diffusion were unaltered by the recombination. Hence we have

$$\delta M = \delta p \frac{n_1}{n} = ann_1\delta t = an_1n_2e^{-\beta t}\delta t.$$

We therefore obtain, by integration from $t = 0$ to $t = \theta$,

$$M = \frac{\alpha n_1 n_2 (1 - e^{-\beta\theta})}{\beta} = \frac{\alpha n_1 (n_2 - n_1)}{\beta}.$$

Substituting for α and β their values, this equation becomes

$$M = \frac{N_2 - N_1}{N_1 N_2} \times \frac{\theta}{T} \cdot \frac{n_1 (n_2 - n_1)}{\log \frac{n_2}{n_1}}.$$

CHARGE ACQUIRED BY THE GAS.

The deflections n_1, which are obtained by collecting the positive ions after the gas has passed through the long tubes T_1, are invariably greater than the corresponding negative deflections n_1', which shows that the gas on issuing from the tubes T_1 has a positive charge proportional to $n_1 - n_1'$. In each of the tubes T_1 there is a small force arising from this charge, which repels the positive ions towards the sides and attracts the negative ions towards the axes of the tubes. It is only when the rays are strong that this force has any appreciable effect on the motion of the ions.

The deflections n_2 are also slightly greater than n_2', but the difference between these numbers is not so great as the difference between n_1 and n_1'. The values which are obtained for the ratios $\frac{n_1 + M}{n_2}$ are greater than the corresponding values of $\frac{n_1' + M}{n_2'}$, which shows that the negative ions diffuse faster than the positive ions.*

It was found that the rates of diffusion of the positive and negative ions differed more when the gases were dry than when they were moist. Two sets of experiments were therefore made with each gas; in one set the gas was passed through large tubes of calcium chloride and entered the diffusion apparatus dry; in the other set the calcium chloride tubes were removed, and long tubes, partly filled with water, were put in their place. In all the experiments the gases passed through plugs of glass wool before entering the tubes A, in order to remove any dust that might be present.

RESULTS OF EXPERIMENTS WITH AIR.

The following tables give the numbers n_1, n_1', n_2, and n_2', which were obtained with different strengths of rays. The intensity of ionization was reduced to any required value by covering the hole in the lead with pieces of aluminium or zinc. Each experiment consists in determining the four electrometer deflections with a constant strength of rays. The positive and negative deflections n_1 and n_1' are given in the same column, the number in the upper line being n_1 and that in the lower n_1'. The corresponding observations n_2 and n_2' are arranged in a similar manner in the next column. M is the correction for recombination, which has to be added to n_1 and n_1' in order to obtain the ratios $\frac{n_1 + M}{n_2}$ and $\frac{n_1' + M}{n_2'}$, which are the required values of y. The values of $\frac{7{\cdot}31K \times 10}{2a^2V}$ are deduced from Curve I., Section I.; a^2V, the square of the radius of the tubing T multiplied by the velocity of the gas, is obtained by observing the rate of escape of the gas from the gasometer. The coefficients of diffusion of the ions into the gas are given in the last column.

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* J. ZELENY, 'Phil. Mag.,' July, 1898, describes an experiment to which he gives a similar interpretation.

Table I. gives the results of experiments with dry air, and Table II. the results obtained with moist air.

TABLE I. for Dry Air.

Experiment.		n_1 and n_1'.	n_2 and n_2'.	M.	y.	$\frac{73{\cdot}1K}{2a^2V}$.	a^2V.	K.
I.	+	18·6	32·5	·5	·588	·495	2·06	·028
	−	13·0	29·1		·464	·76		·043
II.	+	63·6	118	4·0	·573	·525	2·06	·0296
	−	50·1	110		·491	·70		·0395
III.	+	128	262	22·0	·572	·525	2·06	·0296
	−	104	244		·516	·67		·038
							Mean	·0346

We see that, as the strength of the rays is increased, the values of K for the positive ions appear to increase, and the values for the negative ions to diminish. It will be observed that the charge on the gas ($n_1 - n_1'$) varies from 5·6 to 24, and, as already explained, this charge acts in such a manner as to make the calculated value of K for the positive ions too big and that for the negative ions too small. The nearest value for the ratio of the coefficients of diffusion is therefore 1·54, as given by the first experiment.

Experiments were made to see whether consistent results would be obtained by varying the velocity V, and ·036 was obtained for the mean value of K when a^2V was 1·57.

An experiment with air, made with the apparatus arranged for the other gases, gave the mean value of K = ·034.

These values are as consistent as could be expected when all possible sources of error are taken into account.

TABLE II. for Moist Air.

Experiment.		n_1 and n_1'.	n_2 and n_2'.	M.	y.	$\frac{73\cdot1\,K}{2a^2V}$.	a^2V.	K.
I.	+	23·3	43	1·0	·563	·54	2·12	·031
	−	20·3	40		·532	·61		·035
II.	+	57·0	112·5	4·0	·542	·585	2·06	·033
	−	51·0	106		·515	·464		·036
III.	+	122·4	262	24·0	·559	·58	2·11	·032
	−	112	249		·546	·55		·035
							Mean	·0335

These results show that the mean rate of diffusion is only slightly altered by the presence of moisture, but a large change is produced in the ratio of the coefficients of diffusion of positive and negative ions.

An experiment made with the apparatus shown in fig. 2 gave ·034 for the mean value of K.

Oxygen.

The oxygen which was used was taken from a cylinder which contained about 94 per cent. of oxygen, the other 6 per cent. being principally nitrogen. The rates of diffusion of the ions through oxygen and air only differ by about 5 per cent., so that the presence of 10 per cent. of air would only increase the rate of diffusion by ·5 per cent. No correction need therefore be made for the presence of nitrogen in the gas.

The experiments on diffusion gave results exactly similar to those obtained with air, except that the values of K were about 6 per cent. smaller.

The coefficients of diffusion which were obtained for positive and negative ions in dry oxygen are ·025 and ·0396 (mean ·0323).

The corresponding numbers for the moist gas are ·0288 and ·0358 (mean ·0323).

Carbonic Acid.

The carbonic acid which was used was taken from a cylinder. As in the case of oxygen, no correction need be made for the presence of a few per cent. of air mixed with the gas. It would require 3 per cent. of air to make a difference of 1 per cent. in the coefficient of diffusion of the ions into carbonic acid.

The coefficients of diffusion which were obtained for the positive and negative ions in the dry gas are ·023 and ·026.

The corresponding numbers for the moist gas are ·0245 and ·0255.

The most remarkable difference between the diffusion in carbonic acid and other gases is that the rate of diffusion is nearly equal for the positive and negative ions. In oxygen, air, and hydrogen the rates of diffusion of the positive and negative ions differ by as much as 50 per cent. when the gases are dry, whereas in carbonic acid the difference amounts only to 12 per cent.

Hydrogen.

The hydrogen which was used was generated by the action of hydrochloric acid on zinc. The gas was bubbled through three strong solutions of caustic potash and potassium permanganate, in order to remove the acid vapour and the hydrocarbons, and collected in one of the gasometers. The purity of the gas was tested by finding its specific gravity, which is a very sensitive method of detecting the presence of other gases in hydrogen, since the density of the latter is so small. For this purpose a glass flask having a capacity of about 500 cub. centims. was used. Its volume was accurately found, and its loss of weight when dry hydrogen was substituted in it for dry air; from these two measurements the specific gravity of the gas could be calculated.

The presence of 1 per cent., by pressure, of air would alter the density by 14·5 per cent., which can be very easily detected, as 1 per cent. of air in a 500 cub. centim. flask weighs about 6 milligrammes. It was found that the specific gravity of the hydrogen which was prepared did not differ by 2 per cent. from the value ·00009. After being in the gasometers and the diffusion apparatus for a few days the gas rose in density, due to air getting in. It would have been a matter of great difficulty to have made an apparatus, which had so many rubber joints, perfectly gas-tight, and it was considered simpler to find the amount of air in the hydrogen after each experiment, and to make a correction in the observed coefficient of diffusion.

We may here mention an experiment made with the same apparatus as was used for the determination of the rates of diffusion of the ions into oxygen and carbonic acid. The same velocity of gas in the tubing T was used, a^2V being 2·08. The positive and negative deflections obtained after the gas had passed through the tubes T_2 were 29 and 27·5, and the deflections after passing through the tubes T_1 were 6·5 and 2·2. This shows that in the last nine centimetres of the tubes T_1 the mean conductivity of the hydrogen fell from 28·2 to 4·3. An experiment with oxygen made with the same velocity, $a^2V = 2·08$, showed that the mean conductivity of oxygen was reduced from 30·8 to 15·8. The difference in the behaviour of the ions in the two gases cannot be attributed to a greater rate of recombination of ions in hydrogen, as other experiments show that the ions in hydrogen recombine somewhat slower than the ions in oxygen.

Another set of 12 tubes were therefore made, 4 centims. long, and of the same diameter as the tubes T_1, and substituted in the apparatus shown in fig. 2 instead of the tubes T_1. A series of experiments were made, and values of y were obtained which gave the ratios of the conductivities after the gas had passed through tubing 4 centims. and 1 centim. in length. The corresponding values of K were obtained from Curve II., Section I. In correcting for recombination in these experiments it is necessary to take into account the recombination in the tubes 1 centim. long.

The results obtained from experiments with dry hydrogen which contained 1·6 per cent. of air, are:

K = ·117 for the positive ions,

and

K = ·181 for the negative ions.

The coefficients of diffusion in a mixture containing 1·8 per cent. of air, 1·5 per cent. of water vapour, and 96·7 per cent. of hydrogen are: ·121 and ·134 for the positive and negative ions.

These results show that the rates of diffusion in hydrogen are 4·3 times as great as the rates of diffusion in air. In order, therefore, to obtain the coefficients of diffusion in pure hydrogen at atmospheric pressure, the above determinations must be increased by 3·3 per cent. for each per cent. of air in the gas.

Applying this correction, we obtain the following values for the coefficients of diffusion of the ions into hydrogen:

K = ·123 for positive ions in dry hydrogen,

and

K = ·190 for negative ions in dry hydrogen.

The corresponding coefficients for moist hydrogen are:

K = ·128 for positive ions,

and

K = ·142 for negative ions.

The coefficients of diffusion for the four gases which were examined are given in the following tables.

TABLE III.—Coefficients of diffusion of ions in dry gases.

Gas.	K for + ions.	K for − ions.	Mean value of K.	Ratio of the values of K.
Air	·028	·043	·0347	1·54
Oxygen	·025	·0396	·0323	1·58
Carbonic acid	·023	·026	·0245	1·13
Hydrogen	·123	·190	·156	1·54

TABLE IV.—Coefficients of diffusion of ions in moist gases.

Gas.	K for + ions.	K for − ions.	Mean value of K.	Ratio of the values of K.
Air	·032	·035	·0335	1·09
Oxygen	·0288	·0358	·0323	1·24
Carbonic acid . . .	·0245	·0255	·025	1·04
Hydrogen	·128	·142	·135	1·11

REMARKS ON THE EXPERIMENTS.

The values of y which were found in these experiments are probably correct to 3 per cent., but on referring to the curves it will be seen that the error in K is larger than the error in y. For example, considering Curve I. at the point $y = \cdot 5$, it will be seen that a 4 per cent. error in y gives rise to a 6 per cent. error in K. We would therefore expect that the values of K are correct to about 5 per cent. In order to diminish y without changing the apparatus, the velocity of the gas in the tubing has to be diminished, and this has the effect of increasing in importance the correction for recombination. It was therefore considered best to use velocities of the blast which give y about ·5.

It has been assumed that the velocity of the gas is given by the formula $W = \frac{2V}{a^2}(a^2 - r^2)$, and that the motion takes place in straight lines parallel to the axis of the tube. According to Professor REYNOLDS, the motion of a fluid in a tube is not in straight lines when the velocity exceeds a certain critical value, and eddies are produced even when the motion is initially in lines parallel to the axis. When the velocity is less than another critical velocity, any irregular motion will tend to return to the straight line motion. In order to ensure that the motion of any fluid, whose density is ρ, and viscosity μ, should tend to be in straight lines and obey the formula $W = \frac{2V}{a^2}(a^2 - r^2)$, the value of $V\rho a/\mu$ must be less than 700.

In the present experiments $V\rho a/\mu$ is less than 100, so that the velocities used are $\frac{1}{7}$ of the second critical velocity.

THE ATOMIC CHARGE.

The most interesting results which can be deduced from the coefficients of diffusion are obtained by comparing the velocity under an electromotive force with the coefficient of diffusion.

Considering one of the equations of motion

$$\frac{1}{K} pu = -\frac{dp}{dx} + nXe,$$

where e is the charge on the ion of the gas in electrostatic units, n the number of ions per cub. centim., and p their partial pressure, we see that when $dp/dx = 0$, the velocity u due to the electric force X is $\frac{nXeK}{p}$. If the potential gradient is 1 volt per centim., $X = \frac{1}{300}$ in electrostatic units, and the corresponding value of u is

$$u_1 = \frac{Ke}{300} \times \frac{n}{p} .$$

Let N be the number of molecules in a cub. centim. of a gas at pressure P, equal to the atmospheric pressure, and temperature 15° Centigrade, the temperature at which u_1 and K are determined.

The quotient N/P may be substituted for n/p in the above equation, and since the atmospheric pressure P in C.G.S. units is 10^6, we obtain

$$Ne = \frac{3 \times 10^8 u_1}{K} .$$

If we take the values of u_1 from the table of mean velocities given by RUTHERFORD,* and the mean values of K obtained for dry gases, we get the following values of Ne:—

Air . . . . . $Ne_A = 1·35 \times 10^{10}$,
Oxygen . . . . $Ne_O = 1·25 \times 10^{10}$,
Carbonic acid . . $Ne_C = 1·30 \times 10^{10}$,
Hydrogen . . . $Ne_H = 1·00 \times 10^{10}$,

Experiments on electrolysis show that 1 electrodynamic unit of electricity in passing through an electrolyte gives off 1·23 cub. centims. of hydrogen at temperature 15° Centigrade and pressure = 10^6 C.G.S. units. The number of atoms in this volume is 2·46 N, so that if E is the charge on a hydrogen atom in the liquid electrolyte,

$$\begin{aligned} 2·46\ NE &= 1 \text{ electromagnetic unit,} \\ &= 3 \times 10^{10} \text{ electrostatic units.} \end{aligned}$$

Hence

$$NE = 1·22 \times 10^{10},$$

the charge E being expressed in electrostatic units.

Since N is a constant, we conclude that the charges on the ions produced by Röntgen rays in air, oxygen, carbonic acid, and hydrogen are all the same, and equal to the charge on the hydrogen ion in a liquid electrolyte.

Professor THOMSON† has shown that the charge on the ions in hydrogen and oxygen, which have been made conductors by Röntgen rays, is 6×10^{-10} electrostatic unit, and is the same for both gases.

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* E. RUTHERFORD, 'Phil. Mag.,' November, 1897.

† J. J. THOMSON, 'Phil. Mag.,' December, 1898.

Taking this value for the charge e, we obtain

$$N = 2 \times 10^{19}.$$

From this we deduce the weight of a molecule of hydrogen

$$4{\cdot}5 \times 10^{-24} \text{ gramme.}$$

Every step in the theory by which these numbers are obtained is supported by direct experimental evidence.

Since, as we have just shown, the charge on an ion produced by Röntgen rays, is equal to the charge on a hydrogen ion in a liquid electrolyte, this latter charge is also 6×10^{-10} electrostatic unit.

Although the value of Ne for hydrogen is 25 per cent. different from its value for other gases, we are justified in including hydrogen in the above general conclusion, as we would expect the value of u_1 for hydrogen to be too small. RUTHERFORD makes no mention of having corrected for the presence of air in his apparatus, or of having used perfectly dry hydrogen. If we take the mean value of K for moist hydrogen, we obtain

$$Ne_H = 1{\cdot}15 \times 10^{-10}.$$

In order to prove that the charge on the positive ion is equal to the charge on the negative ion, the ratio of the coefficients of diffusion must be shown to be equal to the ratio of the velocities. Professor ZELENY* has shown that the negative ions travel faster under an electromotive force than the positive ions, the ratio of the velocities being 1·24 for air and oxygen, 1·15 for hydrogen, and 1·0 for carbonic acid.

The experiments on diffusion show that the ratio of the velocities would be larger in dry than in moist gases, but as this point has not yet been examined by Professor ZELENY, we cannot expect a very close agreement between the ratios which he gives for the velocities and the ratios of the coefficients of diffusion.

We are led to conclude that the charges on the positive and negative ions are equal from another point of view. It has been proved that the mean charge is the same as the charge on an ion of hydrogen in a liquid electrolyte. If the charges differed, one of them would be less than the charge on the hydrogen ion, whereas experiments on electrolysis show that all ionic charges are either equal to the charge on the hydrogen ion, or an exact multiple of it.

COMPARISON OF THE RATES OF DIFFUSION OF THE IONS WITH THE RATES OF INTERDIFFUSION OF GASES AND VAPOURS.

The coefficients of diffusion of ions into a gas are much smaller than the coefficients of diffusion of gases into each other, but do not differ much from the coefficients of diffusion of vapours into gases.

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* J. ZELENY, 'Phil. Mag.,' July, 1898.

We give here a table of the latter coefficients, so that they may be compared with the numbers given in Tables III. and IV.

TABLE V., giving Coefficients of Diffusion of some Gases and Vapours into Air, Carbonic Acid, and Hydrogen.

Gas or vapour.	Air.	Carbonic acid.	Hydrogen.	Observer.
Oxygen	...	·18	·721	LOSCHMIDT
Carbonic acid . . .	·142	...	·555	
Æther	·077	·055	·29	WINKELMANN
Alcohol	·101	·068	·378	
Water	·198	·132	·687	

The experimental results show that if K is the coefficient of interdiffusion of two gases whose densities are ρ_1 and ρ_2, $K \times \sqrt{\rho_1\rho_2}$ is roughly constant. The rates of diffusion of the ions are roughly inversely proportional to the square roots of the densities of the gases.

Two theories have been suggested to account for the small values which have been found for the rates of diffusion of the ions into a gas.

The effect may be explained if we suppose that a number of molecules surround the ion. The carriers of the charge would then diffuse slowly like a gas made up of large molecules. The mass of the group could be found by comparing the rates of diffusion of the ions with the rates of interdiffusion of gases. A rough calculation shows that the mass of this group should be about 30 times the mass of a molecule of oxygen.

The small values of the coefficients of diffusion may also be explained if we suppose that the carrier is as small as a molecule of the gas, and that the electric force exerted on the molecules which approach it gives rise to encounters which would not have taken place if the carrier were uncharged.

If we adopt the theory that the ions are surrounded by molecules forming a sphere which moves about with the ion, we can apply MAXWELL'S formula for the coefficient of interdiffusion of two gases to find the radius of the sphere.

The coefficient of interdiffusion of two gases, according to the theory founded on the collisions of elastic spheres, is

$$D_{12} = \frac{1}{2\sqrt{6\pi}} \cdot \frac{V}{N} \cdot \sqrt{\frac{1}{w_1} + \frac{1}{w_2}} \cdot \frac{1}{S_{12}^2},^{*}$$

where w_1 and w_2 are the molecular weights of the two gases, that of hydrogen being unity.

---

* J. C. MAXWELL, 'Nature,' vol. 8.

S_{12} is the distance between the centres of the molecules at collision in centimetres.

V is the "velocity of mean square" of a molecule of hydrogen at 0° C.

$$V = \sqrt{(3p/\rho)} = 186000 \text{ centims. per second.}$$

N is the number of molecules in a cubic centimetre at 0° and 760 millims. pressure. Taking the value of N, which we have found, 2×10^{19}, we see that

$$D_{12} = 1{\cdot}1 \times 10^{-15} \sqrt{\left(\frac{1}{w_1} + \frac{1}{w_2}\right)} \frac{1}{S_{12}^2}.$$

If the carrier of the charge is large compared with the molecule, S_{12} will be the radius of the carrier and $1/w_2$ will be small compared with $1/w_1$.

Letting $D_{12} = {\cdot}156$, the coefficient of diffusion of ions into hydrogen, $w_1 = 1$, we obtain for the radius of the ion in hydrogen

$$S = \sqrt{\frac{1{\cdot}1\ 10^{-15}}{{\cdot}156}} = 8{\cdot}2 \times 10^{-8}.$$

A similar calculation shows that the radius of an ion in oxygen is

$$9{\cdot}2 \times 10^{-8}.$$

Adopting the second theory which we have proposed to account for the slowness of the rates of diffusion of the ions, and applying the same formulæ, the values of S will be greater than the above values by the factor $\sqrt{2}$, since the terms $1/w_1$ and $1/w_2$ are of the same order.

The values obtained in this way for S_{12} would denote the distance that a molecule of the gas must approach an ion in order that the electric force should appreciably alter its motion.

RECOMBINATION.

The results of the experiments which were made to determine the rate of recombination are given in the following table. T is the time, in seconds, in which the conductivity falls from N_2 to N_1, and V is the volume, in cub. centims., of gas which was used in each experiment.

TABLE VI.

Gas.	N_2.	N_1.	T.	V.	Correction to be added to N_1.
Air	77	43	·93	1540	3
Oxygen.	59·5	37	·95	1520	2
Carbonic acid. . .	62·5	39	·90	1590	2
Hydrogen	117	85	·275	1360	5

The numbers N_1 and N_2 are the electrometer deflections obtained in the manner described above. In the first three gases the electrode E_1 was at a distance 12 centims. from the window W_1, and the electrode E_2 at a distance 3 centims. from the window W_2. The conductivity therefore fell from N_2 to N_1 while the gas passed through 9 centims. of the tube A_1.

The position of the electrodes had to be altered for hydrogen, as this gas would have lost about 16 per cent. of its conductivity, due to diffusion to the sides alone, in passing along 9 centims. of the wide tubing. The electrodes were therefore put at distances 3 and 6 centims. from the windows, and the strength of the ionization was increased.

The correction given in the tables is to compensate for the loss of conductivity arising from diffusion.

The electrometer was standardised, and it was found that each division corresponded to a charge of ·0042 electrostatic unit. If e is the charge on the ion, the number of ions in a cub. centim. is $\frac{N \times \cdot 0042}{eV}$, which we will denote by v.

From the theory of recombination we have

$$dv/dt = \beta v^2, \quad \text{or} \quad \frac{1}{v_2} - \frac{1}{v_1} = \beta T.$$

From the numbers given in Table VI. we can obtain the values of β for the different gases. We thus find that for air, oxygen, carbonic acid, and hydrogen the values of β are, $3420 \times e$, $3380 \times e$, $3500 \times e$, and $3020 \times e$.

The rates of recombination in air, oxygen, and carbonic acid are practically the same, and about 15 per cent. greater than the recombination in hydrogen.

Substituting for e its value, we obtain for the first three gases $\beta = 2 \times 10^{-6}$, *q.p.*

We can now find how near two ions of opposite sign must approach each other in order to recombine. If there are v positive ions, and v' negative ions, in a cub. centim., the number that recombine in a time δt is $vv'\beta\delta t$.

The number of negative ions that approach within a distance S of positive ions in the same time can be found from the kinetic theory of gases.

MAXWELL has shown* how to calculate the number of times per second a molecule of one gas will come within a distance R of the molecules of another gas.

This number is

$$2n\sqrt{\pi}\sqrt{(\alpha^2+\beta^2)}\,R^2,$$

where n is the number of molecules per cub. centim. of the second gas, $\alpha^2 = \frac{2}{3}v_1^2$, $\beta^2 = \frac{2}{3}v_2^2$, v_1^2 and v_2^2 are the mean squares of the velocities of agitation of the two gases.

We will suppose that an ion has the same mass as a molecule of the gas in which it is produced. The mean square of the velocity of agitation of the ions will, on this hypothesis, be equal to the mean square of the velocity of agitation of the molecules of the gas. $\overline{v_1^2}$ and $\overline{v_2^2}$ will then be equal to $4{\cdot}7 \times 10^4$ for the ions in oxygen.

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* 'Phil. Mag.,' January and July, 1860.

The number of negative ions that approach within a distance R of positive ions in the time δt will be

$$2vv' \sqrt{\pi} \sqrt{\tfrac{2}{3} (\overline{v_1^2} + \overline{v_2^2})}\, R^2\, \delta t.$$

Equating this number to the number that recombine in the same time we obtain

$$R = \tfrac{1}{3}\, 10^{-5} \text{ centim.}$$

At a distance $\frac{1}{3}\, 10^{-5}$ the charge on an ion would exert a force equal to a fall of potential of 16,200 volts. per centim. This force would make two oppositely-charged ions move towards each other with a velocity of 2×10^4 centims. a second.

It would be premature to discuss any further the results which have been obtained, as experiments are being carried out which may throw additional light upon the subject.

In conclusion, I wish to state that I am greatly indebted to Professor THOMSON for his advice and suggestions during the course of these investigations.

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V. *On the Vibrations in the Field round a Theoretical Hertzian Oscillator.*

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(1.) Although HERTZ realised very fully* that his oscillator did not give "perfectly regular and long continued sine-oscillations," and although BJERKNES† determined so long ago as 1891 the general form of the damping, it does not appear that HERTZ's original investigation of the nature of the vibrations in the field round one of his oscillators has hitherto been modified. Indeed, his diagrams of the wave motion have been copied into more than one text-book,‡ and have usually been taken to represent what actually goes on in the surrounding field. Actually not only the diagrams, but a good deal of HERTZ's original theory of interference requires modification, if we are to obtain quantitative accordance between theory and experiment. The object of the present paper is to give a fuller theory of the nature of the vibrations in the field round a typical Hertzian oscillator.

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* "On very rapid Electric Oscillations," 'Wied. Annal.,' vol. 31, p. 421, 'Electric Waves,' p. 49.

† 'Wied. Annal.,' vol. 44, pp. 74, 513–526. The damping of the oscillation in four or five periods is very marked.

‡ For example, ANDREW GRAY, 'Absolute Measurements in Electricity and Magnetism,' vol. 2, p. 734.

(2.) Assuming that BJERKNES' experiments have shown that the Hertzian oscillator vibrates very nearly according to the type :

$$C^{-p_1t} \sin (p_2t + a),$$

we have to find a solution for the equations for the disturbance in the surrounding field on this basis.

Using HERTZ's notation* for the case of symmetry about the axis of z, then if Z and R be the components of electric force parallel and perpendicular to this axis at a point whose coordinates are z and $\rho = \sqrt{x^2 + y^2}$, and P the component of magnetic force perpendicular to the meridian plane through the same point, we have to find a solution of

$$a^2 \frac{d^2\psi}{dt^2} = \nabla^2\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)},$$

suitable to the initial and boundary conditions. If this be done,

$$Z = \frac{1}{\rho} \frac{d}{d\rho}\left(\rho \frac{d\psi}{d\rho}\right), \qquad R = -\frac{1}{\rho} \frac{d}{dz}\left(\rho \frac{d\psi}{d\rho}\right), \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)},$$

$$P = \frac{a}{\rho} \frac{d}{dt}\left(\rho \frac{d\psi}{d\rho}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}.$$

The component of magnetic force parallel to z is zero.

Now if we suppose that at some distance from its centre the oscillator may be looked upon as a "double point" or as producing an oscillation, which has a very small range l, and with poles having $\pm$ E electricity at the maximum, then we may write for ∇^2,

$$\frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{d}{dr}\right).$$

Assume $\psi = f_1(r) e^{pt}$, and we find

$$a^2p^2f_1 = \frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{df_1}{dr}\right).$$

Writing $r' = apr$, and $f_2 = r'f_1$, we have

$$f_2 = \frac{d^2f_2}{dr'^2}.$$

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* See 'Electric Waves,' English edition, p. 140.

Hence,

$$f_2 = A_1 e^{r'} + A_2 e^{-r'},$$

where A_1 and A_2 are constants and

$$\psi = \frac{A_1 e^{p(t+ar)} + A_2 e^{-p(t-ar)}}{apr}.$$

Take only an *outgoing* wave and write $p = -p_1 + p_2\sqrt{-1}$, hence

$$\psi = \frac{B}{r} e^{-p_1(t-ar)} \sin p_2 (t - ar) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

where t must be $> ar$, or $\psi = 0$.

$1/a$ is clearly the wave velocity v. Take $p_2 = \frac{2\pi}{\lambda} v$, then

$$\psi = \frac{B}{r} e^{-p_1(t-ar)} \sin \frac{2\pi}{\lambda}(vt - r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}.$$

For typical oscillators $2\pi p_1/p_2$ seems to vary from ·3 to ·5, hence $p_1 = {\cdot}3$ to ${\cdot}5 \times v/\lambda$, or we see that if r be small as compared with λ, then

$$\psi = \frac{B}{r} e^{-p_1 t} \sin p_2 t.$$

Now, if X, Y, Z be the three components of electric force, we easily find

$$X = -\frac{d}{dx}\left(\frac{d\psi}{dz}\right), \qquad Y = -\frac{d}{dy}\left(\frac{d\psi}{dz}\right), \qquad Z = -\frac{d}{dz}\left(\frac{d\psi}{dz}\right),$$

or they are the three components of a "potential function"

$$V = Be^{-p_1 t} \sin p_2 t \frac{d}{dz}\left(\frac{1}{r}\right).$$

Take $B = -El$, and we see that V is the potential* due to a "double point" of moment oscillating between $+Ele^{-p_1 t}$ and $-Ele^{-p_1 t}$; thus the maximum charges rapidly diminish with the time. In fact, we have a system entirely analogous with that of HERTZ, except for this rapid diminution of the maximum charges with the time. It is in this running down of the maximum charges that the damping effect of the oscillator consists.

(3.) We shall now proceed to find the forces.

In the first place let us find the value of $\rho\, d\psi/d\rho$, which, following HERTZ, we will term Q. Then the components of electric and magnetic force can all be found by simple differentiation of Q, *i.e.*,

$$Z = \frac{1}{\rho}\frac{dQ}{d\rho}, \qquad R = -\frac{1}{\rho}\frac{dQ}{dz}, \quad \text{and} \quad P = \frac{a}{\rho}\frac{dQ}{dt} \quad . \quad . \quad . \quad . \quad \text{(vi)}.$$

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* To use HERTZ'S language, see 'Electric Waves,' p. 142, and compare MAXWELL, vol. 1, § 129.

We have

$$\begin{aligned} \mathrm{Q} &= \rho\frac{d\psi}{d\rho} = \rho\frac{d\psi}{dr}\,\frac{dr}{d\rho} = \frac{\rho^2}{r}\,\frac{d\psi}{dr} \\ &= -\frac{p^2 \mathrm{E}l}{r}\,\frac{d}{dr}\left\{\frac{e^{-p_1(t-ar)}}{r}\sin p_2(t-ar)\right\} \\ &= -\frac{\rho^2}{r^2}\mathrm{E}le^{-p_1(t-ar)}\left\{\left(p_1a-\frac{1}{r}\right)\sin p_2(t-ar) - ap_2\cos p_2(t-ar)\right\}. \end{aligned}$$

Now put $p_1 = \eta \sin \chi$, $p_2 = \eta \cos \chi$; then if $\rho = r \sin \theta$, we have

$$\mathrm{Q} = \mathrm{E}lap_2 \sin^2 \theta e^{-p_1(t-ar)}\left\{\frac{\cos\left[p_2(t-ar)+\chi\right]}{\cos\chi} + \frac{\sin p_2(t-ar)}{rap_2}\right\} \quad . \quad . \quad \text{(vii)}.$$

In our notation HERTZ finds*

$$\mathrm{Q} = \mathrm{E}lap_2 \sin^2 \theta \left\{\cos p_2(t-ar) + \sin \frac{p_2(t-ar)}{rap_2}\right\}. \quad . \quad . \quad . \quad \text{(viii)}.$$

It is clear that the damped wave train, such as actually occurs with the Hertzian oscillator, makes very considerable changes in the form of Q. Thus:

(*a.*) As we might expect the damping factor e^{-p_1t} is introduced, or rather a damping factor $e^{-p_1(t-t_r)}$ where t_r is the time at which the disturbance reaches a distance r from the centre of the oscillator. Clearly the factor e^{p_1ar} will sensibly modify the form of the lines of electric force obtained by tracing the curves

$$\mathrm{Q} = \text{constant.}$$

(*b.*) The first term in the curled brackets is also sensibly modified both as to amplitude and phase.

The reader must not imagine that the difference between the formulæ (vii) and (viii) marks as soon as the vibrations are set up a great difference in the lines of electric force. All we contend for is that it marks a sensible difference in the shape after a few periods, and that this difference increases with the length of time and the distance of the part of the field considered from the oscillator. In fact loops which would remain at the end of each period finite according to formula (viii), have to shrivel up into points and disappear from the field altogether.

Suppose $2\pi p_1/p_2 = \cdot 4$, than $\tan \chi = \cdot 2/\pi$, and we find $\chi = 3^\circ\ 38'\ 33''\cdot 3$, while $\sec \chi = 1\cdot 002,0244$. Hence, the amplitude of the cosine term is increased by about 2 per thousand, and the phase by 3° to 4°. It is the factor e^{p_1ar}, however, which produces most sensible change. For $p_1a = \frac{p_1\,(\text{period})}{(\text{wave length})} = \cdot 4/\lambda$, if λ be the wave length. Now, if λ be 9·60 metres,† which was about its value for one of HERTZ's oscillators

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* It seems better to write $p_2(t-ar)$ than $p_2(ar-t)$ with HERTZ, because ar must be less than t, or $\mathrm{Q} = 0$, 'Electric Waves,' p. 142.

† HERTZ'S λ is $\frac{1}{2}$ (wave-length). Considerable confusion has arisen from this in various translations of his papers. He assumes $\lambda = 4\cdot 8$ metres, *i.e.*, a wave-length $= 9\cdot 6$.

('Electric Waves,' p. 150), we should have at 12 metres distance from the oscillator $e^{p_1 ar} = \sqrt{e} = 1{\cdot}649$, a factor which can very considerably modify Q as a function of r, if t be not equal to ar, *i.e.*, if the disturbance have not just reached that point of the field.

Expressing Q in terms of the wave-length λ, and the period 2τ, we have

$$\frac{Q}{2\pi El/\lambda} = \sin^2\theta e^{-\nu(t/2\tau - r/\lambda)} \left\{ \frac{\cos\left[2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + \chi\right]}{\cos\chi} + \frac{\sin 2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right)}{2\pi r/\lambda} \right\} \quad . \quad \text{(ix)},$$

where $\nu = 2\pi p_1/p_2 = p_1 \times 2\tau$.

Thus we may write

$$\frac{Q}{2\pi El/\lambda} = \phi\left(\theta, \frac{r}{\lambda}, \frac{t}{\tau}\right),$$

where ϕ is a known function.

The next stage was to plot for a reasonable value of ν, the curves $\phi =$ constant for a series of values of Q at different intervals of time. In order to show the decadence of the strength of the field in the neighbourhood of the oscillator, t was given the 56 successive values $\frac{1}{4}\tau$, $\frac{2}{4}\tau$, $\frac{3}{4}\tau$, τ, $\frac{5}{4}\tau$ . . . $13\frac{3}{4}\tau$, 14τ; or the field-changes were traced for seven complete oscillations, at intervals of $\frac{1}{8}$ oscillation. This was done for a sphere of $1\frac{1}{4}$ wave-lengths round the oscillator, or taking the wave-length to be 9·6 metres, for a sphere of 12 metres radius round the centre of the oscillator.*

Values of $Q\lambda/(2\pi El)$ were chosen so as to give eight systems of curves with relative intensities

50, 30, 10, 1, − 1, − 10, − 30, − 50.

In the accompanying diagrams (Plates 1–7) the fine continuous curves† give the ± 50 intensity, the fine dotted curves the ± 30, the heavy continuous curves ± 10, and the heavy dotted curves ± 1. The outermost circle in each case is the boundary of the field explored; the innermost circle corresponds to the boundary of the area round the oscillator, within which it would hardly be legitimate to consider the oscillator a double electric point. The remaining circles are those for which Q = 0, and which separate positive from negative portions of the field. In the spaces between these circles the field must be considered to have alternative positive and negative intensity. It must of course be borne in mind that the curves only give a meridian section of the surfaces of equal intensity.

The work of calculating, plotting, and drawing such a long series of curves was laborious, but as no investigation had been made of the damping out of an electromagnetic field, the work seemed worth undertaking. Some remarks may be made on the methods by which the arithmetic was carried out.

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* HERTZ's diagrams only extend for three-quarters of a wave-length round the oscillator. Our diagrams suffice in extent to show the detachment of the loops preparatory to their outward propagation.

† The actual selection of fine and heavy, continuous and dotted curves to represent the several intensities, is due to the engraver working on the photographs of the original coloured diagrams. For special aid in the preparation of the diagrams and in their reproduction by photography, we have to heartily thank Messrs. E. WRENN and A. WHEELER respectively.

The equation to the curve being written

$$\sin^2\theta = c \times \psi(r),$$

where $c = \mathrm{Q}\lambda/(2\pi\mathrm{E}l)$ and

$$\psi(r) = e^{\nu(t/2\tau - r/\lambda)}\left\{\frac{\cos\left[2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right)\ \chi\ \right]}{\cos\chi} + \frac{\sin 2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right)}{2\pi r/\lambda}\right\}^{-1},$$

c was given the values $\frac{1}{100}$, $\frac{1}{10}$, $\frac{3}{10}$, $\frac{1}{2}$, and the limiting values of r ascertained, for which θ was real. r was then given a series of values between these limits altering by small differences, and the values of $\sin\theta$ calculated; frequent values of r were taken at portions of the curve where it was found to be turning, and thus a close approximation found to its form. The original diagrams on a scale of a metre to the inch were formed by joining up the calculated points and painting in the curves so formed. These were afterwards reduced by photography.*

The dying out of a part of the field of a particular strength is well illustrated in the diagrams. Take the dying out of strengths greater than ± 50 represented by the fine continuous line. In fig. 25 we see the last sensible loop of the field containing greater strengths than this passing away. Up to fig. 33 we can still trace this portion of the field as a dot. But in fig. 29 the oscillator has ceased to give fresh centres even of this strength, and in fig. 34 they have passed away entirely. Figs. 30 to 41 give the shrinking up and final disappearance of parts of the field with a strength greater than ± 30. Figs. 53 to 56 show the outward passage of the last loop of the field with a strength greater than ± 10, and another ten diagrams would have sufficed to show no trace of strengths greater than ± 1.

(4.) We will now proceed to find the electric and magnetic forces from (vi).

First, the electric force, Z, in direction of the axis of the oscillator:

$$\mathrm{Z} = -\frac{\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3 e^{-\nu(t/2\tau - r/\lambda)}}{2\pi r/\lambda}\left[\sin^2\theta\,\frac{\sin\left\{2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right)+2\chi\right\}}{\cos^2\chi} + (2\cos^2\theta - \sin^2\theta)\right.$$
$$\left.\times\frac{\cos\left\{2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right)+\chi\right\}}{(2\pi r/\lambda)\cos\chi} + (2\cos^2\theta - \sin^2\theta)\,\frac{\sin\left\{2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right)\right\}}{(2\pi r/\lambda)^2}\right] \quad \text{(x)}.$$

Second, the electric force, R, perpendicular to the axis of the oscillator:

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* The diagrams have lost very considerably in the process of photography and engraving. It may possibly be that some of the finer loops and dots will appear not at all, or at least unclearly, after the blocks have been somewhat used. We hope shortly to have kinematograph films of the original diagrams available.

$$R = \frac{El\left(\frac{2\pi}{\lambda}\right)^3 e^{-\nu(t/2\tau - r/\lambda)}}{2\pi r/\lambda} \sin\theta\cos\theta \left[-\frac{\sin\left\{2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + 2\chi\right\}}{\cos^2\chi} + \frac{3\cos\left\{2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + \chi\right\}}{(2\pi r/\lambda)\cos\chi} + \frac{3\sin 2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right)}{(2\pi r/\lambda)^2} \right] \quad \text{(xi).}$$

Now let

$$\phi_1(r, t) = \frac{El\left(\frac{2\pi}{\lambda}\right)^3 e^{-\nu(t/2\tau - r/\lambda)}}{2\pi r/\lambda} \left[\frac{\sin\left\{2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + 2\chi\right\}}{\cos^2\chi} - \frac{3\cos\left\{2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + \chi\right\}}{(2\pi r/\lambda)\cos\chi} - \frac{3\sin 2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right)}{(2\pi r/\lambda)^2} \right] \quad \text{(xii),}$$

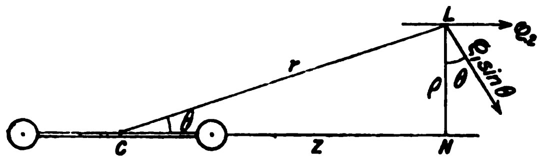
and

$$\phi_2(r, t) = \frac{El\left(\frac{2\pi}{\lambda}\right)^3 e^{-\nu(t/2\tau - r/\lambda)}}{2\pi r/\lambda} \left[\frac{2\cos\left\{2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + \chi\right\}}{(2\pi r/\lambda)\cos\chi} + \frac{2\sin 2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right)}{(2\pi r/\lambda)^2} \right] \quad \text{(xiii).}$$

Then ϕ_1 and ϕ_2 are constant over any spherical surface about the centre of the oscillator at any time, and

$$\left. \begin{aligned} Z &= \phi_1 \sin^2\theta + \phi_2, \\ R &= -\phi_1 \sin\theta\cos\theta, \end{aligned} \right\} \quad \text{(xiv).}$$

The electric force can thus be considered as compounded of a force ϕ_2 parallel to the axis of the oscillator, and depending only on the distance of the point of the field from its centre, and of a force $\phi_1 \sin\theta$, acting in the meridian plane perpendicular to the central distance of any point and towards the oscillator axis.



Clearly along the axis and in the equatorial plane $R = 0$, or the force is parallel to the axis, a result already deduced by HERTZ for a simple harmonic oscillation.* At very great distances we may neglect inverse squares and cubes of r, as compared with first powers, and accordingly ϕ_2 vanishes as compared with ϕ_1. In other words the electric force tends at great distances to become perpendicular to the radius C L, or the propagation to be *purely transverse.*

At a considerable distance from the origin we may take :

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* 'Electric Waves,' pp. 142-3

$$Z = \frac{\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3 e^{-\nu\,(t/2\tau - r/\lambda)}}{2\pi r/\lambda}\sin^2\theta\,\frac{\sin\left\{2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right)+2\chi\right\}}{\cos^2\chi},$$

$$R = -\frac{\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3 e^{-\nu\,(t/2\tau - r/\lambda)}}{2\pi r/\lambda}\sin\theta\cos\theta\,\frac{\sin\left\{2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right)+2\chi\right\}}{\cos^2\chi} \quad . \quad . \quad (\mathrm{xv}).$$

The values in our notation obtained by HERTZ are:

$$Z = \frac{\mathrm{E}l\left(\frac{\pi}{\lambda}\right)^3}{2\pi r/\lambda}\sin^2\theta\sin 2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right),$$

$$R = -\frac{\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3}{2\pi r/\lambda}\sin\theta\cos\theta\sin 2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right).$$

Now it must be remembered that Q and R and Z are all zero until t is $> ar$ or $\frac{t}{2\tau} > \frac{r}{\lambda}$. Hence HERTZ's formulæ imply that at a considerable distance from the origin the intensity of the field increases *gradually* from zero. The formulæ (xv) appear, however, to show that the intensity at a distant point of the field rises abruptly from zero to the definite value:

$$\frac{\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3}{2\pi r/\lambda}\sin\theta\,\frac{\sin 2\chi}{\cos^2\chi} = \frac{2\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3}{2\pi r/\lambda}\sin\theta\tan\chi$$

as the wave reaches it.

The explanation of this is, however, that, while we make the oscillator start from zero charge, yet the initiation is sudden in so far as it requires definite initial values of the electric and magnetic forces. There is an impulsive action at the wave front due to the sudden starting of the oscillator, and the above expression only represents the electric force at a considerable distance from the oscillator, when the impulsive action of the wave front has just passed the point under consideration.

Turning now to the magnetic force P perpendicular to the meridian plane, we have by (vi):

$$P = a\,\frac{1}{\rho}\,\frac{dQ}{dt},$$

or,

$$P = \frac{\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3 e^{-\nu\,(t/2\tau - r/\lambda)}}{2\pi r/\lambda}\sin\theta\left[-\frac{\sin\left\{2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right)+2\chi\right\}}{\cos^2\chi}+\frac{\cos\left\{2\pi\left(\frac{t}{\tau}-\frac{r}{\lambda}\right)+\chi\right\}}{(2\pi r/\lambda)\cos\chi}\right] \quad (\mathrm{xvi}),$$

$$= \phi_3\sin\theta$$

where ϕ_3 is a function of r and t only, and is constant at any time for a spherical surface round the centre of the oscillator.

Clearly P does not, as in HERTZ's formula, appear to gradually rise from a zero value, but suddenly springs to the value

$$\frac{\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3}{2\pi r/\lambda}\sin\theta\left(\frac{1}{2\pi r/\lambda}-2\tan\chi\right),$$

or,

$$=-\frac{2\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3}{2\pi r/\lambda}\sin\theta\tan\chi,$$

at a considerable distance from the oscillator.* This must again be interpreted as representing the value of the magnetic force immediately after the impulsive action at the wave front has passed by.

(5.) We shall now consider what modifications are made in the velocity of transmission owing to the damping of the wave train. HERTZ, GRAY, and others have considered this problem, but have confined their attention to the equatorial plane or the axis, and to a simple harmonic train. There appears to be no real simplicity gained by these limitations. Dealing first with magnetic force in (xvi), we may write the value of P above

$$\mathrm{P}=\mathrm{P}_0(r)\,e^{-\nu(t/2\tau-r/\lambda)}\sin\theta\cos\left\{2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right)+\beta_0\right\}. \quad . \quad . \quad . \quad \text{(xvii)}$$

where we have

$$\mathrm{P}_0(r)\cos\beta_0=\frac{\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3}{2\pi r/\lambda}\left(\frac{1}{2\pi r/\lambda}-2\tan\chi\right),$$

$$\mathrm{P}_0(r)\sin\beta_0=\frac{\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3}{2\pi r/\lambda}\left(\tan\chi\frac{1}{2\pi r/\lambda}+\frac{\cos 2\chi}{\cos^2\chi}\right).$$

Hence

$$\left.\begin{aligned}\mathrm{P}_0(r)&=\frac{\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3}{(2\pi r/\lambda)\cos\chi}\left\{1+\left(\frac{1}{2\pi r/\lambda}-\tan\chi\right)^2\right\}^{\frac{1}{2}},\\ \tan\beta_0&=\frac{\frac{\sin\chi\cos\chi}{2\pi r/\lambda}+\cos 2\chi}{\frac{\cos^2\chi}{2\pi r/\lambda}-\sin 2\chi}\end{aligned}\right\} \quad . \quad . \quad . \quad \text{(xviii)}.$$

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* The apparent equality of the initial electric and magnetic forces arises from the units selected by HERTZ, which have been here adopted for purposes of comparison between the two theories. See 'Electric Waves,' pp. 138–9.

From these we deduce

$$\cot(\beta_0 - \chi) = \frac{1}{2\pi r/\lambda} - \tan\chi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xix)},$$

$$\mathrm{P}_0(r) = \frac{\mathrm{E}l\,(2\pi/\lambda)^3}{(2\pi r/\lambda)\cos\chi\sin(\beta_0 - \chi)} = \frac{\mathrm{E}l\,(2\pi/\lambda)^3\cos(\beta_0 - 2\chi)}{\cos^2\chi\sin^2(\beta_0 - \chi)} \quad . \quad . \quad . \quad \text{(xx)}.$$

From (xix)

$$\frac{1}{\sin^2(\beta_0 - \chi)}\,\frac{d\beta_0}{dr} = \frac{2\pi/\lambda}{(2\pi r/\lambda)^2},$$

or,

$$\frac{d\beta_0}{dr} = \frac{2\pi/\lambda}{(2\pi r/\lambda)^2}\,\frac{1}{\left(\frac{1}{2\pi r/\lambda} - \tan\chi\right)^2 + 1}.$$

To find the wave-speed, we have from (xvii) to find *dr*/*dt* from

$$2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + \beta_0 = \text{constant},$$

i.e.,

$$\frac{\lambda}{2\tau} - \frac{dr}{dt}\left(1 - \frac{\lambda}{2\pi}\,\frac{d\beta_0}{dr}\right) = 0.$$

Let $\lambda/2\tau = v$ as before, and $dr/dt = \mathrm{V}_0$. Then

$$\mathrm{V}_0/v = \frac{1}{1 - \dfrac{\xi}{1 + (\xi - \tan\chi)^2}},$$

where $\xi = \dfrac{1}{2\pi r/\lambda}$. Hence

$$\frac{\mathrm{V}_0 - v}{v} = \frac{\xi^2}{1 + \tan^2\chi - 2\xi\tan\chi} = \frac{1}{2\tan\chi} \times \frac{\xi^2}{\operatorname{cosec} 2\chi - \xi} \quad . \quad . \quad . \quad \text{(xxi)}.$$

Now this result is quite independent of the *direction* of propagation, or the wave moves outwards with the same velocity at the same distance in all directions.

For $\chi = 0$, $\dfrac{\mathrm{V}_0 - v}{v} = \xi^2$. This is the value given by GRAY for propagation in the equatorial plane,* but it is clearly independent of direction. At considerable distances ξ is very small, and therefore $\mathrm{V}_0 = v$. Thus v is the limit of wave velocity as we recede from the source of disturbance.

Now a remarkable result flows from (xxi), which could not in any way be predicted from HERTZ's theory, neglecting the damping. HERTZ tells us that the velocity of propagation at the source is infinite, and GRAY draws the same conclusion, but (xxi) shows us that near the source the velocity of propagation, although very great, is

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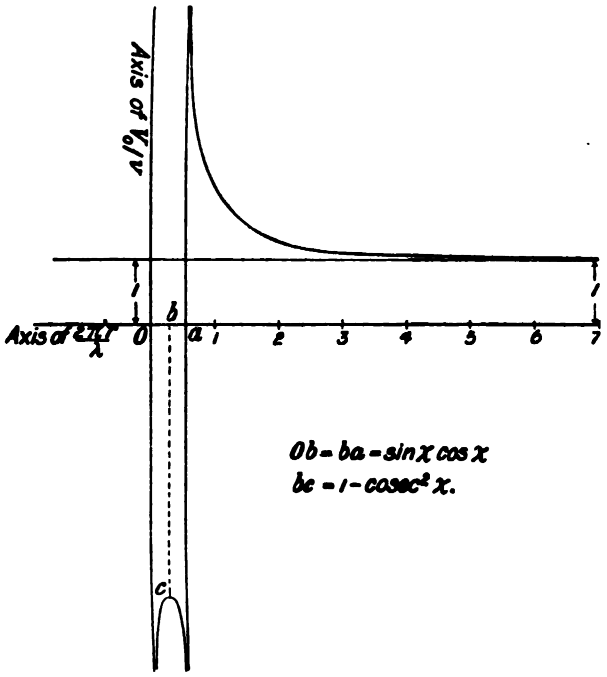
* 'Absolute Measurements,' vol. 2, p. 781

negative. This holds until $\xi = \operatorname{cosec} 2\chi$, and then V_0 become indefinitely great positively. Thus, from the surface of the sphere $r = \frac{\lambda \sin 2\chi}{2\pi}$ the waves move inwards and outwards with indefinitely great velocities. When $\chi = 0$, this sphere closes in on the oscillator, and it will generally be well within the sphere round the oscillator within which it is not legitimate to apply the theory. But for a very rapidly damped wave train and a very considerable wave length, it is possible that its existence could be physically detected.

We may write (xxi)

$$\frac{V_0}{v} = 1 + \frac{\cos^2 \chi}{\frac{2\pi r}{\lambda}\left(\frac{2\pi r}{\lambda} - \sin 2\chi\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxii)}.$$

Hence V_0/v is symmetrical about $2\pi r/\lambda = \frac{1}{2} \sin 2\chi$, and we have the following curve for V_0/v plotted to $2\pi r/\lambda$:



N.B.—This figure is purely diagrammatic. Oa is not really comparable with O1, and bc is possibly 20,000 units of the vertical scale.

The diagram illustrates the rapidity with which the velocity of the magnetic disturbance approximates to v, and shows the minimum negative velocity $v \operatorname{cosec}^2 \chi$ occurring at a distance $r = \frac{\lambda}{4\pi} \sin 2\chi$ from the centre of the oscillator.

(6.) Turning now to the wave-speed for the electric force, we see from the remarks on (xiv) that we can treat the wave as made up of two components corresponding to

ϕ_1 and ϕ_2. This is really a kinematic resolution convenient for analysis of the wave phenomena. We might have resolved the displacement in other ways, possibly with equal advantage. But it must not be supposed that a purely kinematical resolution into wave components can have no physical significance. It is true that neither ϕ_1 nor ϕ_2 could exist by themselves, but this is practically true of all the wave resolutions the mathematical physicist habitually deals with. He is accustomed to consider in electro-magnetism waves of electric and magnetic force, neither have an independent existence; of radial and transverse electric displacement, one is impossible without the other. In the theory of the refraction and reflection of waves at the interface of elastic media, and its applications to the undulatory theory of light, we do not hesitate to separate the waves of transverse from longitudinal displacement and to speak of the former as having an independent existence, yet we are really separating kinematically what can only coexist. Lastly, consider, perhaps, the most familiar case, that of the longitudinal vibrations of rods; here the physicists carry the kinematic resolution so far that they often forget to mention the coexistence of the vibrations perpendicular to the axis of the rod, without which the longitudinal vibrations could not exist at all. The fact is that a purely kinematic resolution is often of first-class physical importance, for one or other of its factors admits of ready physical determination. Thus, in our present case, ϕ_2 is the sole component of electric force in the axis of the oscillator, and determined there it is determined for all points of space at the same distance from the centre. Again, $\phi_1 + \phi_2$ is the component of electric force in the equatorial plane, and with determinations there, ϕ_2 will be known at all points of space. But it was precisely in the direction of the axis and in the equatorial plane that HERTZ made his chief experiments. Thus there seems considerable advantage in reducing the analysis of the electric force to two functions, both independent of the latitudes and varying only with the distance from the oscillator, which can be directly tested in the localities HERTZ found best suited to experimental enquiries. ϕ_1 and ϕ_2 are both independent of the latitude, and give wave speeds V_1 and V_2 having the same values for all points at the same distances from the centre of the oscillator.

The reader will notice that in taking ϕ_2 and $\phi_1 \sin\theta$ as our constituents we have resolved the electric force into two components, one along the axis and one transverse to the ray, and these components will not generally be at right angles. Neither will represent the total force in the given direction; they are transverse and axial components and not total transverse and total axial electric forces.

The total force perpendicular to axis $= -\phi_1 \sin\theta \cos\theta$;
The total axial electric force . . $= \phi_1 \sin^2\theta + \phi_2$;
The total transverse electric force . $= -(\phi_1 + \phi_2)\sin\theta$; and
The total radial electric force . . $= \phi_2 \cos\theta$.

It will be observed that the amplitude of the axial component does not change with θ, but that the amplitude of both the total transverse and the total electric radial forces does. So far as the theory of wave transmission goes, the wave-speed of the total radial force is really discussed under the treatment of ϕ_2, our axial component, in Art. 7. Similarly, the total transverse electric force has a wave-speed determined from $\phi_1 + \phi_2$ and, therefore, it will be found fully discussed under our treatment of the function $\phi_1 + \phi_2$, the equatorial wave in Art. 12. In addition, our analysis enables us to deal separately with that portion of the total transverse force ϕ_1, or the transverse component in our case, which alone is propagated to considerable distances, and which gives at all distances the total force perpendicular to the axis.

(7.) Let us deal first with the case of the component force parallel to the axis or the ϕ_2 factor of the total radial electric force. We may write

$$\phi_2 = \mathrm{P}_2(r)\, e^{-\nu(t/2\tau - r/\lambda)} \sin\left\{2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + \beta_2\right\} \quad . \quad . \quad . \quad . \quad \text{(xxiii)},$$

where

$$\mathrm{P}_2(r) \sin\beta_2 = 2\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3 \xi^2$$

$$\mathrm{P}_2(r) \cos\beta_2 = -\, 2\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3 \xi^2 (\tan\chi - \xi).$$

Thus,

$$\mathrm{P}_2(r) = 2\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3 \xi^2 \{1 + (\tan\chi - \xi)^2\}^{\frac{1}{2}},$$

and,

$$\cot\beta_2 = -\tan\chi + \xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxiv)}.$$

Hence

$$\frac{1}{\sin^2\beta_2}\frac{d\beta_2}{dr} = \frac{\lambda}{2\pi}\frac{1}{r^2} = \frac{2\pi}{\lambda}\xi^2,$$

and differentiating $2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + \beta_2 = 0$ with regard to the time to find $\mathrm{V}_2 = dr/dt$, we have

$$v = \mathrm{V}_2(1 - \xi^2 \sin^2\beta_2).$$

Thus the wave is

$$\phi_2 = \frac{2\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3}{(2\pi r/\lambda)^2}\left\{1 + \left(\tan\chi - \frac{\lambda}{2\pi r}\right)^2\right\}^{\frac{1}{2}} e^{-\nu(t/2\tau - r/\lambda)} \times \sin\left\{2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + \beta_2\right\} \quad \text{(xxv)},$$

and its velocity is given by

$$\mathrm{V}_2/v = 1 + \frac{\cos^2\chi}{\frac{2\pi r}{\lambda}\left(\frac{2\pi r}{\lambda} - \sin 2\chi\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxvi)}.$$

We see at once that $\mathrm{V}_2 = \mathrm{V}_0$, or the magnetic wave and the wave of component electric force parallel to the axis are propagated everywhere with the same velocity at the same distance from the axis. At the same time the amplitude of this electric wave varies inversely as the *square* of the distance from the centre of the oscillator for considerable distances, while the amplitude of the magnetic wave varies under the same conditions inversely as the distance. Thus the effect of the former is rapidly insensible as compared with the latter. Meanwhile it is important to observe how this wave keeps pace with the magnetic wave.

(8.) We may now consider ϕ_1, which gives the transverse electric component $\phi_1 \sin\theta$ and the total electric force perpendicular to the axis, *i.e.*, $-\phi_1 \sin\theta\cos\theta$.*

$$\phi_1 \sin\theta = \mathrm{P}_1(r)\, e^{-\nu(t/2\tau - r/\lambda)} \sin\theta \times \sin\left\{2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + \beta_1\right\} \quad . \quad \text{(xxvii)},$$

where

$$\mathrm{P}_1(r)\cos\beta_1 = \mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3 \xi\,(1 - \tan^2\chi + 3\xi\tan\chi - 3\xi^2),$$

$$\mathrm{P}_1(r)\sin\beta_1 = \mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3 \xi\,(2\tan\chi - 3\xi).$$

Hence

$$\mathrm{P}_1(r) = \frac{\mathrm{E}l\left(\frac{2\pi}{\lambda}\right)^3 \xi}{\cos^2\chi}\,\{1 - 3\xi\sin 2\chi + 3\xi^2\cos^2\chi\,(1 + 4\cos^2\chi)$$
$$- 3\xi^3\sin 2\chi\cos^2\chi + 9\xi^4\cos^4\chi\} \quad . \quad . \quad \text{(xxviii)},$$

and

$$\tan\beta_1 = \frac{2\tan\chi - 3\xi}{1 - \tan^2\chi + 3\xi\tan\chi - 3\xi^2}.$$

From the last equation we deduce, if $\zeta = 1/\xi = \dfrac{2\pi r}{\lambda}$,

$$\tan(\beta_1 - 2\chi) = \frac{-3(1 + \tan^2\chi)(\zeta - \sin 2\chi)}{\zeta^2(1 + \tan^2\chi)^2 - 3\zeta\tan\chi\,(1 + \tan^2\chi) - 3(1 - \tan^2\chi)} = \frac{-3\epsilon}{(\epsilon + \gamma)^2 - 3(1 - \gamma^2)}$$

where

$$\epsilon = (1 + \tan^2\chi)(\zeta - \sin 2\chi), \qquad \gamma = \tfrac{1}{2}\tan\chi.$$

Hence

$$\frac{1}{\cos^2(\beta_1 - 2\chi)}\frac{d\beta_1}{dr} = 3(1 + \tan^2\chi)\frac{2\pi}{\lambda} \times \frac{3 + \epsilon^2 - 4\gamma^2}{\{(\epsilon + \gamma)^2 - 3(1 - \gamma^2)\}^2},$$

$$\frac{d\beta_1}{dr} = \frac{2\pi}{\lambda}\,\frac{3(1 + 4\gamma^2)(3 + \epsilon^2 - 4\gamma^2)}{(\epsilon^2 + 2\gamma\epsilon + 4\gamma^2 - 3)^2 + 9\epsilon^2}.$$

---

* This again is an important physical significance for ϕ_1, and enables it to be readily differentiated physically from ϕ_2 and $\phi_1 + \phi_2$.

We must now differentiate $\frac{2\pi}{\lambda}(vt - r) + \beta_1 = 0$, to get the velocity V_1, and we find

$$v = V_1\left(1 - \frac{\lambda}{2\pi}\frac{d\beta_1}{dr}\right),$$

or,

$$\frac{V_1}{v} = 1 + \frac{1}{\frac{2\pi}{\lambda}\frac{1}{d\beta/dr} - 1}.$$

Substituting for $d\beta_1/dr$, we have, after reductions :

$$\frac{V_1}{v} = 1 + \frac{3(1 + 4\gamma^2)\{\epsilon^2 - (4\gamma^2 - 3)\}}{\epsilon^4 + 4\gamma\epsilon^3 + 4\gamma(4\gamma^2 - 3)\epsilon + 16\gamma^2(4\gamma^2 - 3)}.$$

Now let

$$\left.\begin{aligned}\zeta_0 &= \sin 2\chi\\ \zeta_1^2 &= \cos^2\chi(3\cos^2\chi - \sin^2\chi)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxix}).$$

Then

$$\zeta_0 = \frac{4\gamma}{1 + 4\gamma^2}, \qquad \zeta_1^2 = \frac{3 - 4\gamma^2}{(1 + 4\gamma^2)^2},$$

and

$$\begin{aligned}\frac{V_1}{v} &= 1 + \frac{3\cos^2\chi\{(\zeta - \zeta_0)^2 + \zeta_1^2\}}{(\zeta - \zeta_0)^4 + \zeta_0(\zeta - \zeta_0)^3 - \zeta_0\zeta_1^2(\zeta - \zeta_0) - \zeta_0^2\zeta_1^2}\\ &= 1 + 3\cos^2\chi\frac{\{(\zeta - \zeta_0)^2 + \zeta_1^2\}}{\zeta\{(\zeta - \zeta_0)^3 - \zeta_0\zeta_1^2\}}.\end{aligned}$$

Let $\zeta_0 = 2\pi r_0/\lambda$, $\zeta_1 = 2\pi r_1/\lambda$. Then

$$\frac{V_1}{v} = 1 + \frac{3\cos^2\chi}{(2\pi/\lambda)^2}\,\frac{(r - r_0)^2 + r_1^2}{r\{(r - r_0)^3 - r_0 r_1^2\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxx}).$$

This gives the velocity V_1 at each distance r from the origin of the transverse electric wave. Its amplitude is given by (xxviii).

When r is great, the amplitude approaches the value

$$\frac{El(2\pi/\lambda)^3}{\cos^2\chi(2\pi r/\lambda)}.$$

Therefore at such distances

$$\phi_2/\phi_1 = 2(\lambda/2\pi r),$$

or ϕ_2 is insensible as compared with ϕ_1. Even at distances 5 to 10 times the wave length, ϕ_2 will be very small as compared with ϕ_1; that is to say, the electric vibrations at comparatively short distances are to all intents and purposes *transverse*.

Turning now to the velocity V_1, we will first endeavour to trace its changes. Let us write $r_0/r_1 = q^3$, where q is generally a small quantity. Then the denominator of the expression for $V_1 - v$ in terms of r (see xxx) may be written

$$r(r - r_0 - qr_1)\{(r - r_0)^2 + (r - r_0)qr_1 + q^2r_1^2\}.$$

The second factor is negative for $r < r_0 + qr_1$, then vanishes for $r = r_0 + qr_1$, and is ever afterwards positive.

The third factor would vanish for imaginary values only, and since it is positive for $r = r_0$, it remains positive always. This supposes that r_1^2 is positive. Similarly the numerator in the value of ϕ_1, or $(r - r_0)^2 + r_1^2$ will always be positive, if r_1^2 be positive. Now by (xxix), r_1^2 cannot be negative unless $\tan\chi > \sqrt{3}$ or $\chi > 60°$. This corresponds to a degree of damping in which the amplitude would be reduced to ·000019 of itself every period, or a degree immensely higher than that of the usual Hertzian oscillator. Hence both the numerator and the third factor of the denominator are always positive.

Accordingly $V_1 - v$ is negatively infinite for $r = 0$, becomes negatively finite, but very large, until $r = r_0 + qr_1$ when it again becomes negatively infinite, after this it becomes positively infinite, and rapidly decreases to zero as r increases. Thus, as in the case of the other waves, there is a sphere round the oscillator within which the waves move inwards for transverse electric vibrations. This sphere is of somewhat larger radius ($r_0 + qr_1$ as compared with r_0) than in the case of the magnetic wave. In terms of χ the radius of this sphere is given by

$$\frac{\lambda}{2\pi}(\sin 2\chi)\{(\tfrac{3}{4}\operatorname{cosec}^2\chi - 1)^{1/3} + 1\}.$$

For example, when $\tan\chi = {\cdot}2/\pi$ or $\chi = 3° \, 38' \, 33''{\cdot}5$, we find for the radius of this sphere, $6{\cdot}69604\, r_0$, or between six and seven times the radius of the sphere within which the velocity of the magnetic wave is negative. Substituting the value of r_0, the radius $= {\cdot}135\lambda$. Hence, for a small oscillator, say $\frac{1}{10}$ of a wave-length long, this sphere would be well outside the sphere ·05 of a wave-length circumscribing the oscillator, and thus practically within the field to which our theory might be approximately applied. The inward moving wave of transverse electric vibrations ought thus to be capable of experimental demonstration.

We have, in the next place, to find the minimum negative velocity, and its distance from the centre.

Writing $r - r_0 = u$ and $r_0 = q^3 r_1$, we have to find the maxima or minima of

$$\frac{u^2 + r_1^2}{(u + q^3r_1)(u^3 - q^3r_1^3)}.$$

Differentiating, we find for the required values

$$2\left(\frac{u}{r_1}\right)^5 + q^3\left(\frac{u}{r_1}\right)^4 + 4\left(\frac{u}{r_1}\right)^3 + 4q^3\left(\frac{u}{r_1}\right)^2 + 2q^6\left(\frac{u}{r_1}\right) - q^3 = 0 \quad . \quad \text{(xxxi).}$$

Now, at $r = r_0$ it is easy to see that the curve in which $\frac{V_1 - v}{v}$ is plotted to r is still sloping towards the horizontal. For if $\frac{V_1 - v}{v} = \eta$ and $r - r_0 = \xi$, its equation is approximately

$$\eta = -\frac{3 \cos^2 \chi}{(2\pi/\lambda)^2} \frac{1}{r_0 (r_0 + \xi)},$$

or, η decreases with increase of ξ hyperbolically.

Hence the minimum negative value sought must lie between r_0 and $r_0 + qr_1$, say at $r_0 + \eta q r_1$, where η is less than unity but not necessarily very small. Thus we must put $u/r_1 = \eta q_1$ where q is small, and endeavour to solve (xxxi). Substituting, we have

$$2q^5\eta^5 + q^7\eta^4 + 4q^3\eta^3 + 4q^5\eta^2 + 2q^7\eta - q^3 = 0,$$

or,

$$2q^2\eta^5 + q^4\eta^4 + 4\eta^3 + 4q^2\eta^2 + 2q^4\eta - 1 = 0.$$

Clearly, η must be of the form $\eta_0 + \eta_1 q^2 + \eta_2 q^4 + \eta_3 q^6 + \&c.$ Let us substitute this value and equate the successive powers of q^2 as well as the constant term to zero. We find

$$4\eta_0^3 - 1 = 0,$$
$$\eta_0^5 + 6\eta_1\eta_0^2 + 2\eta_0^2 = 0,$$
$$10\eta_0^4\eta_1 + \eta_0^4 + 12\eta_0^2\eta_2 + 12\eta_0\eta_1^2 + 8\eta_0\eta_1 + 2\eta_0 = 0,$$
$$10\eta_0^4\eta_2 + 20\eta_0^3\eta_1^2 + 4\eta_0^3\eta_1 + 12\eta_0^2\eta_3 + 24\eta_0\eta_1\eta_2 + 4\eta_1^3 + 8\eta_0\eta_2 + 4\eta_1^2 + 2\eta_1 = 0.$$

These equations give us

$$\left.\begin{aligned} \eta_0 &= (\tfrac{1}{4})^{1,3} = \cdot 62996 \\ \eta_1 &= -\tfrac{3}{8} = -\cdot 375 \\ \eta_2 &= 0 \\ \eta_3 &= \cdot 01476 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad \text{(xxxii)}.$$

Thus

$$\eta = \cdot 62996 - \cdot 375 q^2 + \cdot 01476 q^6.$$

We find at once that the required radius of the sphere at which the velocity of the electric transverse wave takes a minimum negative value

$$= r_0 + \eta q r_1 = r_0 (1 + \eta/q^2)$$
$$= r_0 \left(\frac{\cdot 62996}{q_2} + \cdot 625 + \cdot 01476 q^4\right) \quad \ldots\ldots \quad \text{(xxxiii)}.$$

The next term in the expression for the radius would involve q^6 and may generally be neglected. If the point of minimum velocity were half-way between the two points of infinite negative velocity, we should have the radius $= \frac{1}{2}(r_0 + qr_1)$ $= r_0\left(\frac{\cdot 5}{q^2} + \cdot 5\right)$. Hence, the point of minimum velocity is more than half-way towards the outer point of infinite velocity. For the particular oscillator referred to on p. 162, $q^2 = \cdot 17556$, and the radius of the sphere of minimum velocity $= 4\cdot 21\, r_0$ nearly, while the sphere of infinite negative velocity has $6\cdot 70\, r_0$ for radius nearly (see p. 174).

At a distance from the centre of the oscillator we have very approximately

$$\frac{V_1 - v}{v} = \frac{3\cos^2\chi}{(2\pi r/\lambda)^2},$$

or,

$$V_1 - v = 3(V_2 - v).$$

Thus at some distance from the centre of the oscillator, the excess of the velocity of this component transverse electric wave (or of the total electric wave perpendicular to the axis) over the velocity of light is three times the excess of the velocity of the magnetic wave.

In order to find the distance from the centre at which this component transverse electric and the magnetic waves have equal negative velocities we must make

$$\frac{1}{r(r - r_0)} = \frac{3\{(r - r_0)^2 + r_1^2\}}{r\{(r - r_0)^3 - r_0 r_1^2\}},$$

or, putting $r - r_0 = u$, solve the cubic

$$2u^3 + 3r_1^2 u + r_0 r_1^2 = 0.$$

Take $u = -\eta r_0 = -\eta q^3 r_1$; hence

$$2\eta^3 q^6 + 3\eta - 1 = 0.$$

Thus,

$$\eta = \tfrac{1}{3} - \tfrac{2}{81} q^6, \text{ nearly} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(xxxiv).}$$

We conclude accordingly that the two velocities are negatively equal at a distance from the centre equal to $\frac{2}{3} r_0 (1 + \frac{1}{27} q^6) = \frac{2}{3} r_0$, nearly.

This equal negative velocity is $v\left(1 - \frac{1\cdot 125}{\sin^2\chi}\right)$, nearly. Similarly, the minimum negative velocity of the electric transverse wave may be shown to be

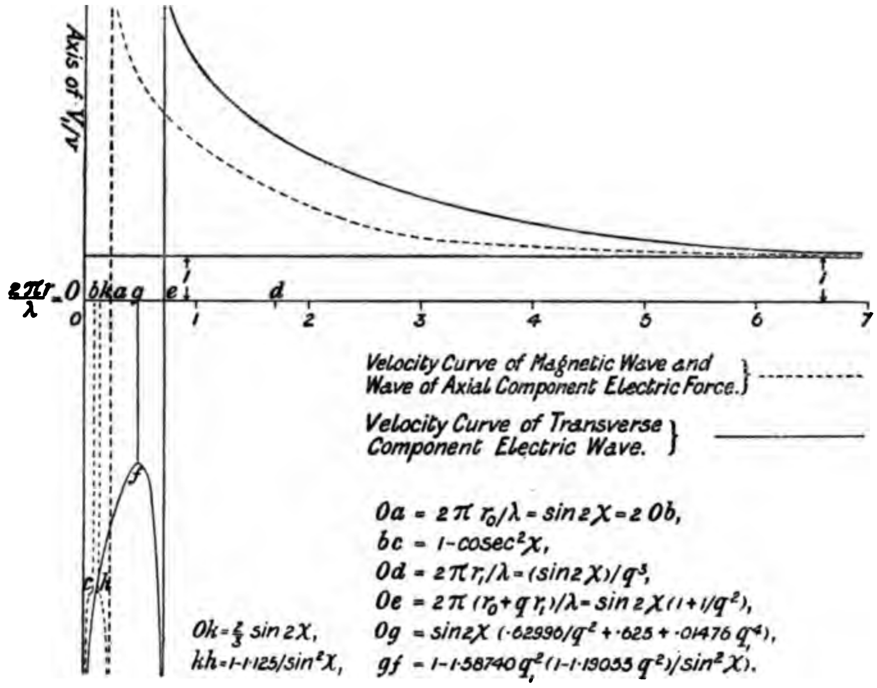
$$v\left\{1 - \frac{1\cdot 58740 q^2}{\sin^2\chi}(1 - 1\cdot 19055 q^2 + 2\cdot 12601 q^4)\right\} \quad . \quad . \quad . \quad . \qquad \text{(xxxv),}$$

where $q^2 = \dfrac{1}{\{\frac{3}{4}\operatorname{cosec}^2\chi - 1\}^{1/3}}$ as before.

For the special oscillator discussed above it equals $-$ 59·545, or is slightly less than 60 times the velocity of light. The minimum negative velocity of the transverse magnetic wave $= -248 \cdot 46v$, and the equal negative velocity of both waves $= -279 \cdot 64v$. Thus again it appears more feasible to test experimentally the negative velocity of this transverse electric wave than that of the magnetic wave.

With regard to these negative waves, we can only cite what HERTZ has remarked on the infinite motion which occurs with a steady oscillator ('Electric Waves,' p. 146): "At an infinitesimal distance from the origin the velocity of propagation is even infinite. This is the phenomenon which, according to the old mode of expression, is represented by the statement that upon the electromagnetic action, which travels with velocity $^1/A$ [our v], there is superposed an electrostatic force travelling with infinite velocity. In the sense of our theory we more correctly represent the phenomenon by saying that fundamentally the waves which are being developed do not owe their formation solely to processes at the origin, but arise out of the conditions of the whole of the surrounding space, which latter, according to our theory, is the true seat of the energy."

The following figure gives the velocity curve of the transverse component electric wave or of the total electric wave perpendicular to the axis *diagrammatically*.



(9.) It is noteworthy that HERTZ, speaking of the electric force in the equatorial plane, *i.e.*,

$$Z = \phi_1 + \phi_2,$$

says that it cannot *possibly* be broken up into two simple waves travelling with different velocities ('Electric Waves,' p. 151). The explanation of HERTZ's difficulty is, perhaps, simple; for, having put $\theta = \pi/2$, the distinction between ϕ_1 and ϕ_2 was lost in his theory. We may consider these to be two separate waves travelling in the equatorial plane, both indeed transverse just for this plane, but one ϕ_1 travelling with the velocity of the transverse component electric wave and the other with the velocity of the magnetic wave, or that of the wave of axial component electric force. Thus HERTZ's original explanation of the irregularity of the interferences which did not succeed each other at equal distances, but with more rapid changes in the neighbourhood of the oscillator, seems justified by the fuller theory. Namely, he explained this behaviour by the supposition "that the total force might be split up into two parts, of which the one, the electromagnetic, was propagated with the velocity of light, while the other, the electrostatic, was propagated with greater and perhaps infinite velocity." Actually we may resolve into two waves; the transverse component electric wave is propagated with greater velocity than the wave of axial component electric force, which has the velocity of the magnetic wave. Many of HERTZ's experiments were made at a comparatively short distance from the oscillator, and some of the discrepancies he noted between his theory and experiment seem capable of explanation by aid of the velocity diagram given above.

(10.) There is another quantity which HERTZ considers at length, namely, the rate of change of the magnetic force, which gives the integral force of induction round a small circle in a plane perpendicular to the magnetic force. We shall now accordingly consider the expression for dP/dt from (xvii):

$$\begin{aligned}\frac{dP}{dt} &= P_0(r)\sin\theta\, e^{-\nu(t/2\tau - r/\lambda)}\left\{-p_1\cos\left(2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right)+\beta_0\right)\right.\\ &\qquad\left.-p_2\sin\left(2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right)+\beta_0\right)\right\}\\ &= -\frac{P_0(r)\,v\,(2\pi/\lambda)\sin\theta}{\cos\chi}\,e^{-\nu(t/2\tau - r/\lambda)}\sin\left(2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right)+\beta_0+\chi\right)\\ &= -\frac{P_0(r)\,v\,(2\pi/\lambda)\sin\theta}{\cos\chi}\,e^{-\nu(t/2\tau - r/\lambda)}\sin\left(2\pi\left(\frac{t}{2\tau}-\frac{r}{\lambda}\right)+\beta_3\right) \quad . \quad \text{(xxxvi)},\end{aligned}$$

where $\beta_3 = \beta_0 + \chi$, and therefore from (xix),

$$\cot(\beta_3 - 2\chi) = \frac{1}{2\pi r/\lambda} - \tan\chi.$$

A more convenient form is easily deduced, namely:

$$\tan\left(\beta_3 - 3\chi\right) = \frac{2\pi}{\lambda}\left(r - \tfrac{1}{2}r_0\right)\Big/\cos^2\chi \quad . \quad . \quad . \quad . \quad \text{(xxxvii)},$$

where $r_0 = \frac{\lambda}{2\pi}\sin\chi$ as before.

Now let δ_3 give the phase of the sine term in (xxxvi), or :

$$\delta_3 = \frac{2\pi r}{\lambda} - \beta_3.$$

Comparing with (xxiv), we see $\beta_3 = \beta_2 + 2\chi$, or :

$$\delta_3 = \delta_2 - 2\chi.$$

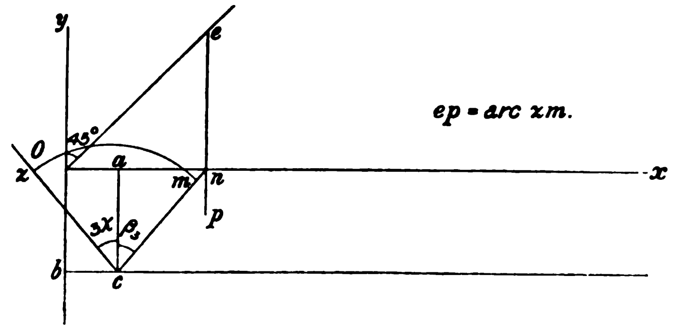
Thus the phase of the wave of magnetic induction always differs by a constant from that of the wave of axial component electric force.

Since $d\beta_3/dr = d\beta_2/dr$, the velocity of the wave of magnetic induction equals that of the wave of axial component electric force, and is given by the V_2 of (xxvi), or the diagram of p. 169.

(11.) In our diagram (p. 184) we have plotted δ_3 from the calculated curve for δ_2, but it is interesting to note the following graphical construction for finding it directly. Let $\zeta = 2\pi r/\lambda$ as before.

Let Ox be the axis of ζ, Oy of δ_3, then $\delta_3 = \zeta - \beta_3$. Take $Ob = \cos^2\chi$, $Oa = \frac{1}{2}(2\pi r_0/\lambda) = \frac{1}{2}\zeta_0$, and let the vertical through a meet the horizontal through b in c. Let On equal ζ, then $an = \zeta - \frac{1}{2}\zeta_0$ and $an/ca = (\zeta - \frac{1}{2}\zeta_0)/\cos^2\chi = \tan(\beta_3 - 3\chi)$, or the $\angle\, acn = \beta_3 - 3\chi$.

Take cz, so that $\angle\, zca = 3\chi$, then $\beta_3 = \angle\, zcn$. If a circle be described round c with radius $cz = 1$, the arc zm corresponding to the angle zcn is the required length β_3. Subtract this arc (rectified, say, by RANKINE's rule) from the corresponding ordinate ne of the 45° line through the origin ($\delta = \zeta$), and we find the true value of $\delta_3 = np$. This construction was actually gone through and compared with the calculated values for verification.



N.B.—The angle χ has been much exaggerated for diagrammatic purposes.

(12.) Before we refer the reader to our diagram giving the phases of the various waves, and describe how they were obtained, it seems well to consider the combined wave of total transverse electrical force in the equatorial plane. The resultant electrical force in this plane may be considered, as we have already seen, to consist of two parts, having different wave-speeds; but, possibly, if a receiver were sufficiently small, it might not be possible to differentiate them, and HERTZ's theoretical view of their unity seems finally to have mastered his experimental suspicion as to the possible coexistence of two waves in this plane (see our p. 177). Accordingly we have worked out both the velocity and phase of this compound wave, which at the same time determines the total transverse wave (see p. 170), in order that a comparison may be made with HERTZ's results for an undamped single wave in the equatorial plane.

Returning to the formula (x) and putting $\theta = \frac{1}{2}\pi$, it may be read as:

$$Z = Z_0(r)\, e^{-\nu(t/2\tau - r/\lambda)} \sin\left\{2\pi\left(\frac{t}{2\tau} - \frac{r}{\lambda}\right) + \beta_4\right\} \quad . \quad . \quad . \quad \text{(xxxviii)},$$

where if $\xi = \dfrac{1}{2\pi r/\lambda}$:

$$Z_0(r) \sin \beta_4 = (2 \tan \chi - \xi)\, \mathrm{E}l\, (2\pi/\lambda)^3\, \xi,$$

$$Z_0(r) \cos \beta_4 = (1 - \tan^2 \chi + \xi \tan \chi - \xi^2)\, \mathrm{E}l\, (2\pi/\lambda)^3\, \xi.$$

Hence,

$$\tan \beta_4 = (2 \tan \chi - \xi) \big/ (1 - \tan^2 \chi + \xi \tan \chi - \xi^2) \quad . \quad . \quad . \quad \text{(xxxix)}.$$

From this we deduce:

$$\tan(\beta_4 - 2\chi) = \frac{-\cos^2 \chi \left(\dfrac{2\pi r}{\lambda} - \sin 2\chi\right)}{\left(\dfrac{2\pi r}{\lambda} - \frac{1}{4} \sin 2\chi\right)^2 - \cos^2 \chi\, (\cos^2 \chi - \frac{3}{4} \sin^2 \chi)}$$

$$= \frac{-c_0(\zeta - \zeta_0)}{(\zeta - \frac{1}{4}\zeta_0)^2 - c_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xl)},$$

where

$$c_0 = \cos^2 \chi, \qquad \zeta_0 = \sin 2\chi, \qquad c_1^2 = \cos^2 \chi\, (\cos^2 \chi - \tfrac{3}{4} \sin^2 \chi).$$

This formula is fairly well adapted for logarithmic calculation. The phase may be obtained by taking

$$\delta_4 = \zeta - \beta_4.$$

It may be compared with the value for δ_1, obtained from β_1, on p. 172, where we easily find

$$\tan(\beta_1 - 2\chi) = \frac{-3 \cos^2 \chi \left(\dfrac{2\pi r}{\lambda} - \sin 2\chi\right)}{\left(\dfrac{2\pi r}{\lambda} - \frac{3}{4} \sin 2\chi\right)^2 - 3 \cos^2 \chi\, (\cos^2 \chi - \frac{1}{4} \sin^2 \chi)}$$

$$= \frac{-3c_0(\zeta - \zeta_0)}{(\zeta - \frac{3}{4}\zeta_0)^2 - c_2^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xli)},$$

if

$$c_2^2 = 3 \cos^2 \chi\, (\cos^2 \chi - \tfrac{1}{4} \sin^2 \chi).$$

As before

$$\delta_1 = \zeta - \beta_1.$$

This again readily lends itself to logarithmic calculation.

Both have for asymptote the same 45° line, namely,

$$\delta = \zeta - (\pi + 2\chi).$$

We now proceed to trace the velocity of the electrical disturbance in the equatorial plane.

We have

$$\tan\beta_4 = \frac{2\tan\chi - \xi}{1 - \tan^2\chi + \xi\tan\chi - \xi^2},$$

whence, by differentiation, we find

$$\frac{\lambda}{2\pi}\frac{d\beta_4}{dr} = \frac{\xi^2\{1 + \tan^2\chi - 4\xi\tan\chi + \xi^2\}}{(1 + \tan^2\chi)^2 - 2\xi\tan\chi(1 + \tan^2\chi) - \xi^2(1 - 3\tan^2\chi) - 2\xi^3\tan\chi + \xi^4}.$$

Hence, if V_4 be the velocity and $v = \lambda/(2\tau)$, as before

$$\begin{aligned}\frac{V_4}{v} &= 1 + \frac{(\lambda/2\pi).(d\beta_4/dr)}{1 - (\lambda/2\pi).(d\beta_4/dr)}\\ &= 1 + \frac{\xi^2(1 + \tan^2\chi - 4\xi\tan\chi + \xi^2)}{(1 + \tan^2\chi)^2 - 2\xi\tan\chi(1 + \tan^2\chi) - 2\xi^2(1 - \tan^2\chi) + 2\xi^3\tan\chi}.\end{aligned}$$

Now put

$$\zeta = 1/\xi = 2\pi r/\lambda,$$

$$\frac{V_4}{v} = 1 + \frac{\cos^2\chi\left\{\left(\zeta - \dfrac{2\tan\chi}{1 + \tan^2\chi}\right)^2 + \dfrac{1 - 3\tan^2\chi}{(1 + \tan^2\chi)^2}\right\}}{\zeta\left\{\zeta^3 - \zeta^2\dfrac{2\tan^2\chi}{1 + \tan^2\chi} - 2\zeta\dfrac{1 - \tan^2\chi}{(1 + \tan^2\chi)^2} + \dfrac{2\tan\chi}{(1 + \tan^2\chi)^2}\right\}}.$$

Again put

$$\zeta_0 = \frac{2\tan\chi}{1 + \tan^2\chi} = \sin^2\chi = \frac{2\pi}{\lambda}r_0,$$

$$\zeta_2^2 = \frac{1 - 3\tan^2\chi}{(1 + \tan^2\chi)^2} = \cos^2\chi(\cos^2\chi - 3\sin^2\chi) = \frac{4\pi^2}{\lambda^2}r_2^2.$$

Then

$$\zeta_0^2 + \zeta_2^2 = \cos^2\chi, \qquad \zeta_0(\zeta_0^2 + \zeta_2^2) = 2\tan\chi\Big/(1 + \tan^2\chi)^2,$$

$$\zeta_2^2 + \tfrac{1}{2}\zeta_0^2 = \cos^2\chi(\cos^2\chi - \sin^2\chi) = (1 - \tan^2\chi)\Big/(1 + \tan^2\chi)^2.$$

Thus we may write the above result

$$\begin{aligned}\frac{V_4}{v} &= 1 + \frac{\cos^2\chi\{(\zeta - \zeta_0)^2 + \zeta_2^2\}}{\zeta\{\zeta^3 - \zeta_0\zeta^2 - (\zeta_0^2 + 2\zeta_2^2)\zeta + \zeta_0(\zeta_0^2 + \zeta_2^2)\}}\\ &= 1 + \frac{\cos^2\chi\{(\zeta - \zeta_0)^2 + \zeta_2^2\}}{\zeta\{(\zeta - \zeta_0)^2(\zeta + \zeta_0) - 2\zeta_2^2(\zeta - \zeta_0) - \zeta_2^2\zeta_0\}}\\ &= 1 + \frac{\cos^2\chi\{(r - r_0)^2 + r_2^2\}}{(2\pi/\lambda)^2 r\{(r - r_0)^2(r + r_0) - 2r_2^2(r - r_0) - r_2^2 r_0\}} \quad . \quad . \quad . \quad \text{(xlii)}.\end{aligned}$$

This is a form directly comparable with (xxx).

If the wave be not damped, $\zeta_0 = 0$, $\zeta_2 = 1$, which gives

$$\frac{V_4}{v} = 1 + \frac{1+\zeta^2}{\zeta(\zeta^3 - 2\zeta)} = \frac{1 - \zeta^2 + \zeta^4}{\zeta^2(\zeta^2 - 2)},$$

a value identical with that given by **Gray**.*

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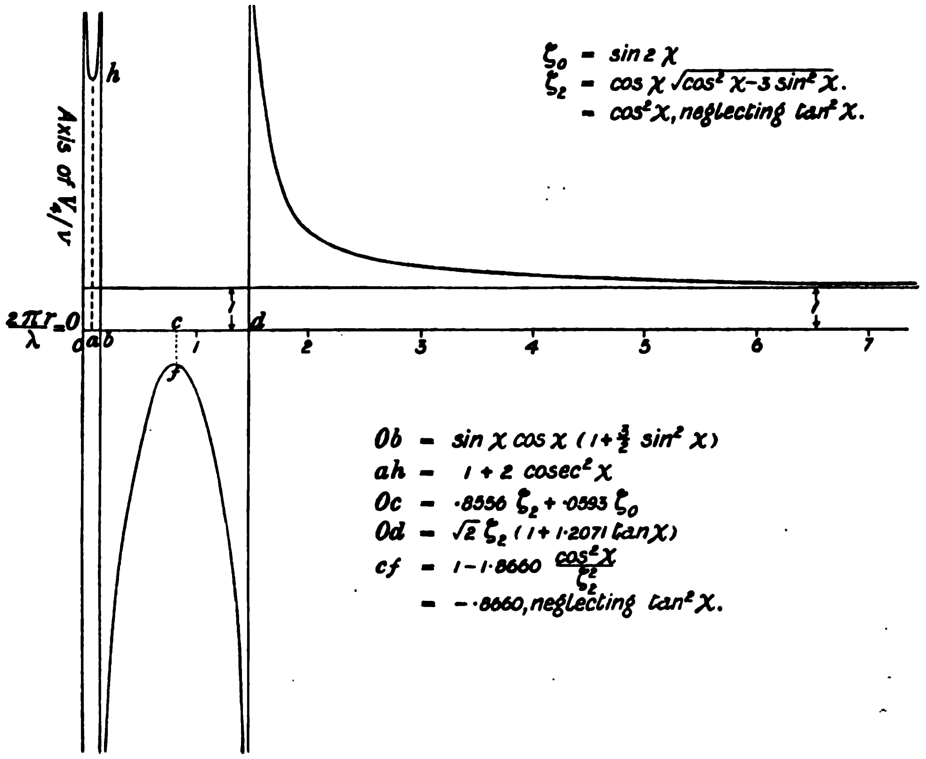
* 'Absolute Measurements,' vol. 2, Part II.,

The curve giving the velocity of this wave is found to be even more complex than that for either the wave of transverse or of axial component electric force alone (p. 177). In the neighbourhood of the origin the wave starts with infinite positive velocity; this falls to a finite but very great positive velocity at about $\frac{1}{4}\zeta_0$, and then rises again and becomes infinite at about $\frac{1}{2}\zeta_0$. Beyond this we have a region of negative velocity, starting with an infinite negative velocity, and falling to a velocity even less than that of light at about $\cdot 8556\,\zeta_2$. Then the velocity increases again, becoming infinite at about $\sqrt{2}\zeta_2$, after which it takes an infinite positive value and falls rapidly till it becomes equal to the velocity of light. Thus there are two centres, one at about $\frac{1}{2}\zeta_0$, and another at about $\sqrt{2}\zeta_2$ from the oscillator (more exact values are given below), from or towards which the waves appear to move with varying velocities. In most cases the $\frac{1}{2}\zeta_0$ centre would be far too near the oscillator for experiment, but the $\sqrt{2}\zeta_2$ centre, and even the point of negative velocity less than that of light, appear to be well within the field of experiment, and actually within the area inside which HERTZ made a good many experiments. Considering the complex character of the velocity of this wave of electrical disturbance in the equatorial plane, we feel doubtful as to what interpretation can possibly be put on HERTZ's interference experiments in this plane, especially on those made at a moderate distance from the oscillator.

The following figure gives the velocity curve of the equatorial electric wave *diagrammatically*. The approximate values of the chief quantities concerned have been calculated in the same manner as those on pp. 173–176, but it does not seem necessary to reproduce the calculations.



Now it is clear, both from this diagram and from that on p. 177, that we must go to $\frac{2\pi r}{\lambda} = 6$ about to get fairly constant velocity of transmission. But λ in HERTZ's experiments was about 9·6 metres (see p. 162), or the experiments, supposing a constant velocity of transmission desirable, ought to have been made at some 9 metres or more from the oscillator. HERTZ's first series are at less than 8, his third series at less than 4, and his second, which do go up to 12, he states "required rather an effort."* Within these limits the exact nature of the interference and the points at which it may be expected to occur seem open to some criticism, even in addition to the difficulties which have been raised by the problem of "multiple resonance."† The views expressed above will be strengthened by an examination of the diagram giving the phases of the waves, the velocities of which have been already discussed.

(13.) The phases have been plotted from the formulæ given above, as follows :—

I. *Transverse Component Electric Wave and Wave of Total Force perpendicular to Axis* (ϕ_1). $\delta_1 = \zeta - \beta_1$.

The formula is (xli). For the special oscillator for which our results are plotted,

$$\zeta_0 = \sin 2\chi = \cdot 126{,}8098, \qquad c_0 = \cdot 995{,}964,$$
$$\chi = \cdot 063{,}5760, \qquad c_2^2 = 2\cdot 972{,}821.$$

The asymptote is $\delta_1 = \zeta - 3\cdot 2687$ (*i.e.*, $\delta_1 = \zeta - (\pi + 2\chi)$).

We see from the diagram that the phase does not approach very rapidly to its asymptotic value, being still about 5 per cent. of its value from it, when $\zeta = 10$.

We notice that δ_1 starts from zero at the origin, and after high contact becomes small and negative. This negative portion of the phase is too small to be seen on the large diagram, but an enlarged inset figure is added. The curve cuts the axis and becomes positive at about 1·165, or at about 1·42 metres from the origin, a distance within which some of HERTZ's experiments on interference were made. The range of negative phase could be considerably increased with a more rapidly damped oscillator.

The approximate value‡ of Oa for any value of χ is

$$= (15 \tan \chi)^{\frac{1}{3}} + \frac{20}{7} \tan \chi + \frac{2000}{1176} \tan \chi \left(\frac{\tan^2 \chi}{15}\right)^{\frac{1}{3}} + \ldots\ldots \quad \text{(xliii)}.$$

II. *Axial Component Electric Wave and Wave of Total Radial Force* (ϕ_2). $\delta_2 = \zeta - \beta_2$.

In this case the formula used was deduced from (xxiv). This may be altered to

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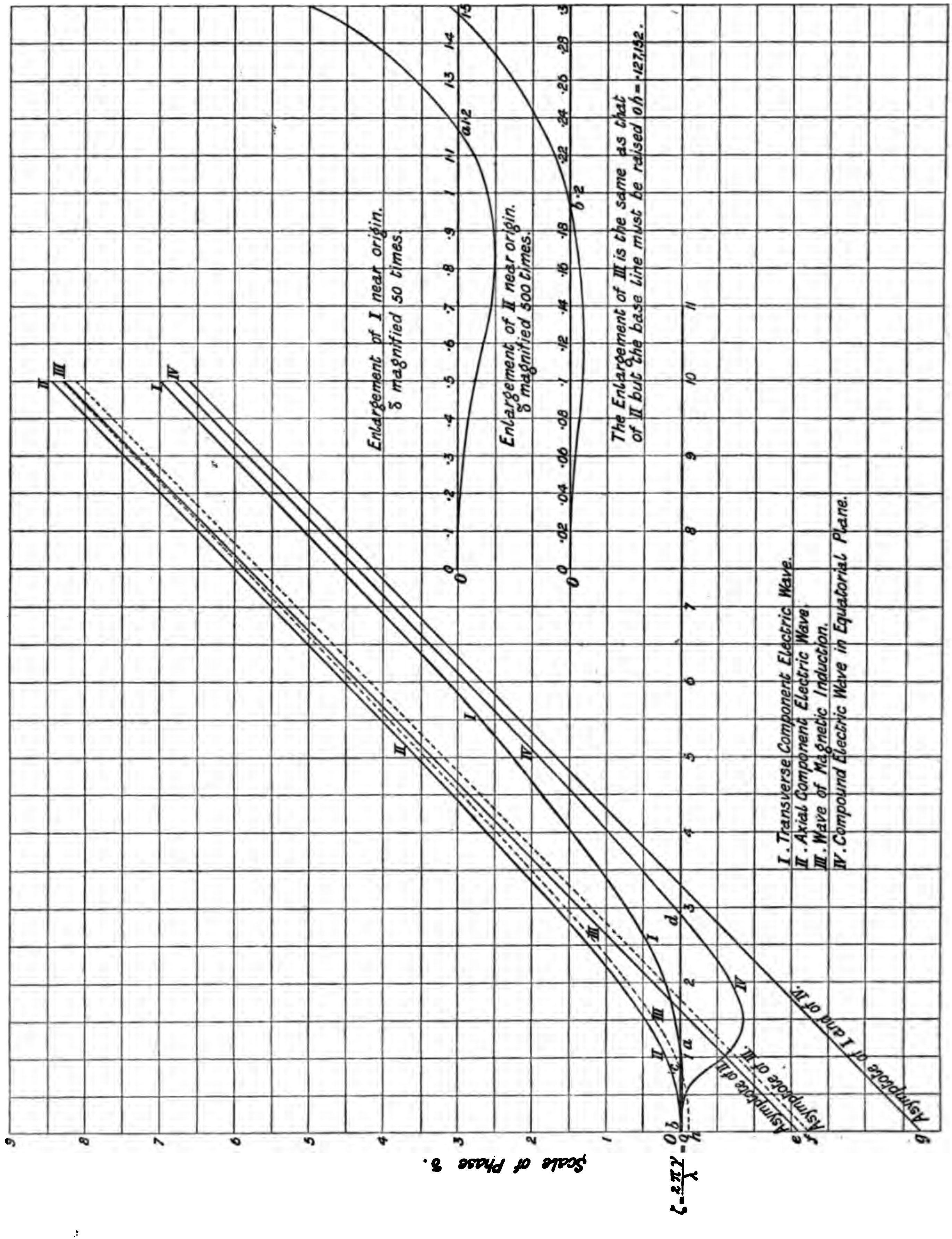
* 'Electric Waves,' pp. 118, 120, and 119 respectively.

† POINCARÉ, 'Électricité et Optique,' vol. 2, p. 195 *et seq.*, and p. 249 *et seq.*

‡ We owe this general solution to the kindness of Mr. L. N. G. FILON, M.A.


Enlargement of I near origin.
δ magnified 50 times.
Enlargement of II near origin.
δ magnified 500 times.
The Enlargement of III is the same as that of II but the base line must be raised oh = ·127,132.
I. Transverse Component Electric Wave.
II. Axial Component Electric Wave.
III. Wave of Magnetic Induction.
IV. Compound Electric Wave in Equatorial Plane.
Asymptote of I and of IV.
Asymptote of III.
Asymptote of II.
I
II
III
IV
a
b
c
d
e
f
g
h
a 1·2
b·2
0
1
2
3
4
5
6
7
8
9
10
11
·1
·2
·3
·4
·5
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·16
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·22
·24
·26
·28
·3
$\zeta = \frac{2\pi r}{\lambda}$
Scale of Phase δ.

$$\tan(\beta_2 - \chi) = \frac{\zeta - \frac{1}{2}\zeta_0}{\cos^2\chi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xliv)},$$

which serves readily for logarithmic calculation.

The portion of the curve for which δ is negative now lies between 0 and ·2, and we must multiply it some 500 times to render the small negative values of δ visible.

The approximate value of Ob for any value of χ is given by

$$\text{O}b = \tfrac{3}{2}\zeta_0,$$

and thus = ·19021 for our special case.

The asymptote is given by the 45° line

$$\delta = \zeta - \left(\frac{\pi}{2} + \chi\right),$$

or, for our special case Oe = 1·63437.

The curve for this wave has a far less sensible negative portion, and approaches much more rapidly to its asymptote, being in our special case at a distance less than 2 per cent. of the value of δ from its asymptote when $\zeta = 10$.

Thus we see that there is a marked difference between the phases of the component transverse and component axial electric waves. The first appears to proceed from a centre at distance Oa given by (xliii) from the centre, and the second from a distance Ob equal $\frac{3}{2}\zeta_0$, or practically from a point so close to the centre of the oscillator as not to be sensible in the case of any but very rapidly damped wave trains. The ultimate phase difference of the two waves $= \frac{\pi}{2} + \chi$.

III. *Wave of Magnetic Induction.* $\delta_3 = \zeta - \beta_3$.

Here we can either use the graphical method of p. 179, or the simple relation $\delta_3 = \delta_2 - 2\chi$.

The latter method shows us that the asymptote is

$$\delta = \zeta - \left(\frac{\pi}{2} + 3\chi\right),$$

or in our special case

$$\delta = \zeta - 1{\cdot}7615.$$

Thus the ultimate difference of phase between the transverse component electric and the magnetic waves is $\frac{1}{2}\pi - \chi$, and between the axial electric and magnetic waves is -2χ.

It will be seen that this wave of magnetic induction for a damped wave differs considerably from that given by HERTZ.

The most notable difference is the ***finite negative*** value of δ_3 at the origin, and this negative value of the phase continues until $\frac{2\pi r}{\lambda} = \text{O}c$; for the general case we may take approximately

$$\text{O}c^* = (6\chi)^{\frac{1}{3}}(1 + \tfrac{1}{15}(6\chi)^{\frac{2}{3}}).$$

For the special value of χ used above, *i.e.*, $\chi = 3^\circ\ 38'\ 33{\cdot}5''$, we find

$$\text{O}c = {\cdot}8664.$$

This is at a distance from the source of

$$r = {\cdot}1379\lambda,$$

or about 1·324 metres. Thus we see that the phase of this action does not, as HERTZ states ('Electric Waves,' p. 154), "increase continuously from the origin itself." In discussing the velocity we have seen that $d\beta_3/dr = d\beta_0/dr$, so that the velocity of propagation is identical with that given by (xxi). Thus it becomes indefinitely great when $2\pi r/\lambda = \sin 2\chi = {\cdot}1268$ for the above values of the constants. The point, therefore, of infinitely great velocity is much closer to the origin than that of zero phase. We think HERTZ considered these two points to be identical. At any rate, he held that the wave moved with infinite velocity up to the point of zero phase. For he argues from his conclusion that the phase increases continuously from the origin that:

"The phenomena which point to a finite rate of propagation must in the case of these interferences† make themselves felt even close to the oscillator. This was indeed apparent in the experiments, and therein lay the advantage of this kind of interference. But, contrary to the experiment, the apparent velocity near to the oscillator comes out greater than at a distance from it . . ."‡

As a matter of fact, the alterations in the velocity of transmission of either magnetic wave or transverse component electric wave are very complex in the neighbourhood of the oscillator, and these variations do not bear a simple relation to the points of zero phase, from which points HERTZ assumes the outward moving wave to start. Something of the difficulties HERTZ met with in his experiments on "interferences of the second kind" made close to the oscillator may well have been due to this complexity of result arising from the use of a damped wave-train.

IV. *The Compound Electric Wave in the Equatorial Plane, and the Wave of Total Transverse Electric Force* $(\phi_1 + \phi_2)$. $\delta_4 = \zeta - \beta_4$.

The formula is now (xl).

For the special oscillator for which our curve is drawn, $c_0 = {\cdot}995{,}9635$, $c_1^2 = {\cdot}988{,}928$, $\zeta_0 = {\cdot}126{,}8098$, and the asymptote is $\delta = \zeta - 3{\cdot}2687$.

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* This must only be taken as the basis to find a second approximation, as the series converges slowly.

† Interferences of the "second kind": see 'Electric Waves,' p. 119.

‡ 'Electric Waves,' p. 154.

The general value for Od, giving the point of zero phase, is O$d = 2{\cdot}743{,}707 + {\cdot}112{,}193\ \zeta_0$, approximately. For our special case, O$d = 2{\cdot}9079$.

It will be seen that our curve approaches closer to its asymptote than the curve for the true wave of transverse vibrations. Further, the centre, d, from which the wave with positive velocity may be supposed to start, is moved far further from the origin. It is rather further from the origin than it would be in the case of an undamped train.* There seems, therefore, considerable doubt as to what HERTZ was really measuring in the case of his interference experiments in the equatorial plane. Was he really measuring IV., or possibly I., obscured in its effect by the superposition of II.? It is, we think, needful to re-examine the whole matter physically, endeavouring to distinguish between the components ϕ_1 and ϕ_2. For this purpose the direction of the axis, rather than the plane of the equator, seems the more suitable for experiment. For in this direction the magnetic induction and the transverse electric component wave disappear, and we have the axial component electric wave only to deal with, *i.e.*, II. only, I. and III. being not extant.

Further, the numerous theoretical singular points to which we have referred seem to deserve, if possible, physical investigation. To do this we require to emphasise as much as possible their distance from the oscillator. But this means that we must get from a small oscillator a long wave-length, if the rate of damping (χ) be fixed, or, if the wave-length be fixed, we require very rapid damping. Neither of these conditions seem easy of physical realisation. Still something might be done by way of striking a balance, and, at any rate, we must not disregard the fact that a considerable portion of HERTZ's experiments were made inside that portion of the field where the influence of these singular points and of the damping should at least theoretically make themselves felt.

(14.) *Conclusions.*—We may draw the following general conclusions:—

(i.) The effect of damping makes itself very sensible in modifying the form of the wave-surface as propagated into space from a theoretical oscillator. The typical Hertzian wave-diagrams require to be replaced by the fuller series accompanying this memoir.

(ii.) *Three* waves of electro-magnetic force may be considered as sent out from the oscillator, and these waves we believe capable of physical identification.

First, a component wave of transverse electric force, determined by ϕ_1. This also gives the wave-speed and phase of force perpendicular to the oscillator axis.

Secondly, a component wave of electric force parallel to the axis, determined by ϕ_2. This also gives the wave-speed and phase of force radial to the oscillator.

Thirdly, a wave of magnetic force.

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* In this case Od = 2·743,707 correct to the last place. In the general approximate value given above, we have neglected terms of the order χ^2, and the approximation is not nearly as accurate as this.

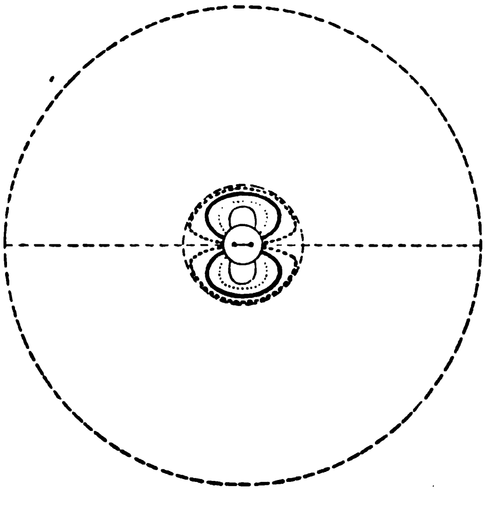
The waves of magnetic force and of component axial electric force move outwards each with the same velocity at all points, and this velocity is equal for all points at the same distance from the oscillator. The intensity of the first force for points on the same sphere varies as the cosine of the latitude, but that of the second force is constant. The wave of component transverse electric force moves outwards with equal velocity for all points at the same distance from the oscillator, and its amplitude varies as the cosine of the latitude. Its velocity after it has reached a certain distance from the origin, is always greater than that of the waves of component axial electric force, and of magnetic force, and its excess over the velocity of light tends to become three times the excess of the velocity of the wave of magnetic force over the velocity of light.

(iii.) The velocities of these waves undergo remarkable changes in the neighbourhood of the oscillator, but still at distances such as HERTZ experimented at, and which seem indeed to some extent within the field of possible physical investigation.

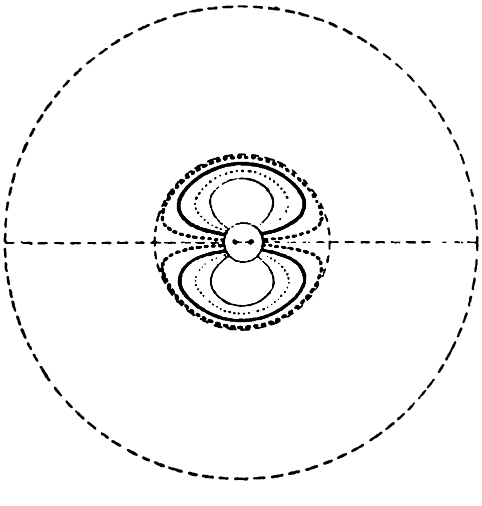
(iv.) The point of zero phase for both transverse and axial component electric waves does not coincide with the centre of the oscillator, so that these waves appear to start from a sphere of small but finite radius round the oscillator. A fourth wave dealt with by HERTZ, the wave of magnetic induction, does not, as he supposes, start from the centre of the oscillator with zero phase, but in the case of a damped wave train with a small but finite phase.

(v.) Our analysis of these waves and of their singular points in the neighbourhood of the oscillator appears to add something to HERTZ's discussion; it is possible that it may throw light on the difficulties which arise in connecting with some of his interference experiments. It would seem to us that all interference experiments ought to be made at distances greater than 6 to 7 $(\lambda/2\pi)$ from the centre of the oscillator, roughly about a wave-length from the oscillator, whereas HERTZ rather terminated than started his experiments at this distance. At such distances the phase curves are approximately parallel to their asymptotes.

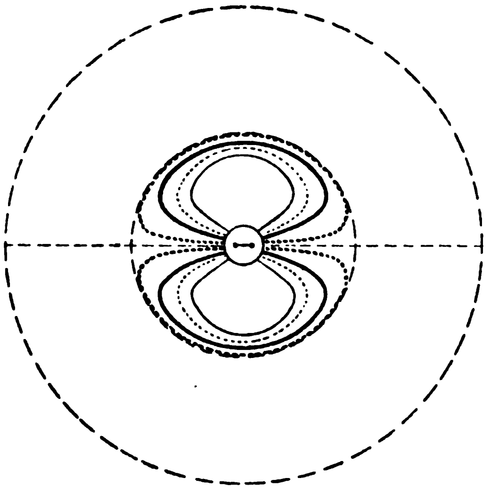
Finally, we are not unaware of the physical difficulties attending experiments, at such distances, and wish in conclusion to again emphasise the fact that our analysis only applies to a *theoretical* type of oscillator. It is, however, the type for which HERTZ himself endeavoured to provide a mathematical investigation.



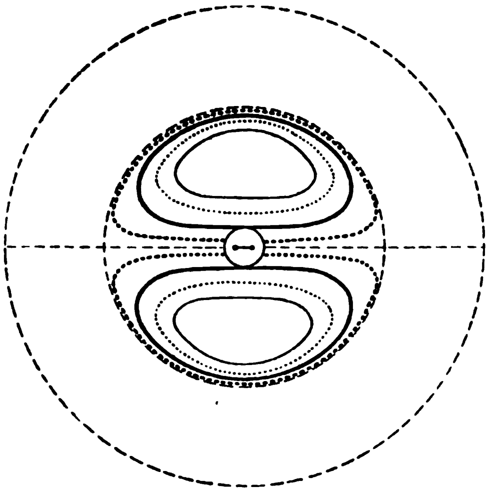
1. Time = $\frac{1}{8}$ Period = $\frac{1}{8}(2T)$



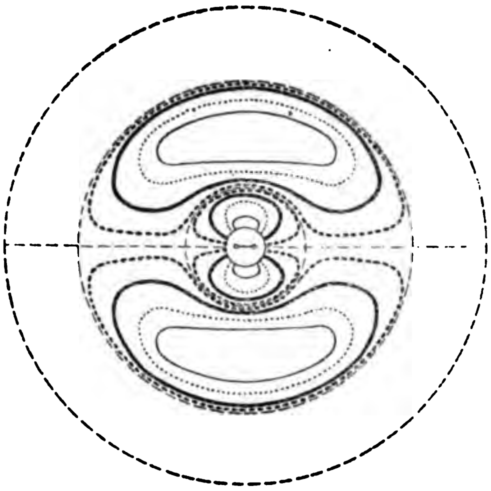
2. = $\frac{1}{4}(2T)$



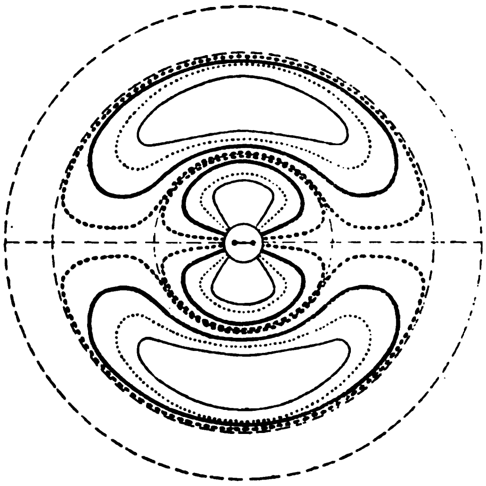
3. = $\frac{3}{8}(2T)$



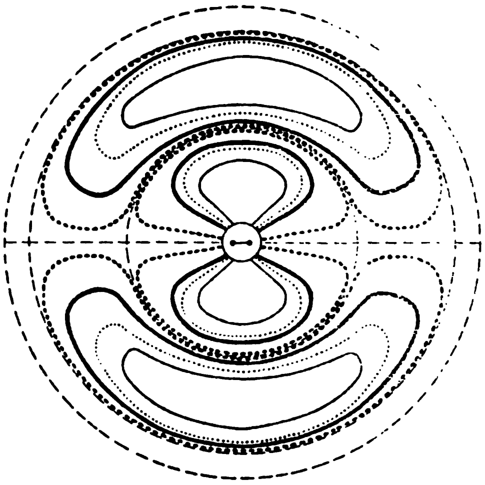
4. = $\frac{1}{2}(2T)$



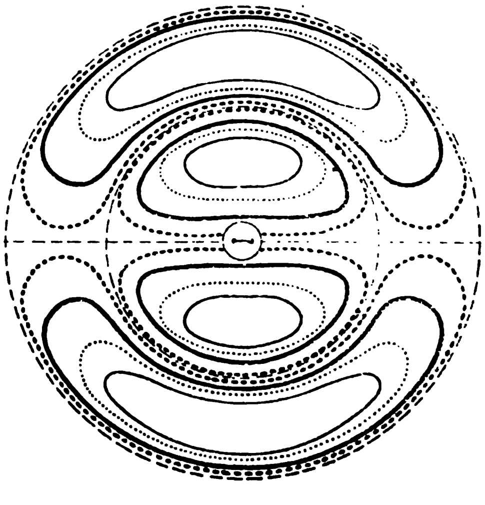
5. = $\frac{5}{8}(2T)$



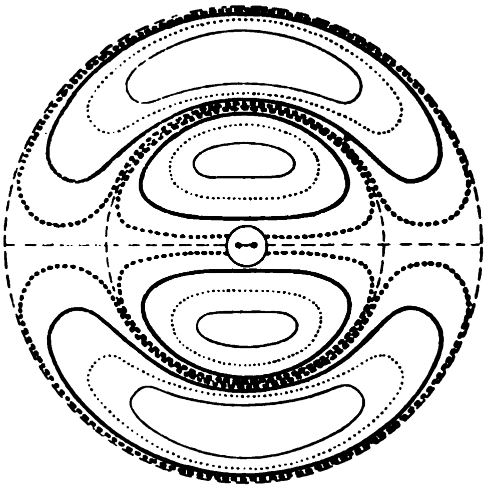
6. = $\frac{6}{8}(2T)$



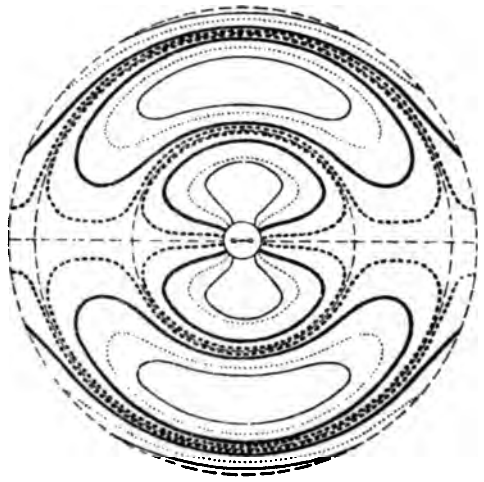
7. = $\frac{7}{8}$ 2T



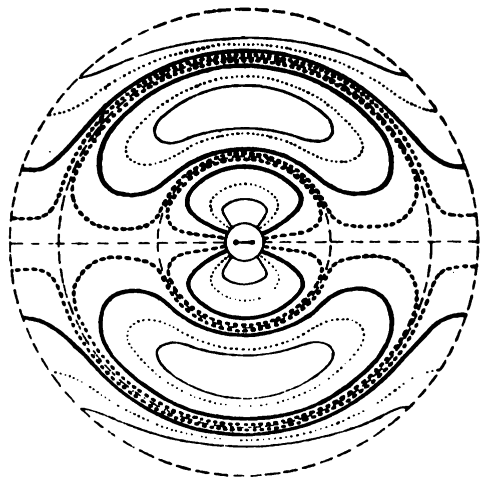
8. Time = 1 (2T)



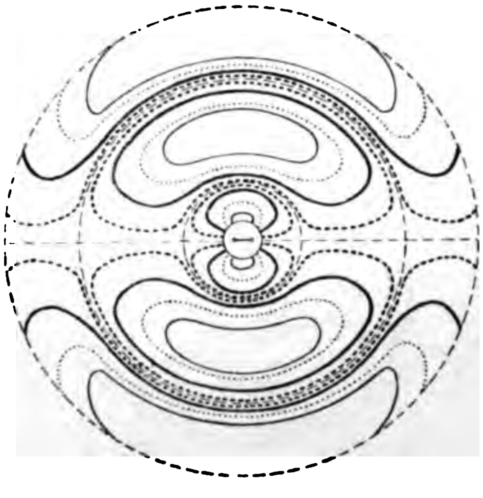
12. = 1½ (2T)



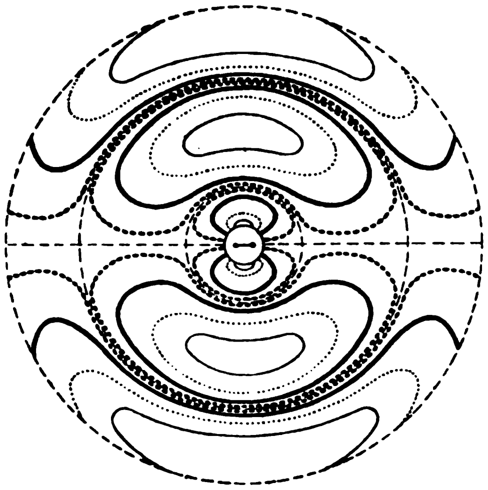
11. = 1⅜ (2T)



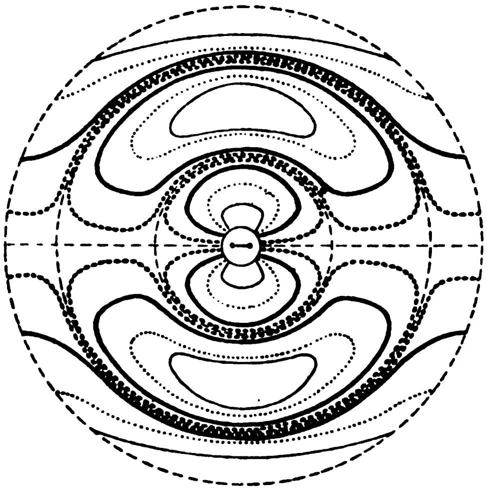
10. = 1¼ (2T)



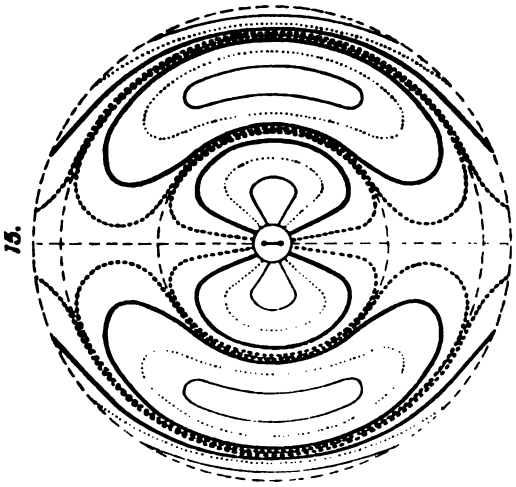
9. Time = 1⅛ (2T)



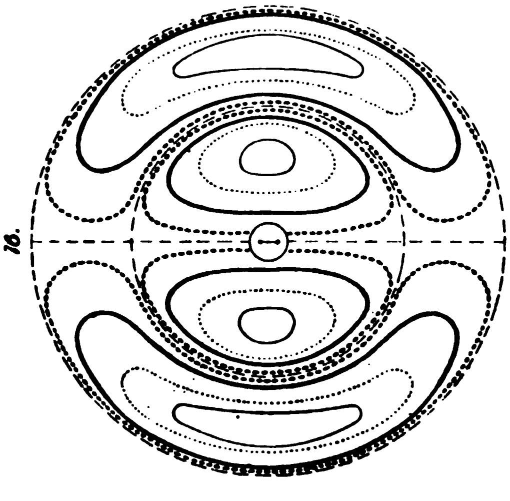
13. = 1⅝ (2T)



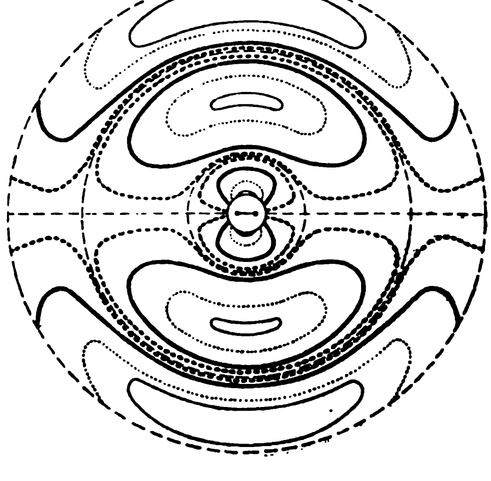
14. = 1¾ (2T)



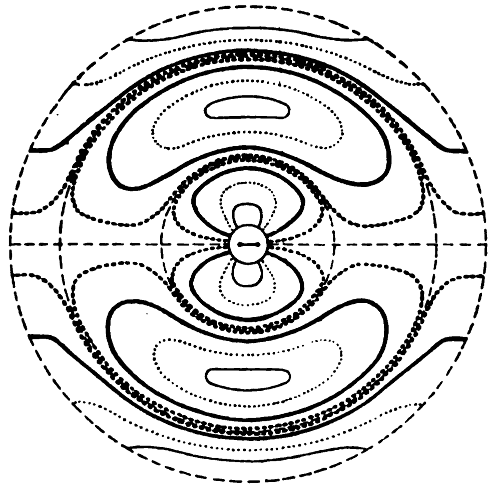
15. = 1⅞ (2T)



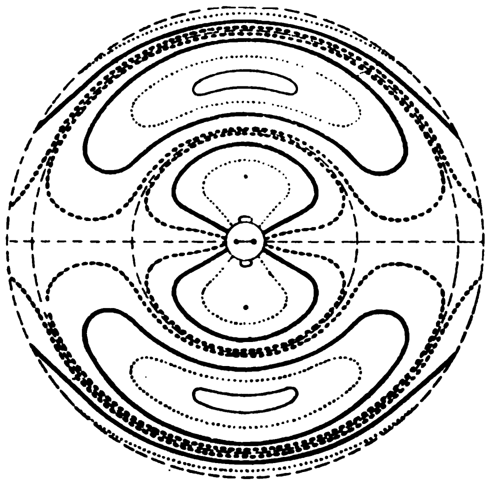
16. Time = 2 (2T)



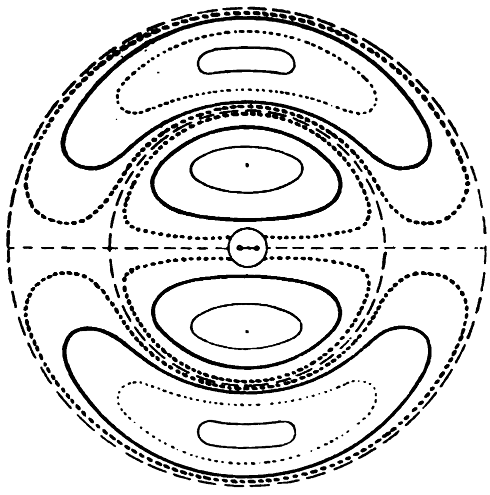
17. *Time* $= 2\frac{1}{8}$ (2T)



18. $= 2\frac{1}{4}$ (2T)



19. $= 2\frac{3}{8}$ (2T)



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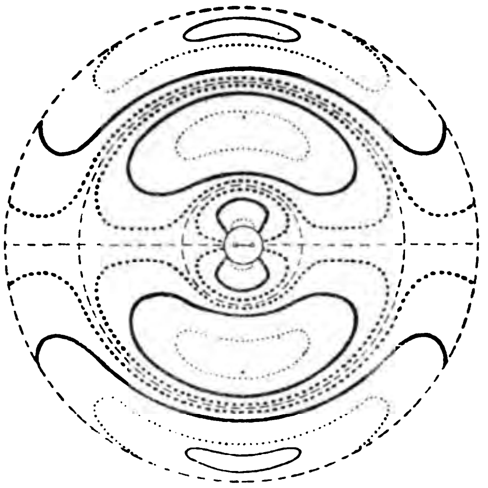
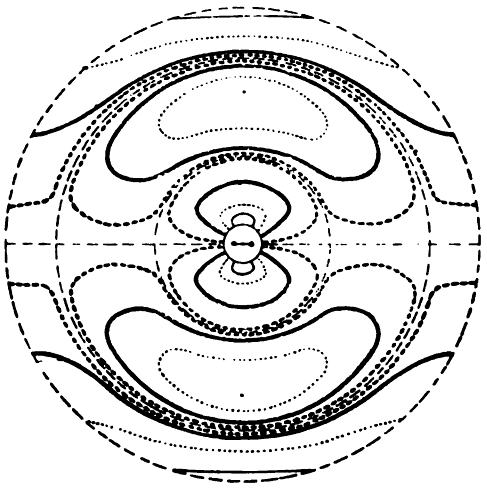
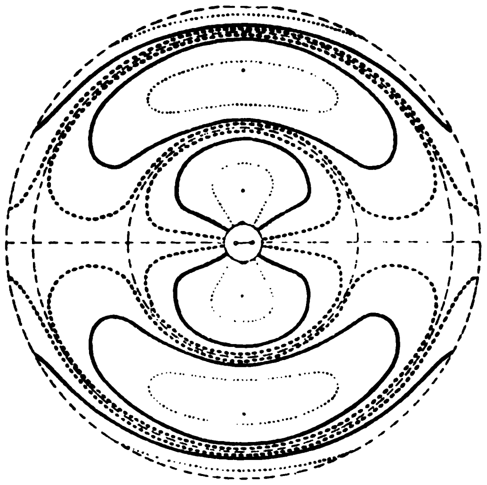


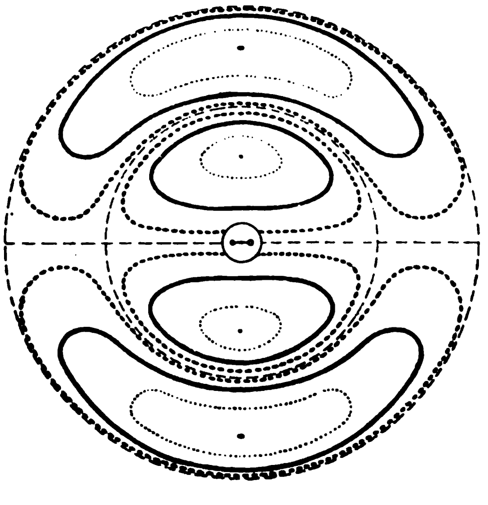
21. $= 2\frac{5}{8}$ (2T)



22. $= 2\frac{3}{4}$ (2T)

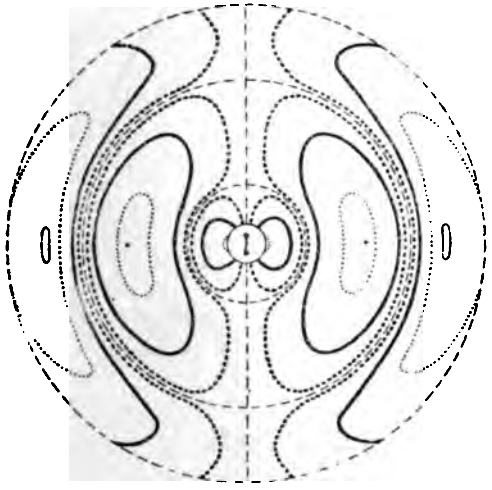


23. $= 2\frac{7}{8}$ (2T)



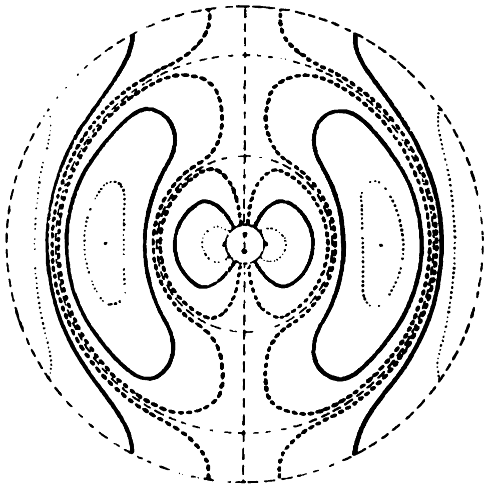
24. *Time* $= 3$ (2T)

25.



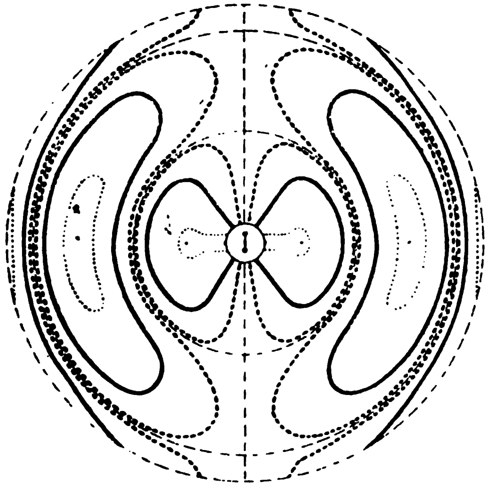
Time = $3\frac{1}{8}$ (2T)

26.



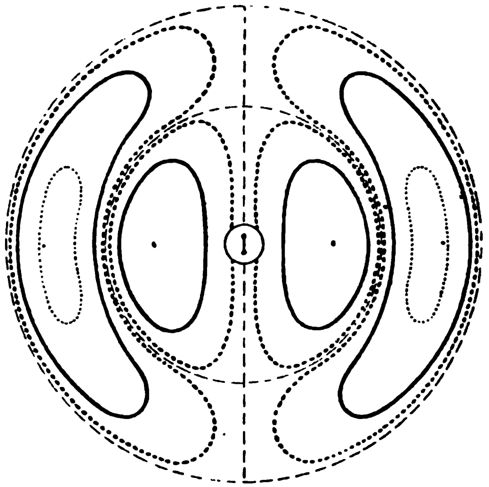
= $3\frac{1}{4}$ (2T)

27.



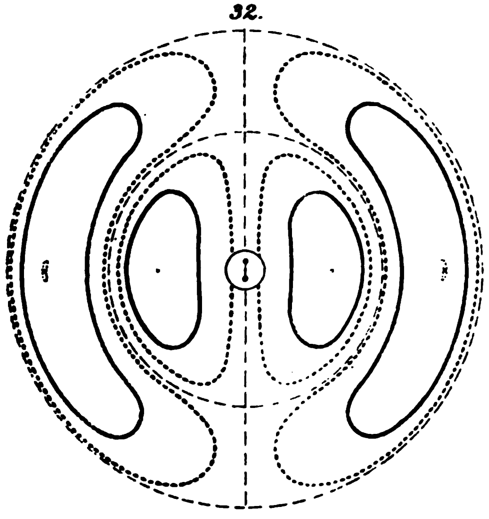
= $3\frac{3}{8}$ (2T)

28.



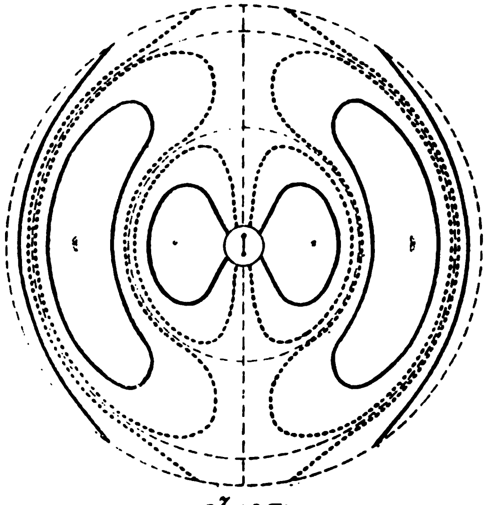
= $3\frac{1}{2}$ (2T)

32.



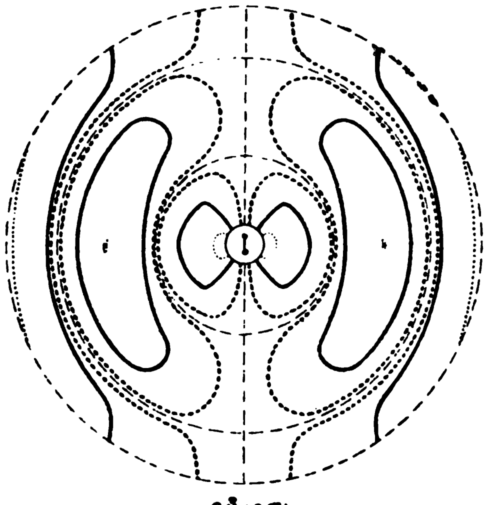
Time = 4 (2T)

31.



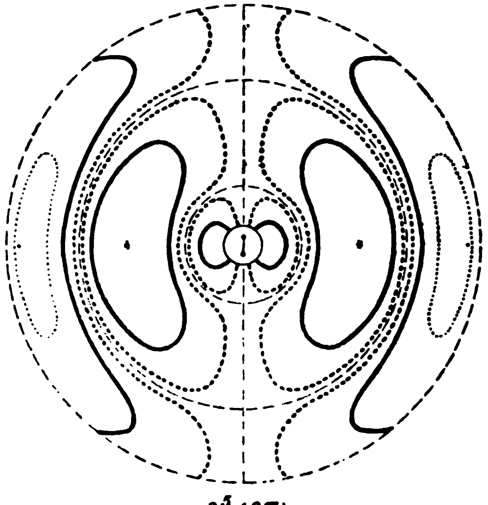
= $3\frac{7}{8}$ (2T)

30.



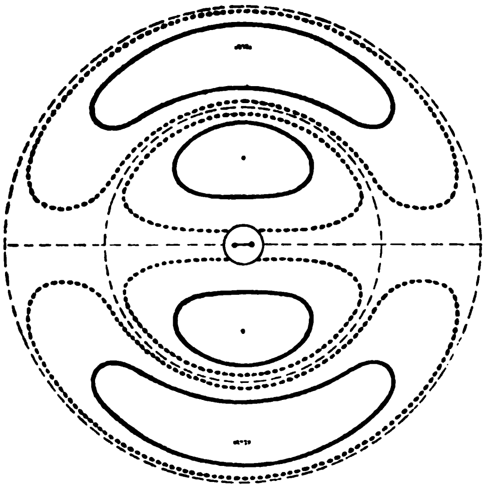
= $3\frac{3}{4}$ (2T)

29.



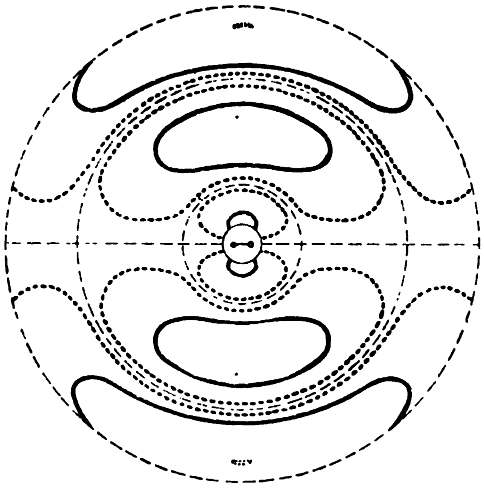
= $3\frac{5}{8}$ (2T)

36.



$=4\frac{1}{2}(2T)$

37.



$=4\frac{5}{8}(2T)$

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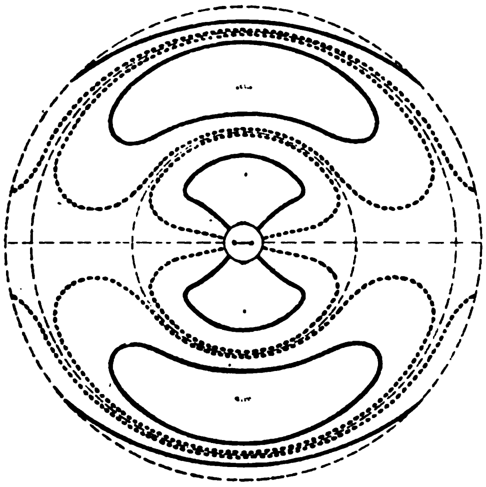

$=4\frac{3}{8}(2T)$

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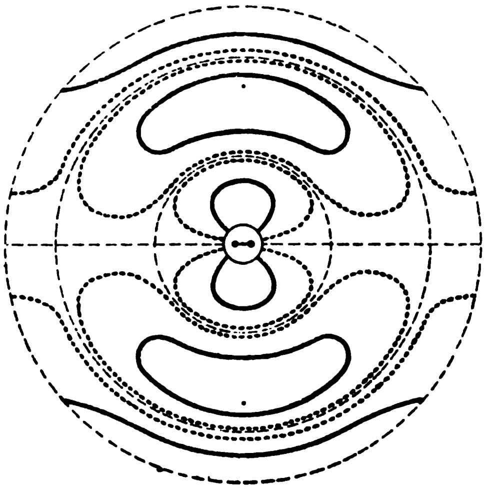

$=4\frac{3}{4}(2T)$

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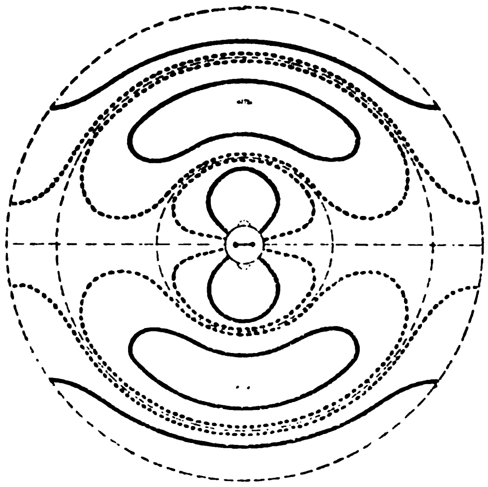
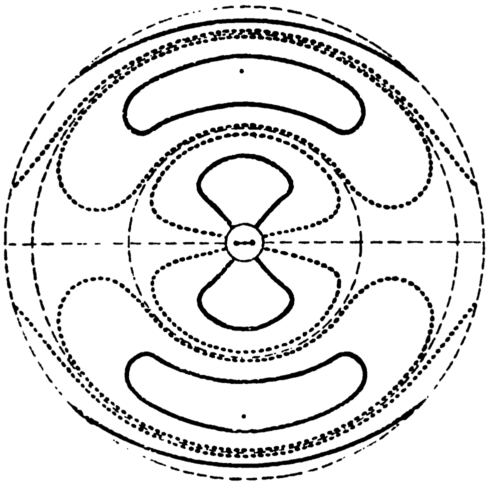


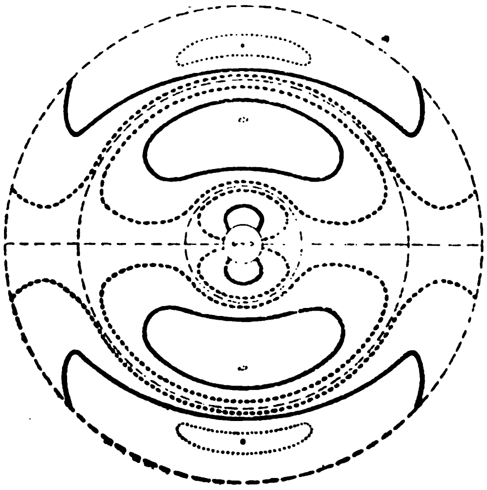
$=4\frac{1}{4}(2T)$

39.



$=4\frac{7}{8}(2T)$

33.



Time $=4\frac{1}{8}(2T)$

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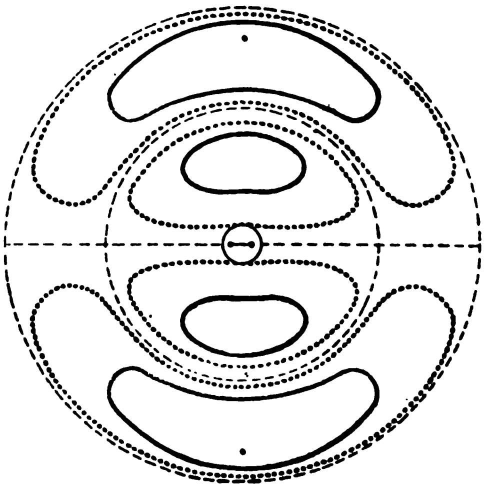
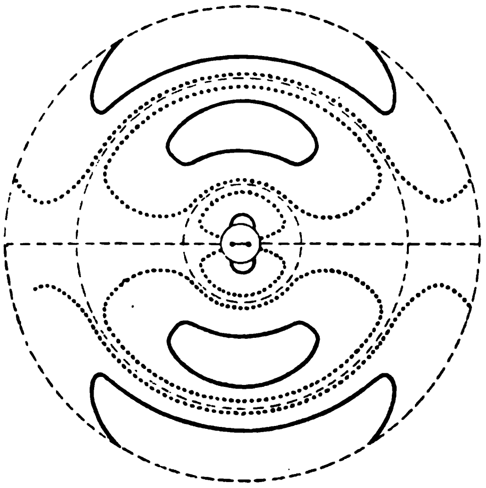
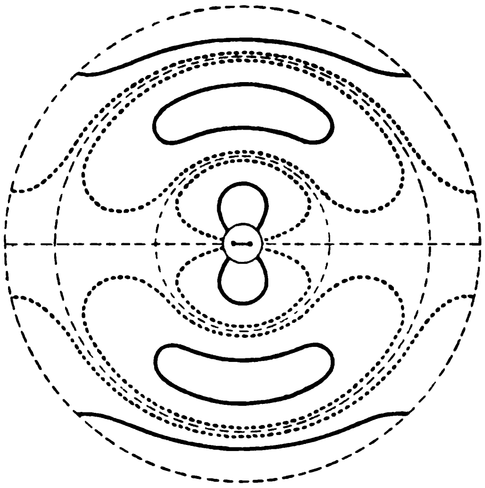


Time $=5(2T)$



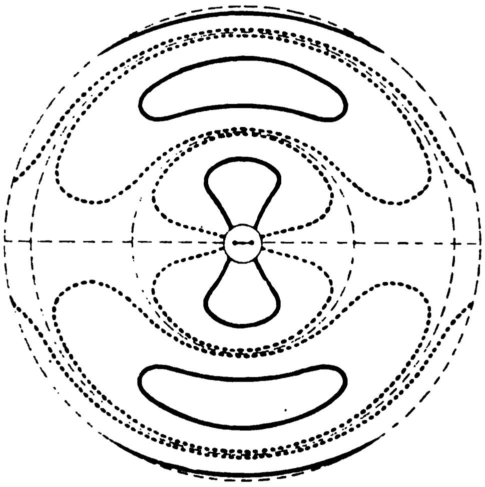
41.

Time $= 5\frac{1}{8}$ (2T)



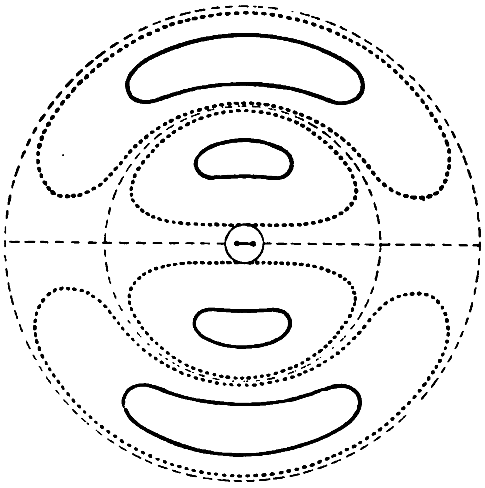
42.

$= 5\frac{1}{4}$ (2T)



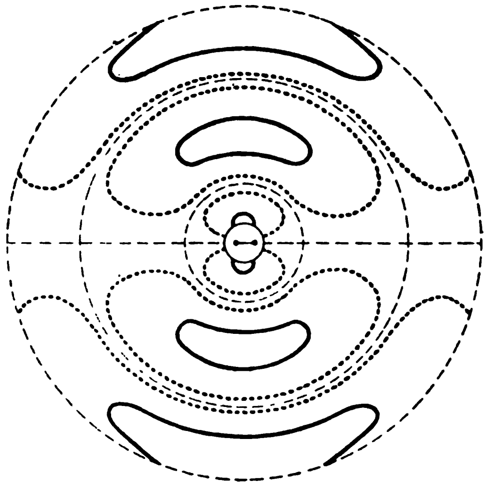
43.

$= 5\frac{3}{8}$ (2T)



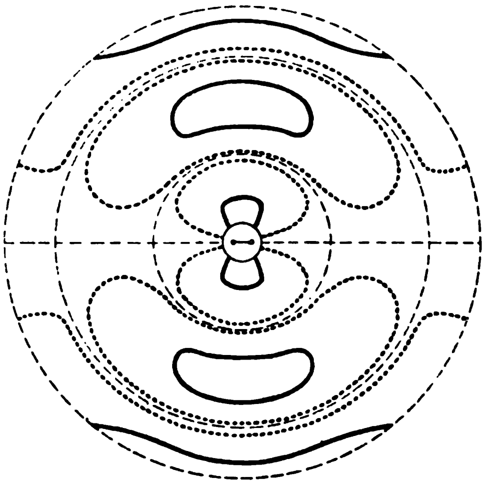
44.

$= 5\frac{1}{2}$ (2T)



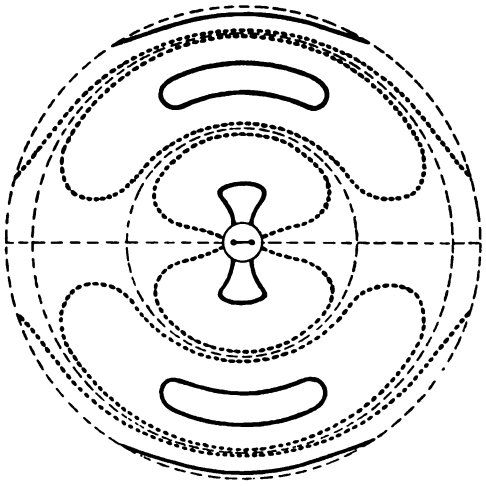
45.

$= 5\frac{5}{8}$ (2T)



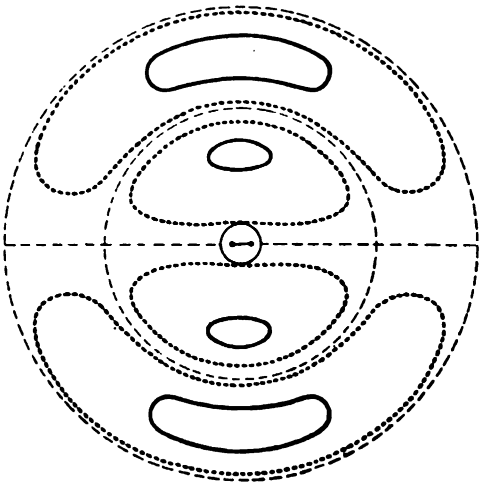
46.

$= 5\frac{3}{4}$ (2T)



47.

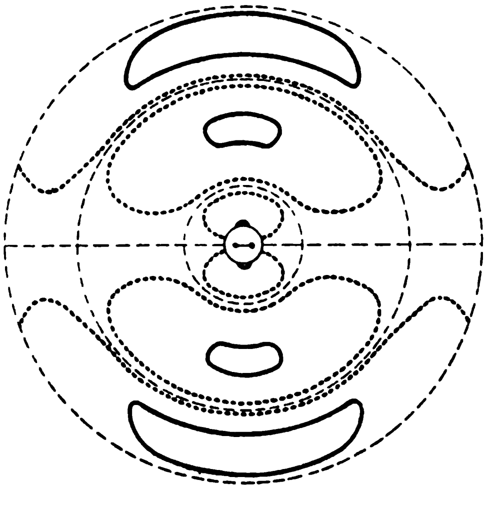
$= 5\frac{7}{8}$ (2T)



48.

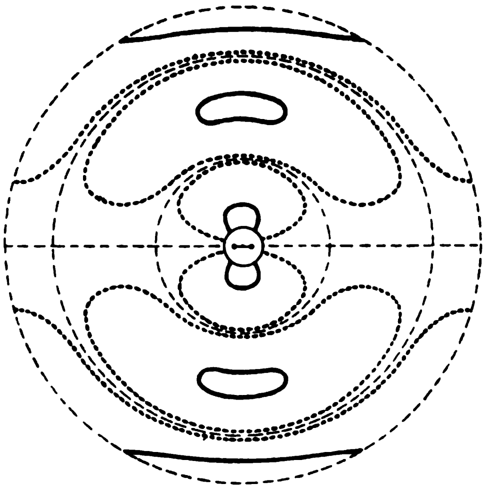
Time $= 6$ (2T)

49.



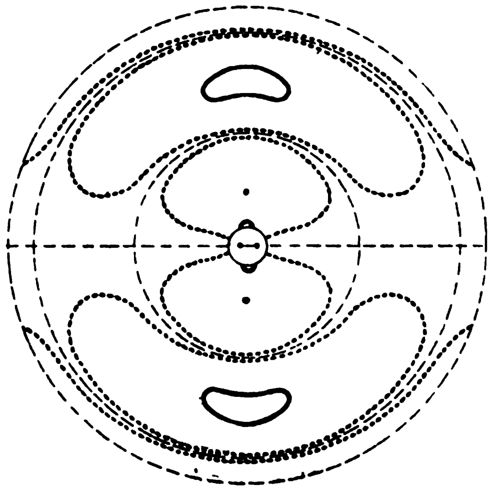
Time = $6\frac{1}{8}$ (2T)

50.



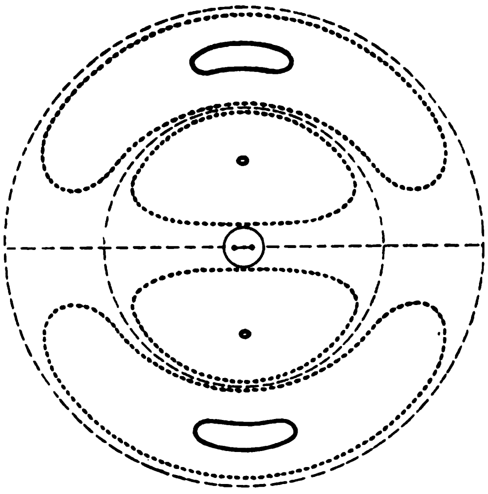
= $6\frac{1}{4}$ (2T)

51.



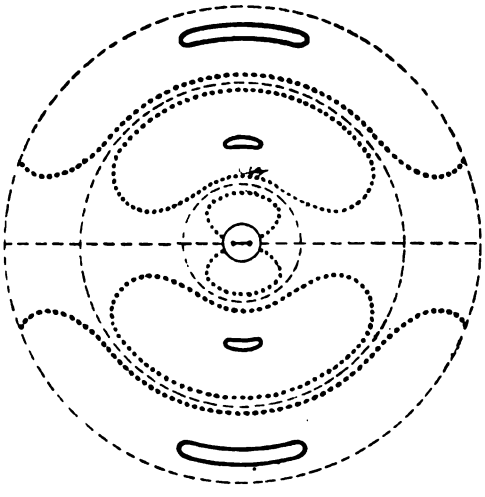
= $6\frac{3}{8}$ (2T)

52.



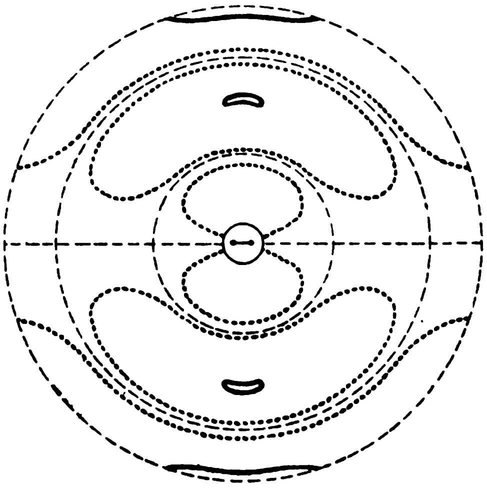
= $6\frac{1}{2}$ (2T)

53.



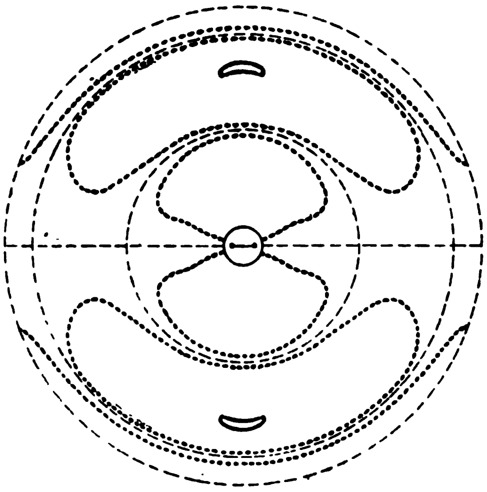
= $6\frac{5}{8}$ (2T)

54.



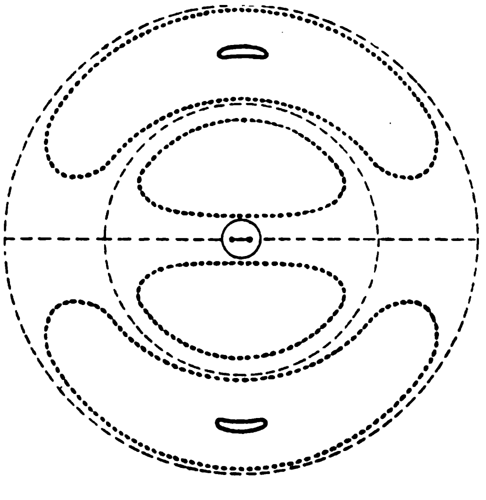
= $6\frac{3}{4}$ (2T)

55.



= $6\frac{7}{8}$ (2T)

56.



Time = 7 (2T)

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VI. *On the Constitution of the Electric Spark.*

By ARTHUR SCHUSTER, *F.R.S.*, *and* GUSTAV HEMSALECH.

Received February 2,—Read February 2, 1899.

[PLATES 8–12.]

1. *Object of Research and Description of Apparatus.*

WHEN an electric spark passes between metallic electrodes, the spectrum of the metal appears not only in immediate contact with the electrodes, but stretches often across from pole to pole. It follows that, during the short time of the duration of the spark, the metal vapours must be able to diffuse through measurable distances.

The following investigation was undertaken primarily to measure this velocity of diffusion, with the special view of comparing different metals and different lines of the same metal.

Dr. FEDDERSEN* published in the year 1862 an interesting research in which photographs of sparks were taken after reflection from a rotating mirror. He was able to draw some important conclusions from his experiments, but it was necessary for the more detailed examination we had in view, to analyse the light by a spectroscope, so as to distinguish between the luminous particles of air and those of the metal poles.

Attempts to measure the required velocity, using rotating mirrors either between the spark and the slit of a spectroscope or between the prism and telescope, were made by one of us at various times during the last fifteen years. They failed because the method requires that the spark should pass when the mirror is in the same position, and no satisfactory device could be found to secure this object, without at the same time complicating the spark circuit, which it was necessary to confine as much as possible to the electrostatic capacity and spark gap.

Instead of using a rotating mirror, it is possible to secure the same object by taking photographs in a rapidly moving film. An arrangement of this kind was employed by Professor H. DIXON in his experiments on explosions, and proved at once successful.

The principle of the method is extremely simple, and may be employed in all cases where each spark is of sufficient intensity to give a good spectrum. A sensitive film is attached to a spinning wheel and the spark image formed on it. Were the spark absolutely instantaneous, the images taken on the rotating wheel would be identical with that taken on it when stationary, but on trial this is found not to be the case. The metal lines are found to be inclined and curved, and their inclination serves to measure the rate of diffusion of the metallic particles.

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* 'Pogg. Ann.,' vol. 116, p. 132 (1862).

The interpretation of the photographs presents no difficulties if the velocity of the film is everywhere at right angles to the direction of the slit. Such is the case in Professor DIXON'S experiment, and our first wheel was nearly a copy of that used by him. Fig. 1 shows the section of this wheel; the axle bears at A and B against strong screws fixed in a cast-iron support, not shown in the figure. The rim of the wheel had a width of 4 centims., its diameter was 30 centims. The photographic film was wound round the rim, and, although good results were obtained with this wheel, some difficulty arose from the tendency of the film to fly off. Several methods of fastening it to the rim were tried, but the only one which proved satisfactory was to tie it down by two strong steel wires, stretched tightly round the whole circumference. Velocities of about 80 revolutions a second could be obtained, but at the high speeds the film lifted up along the central line, just where the spark images fell.

Fig. 1.

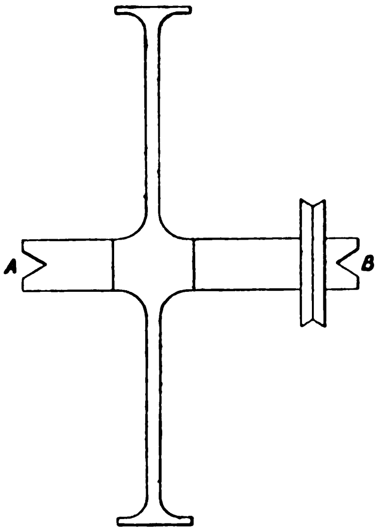
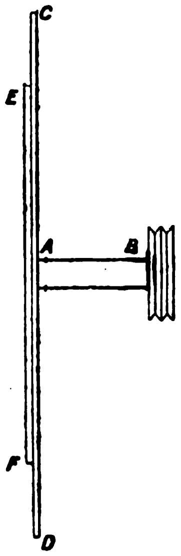


Fig. 2.



A new apparatus was now constructed for us by the Cambridge Scientific Instrument Company, and is illustrated by figs. 3, 4, 5 (Plate 8), while fig. 2 shows a section of the moving parts. The film is placed flat against a steel disc, CD, and is kept in place by a second steel disc, EF, which presses against it, being secured by small screws, shown in fig. 4. The ball-bearing, AB, consists of the hub and axle of a bicycle pedal, the hub being supported by two sets of three strong twisted cords, stretched and fixed as shown in fig. 3, which gives a back view of the apparatus. The same figure also shows a narrow vertical tube for oiling the bearing. The moving parts are surrounded by a circular cast-iron box, which can be covered both in front and at the back. The light forming the image of the spectrum passes through a rectangular aperture cut into the front cover. Fig. 4 (Plate 8) gives a front view with the smaller disc detached and placed in front of the cover. Fig. 5 (Plate 8) shows the whole apparatus ready for experiment. The driving power was a half h.-p. electric motor capable of revolving 33 times a second, and carrying a disc with three grooves. The disc was provided with two pulleys of equal size, so that at high speeds it might be driven from two motors at opposite sides, thus avoiding the side pull. The motor and spinning wheel were tightly clamped to the table of a lathe, in such a way that the apparatus could be dismounted and put together again, with all parts occupying the same position. This was of importance, the method of focussing adopted rendering the removal of the disc necessary during part of the operation. The diameters of the two steel discs were 33 and 22·2 centims., the photographs being taken in the annular space of 10·8 centims., left free when the smaller disc was placed on the larger one.

In all the experiments the driving cord was passed round the second pulley of the motor, which had a diameter of 21·7 centims., while the groove of the pulley on the spinning-wheel was cut to 2 inches, or 5·1 centims. The speed of the motor was measured by an indicator of Elliott Brothers, which was tested and found correct. The ratio of the angular velocities of motor and disc might be obtained approximately by calculation from the diameter of the pulleys, allowance being made for the thickness of the cord, or we might turn the motor slowly by hand, counting the number of turns of the disc at the same time. For our purpose this would have been sufficient, as the other uncertainties of the experiments do not at present allow a very great accuracy, but in order to avoid any doubt, and on account of the interest which attaches to the amount of slipping which takes place at high speeds, an independent stroboscopic method was employed to determine the ratio of the angular velocities. We were surprised to find no measurable slip, except at the highest speeds, such as were never used by us in our experiments.

In our experiments the disc revolved generally about 120 times a second, giving a linear velocity of between 90 and 100 metres per second for that part of the film on which the photograph was taken. When the cord was passed over the largest pulley of the motor we could spin the disc over 160 times a second; this was tried and the speeds were tested, but no photographs were taken, as the smaller velocities gave us sufficiently good results.

Our first successful experiments were made with a single-plate Voss machine, the discharge being taken from four Leyden jars in the circuit, but the beauty of the photographs was considerably increased when a much more powerful battery was introduced, and charged from an induction machine constructed for us by Mr. H. C. WIMSHURST. This machine has 12 plates, of 62 centims. diameter, and gives, without jars, sparks which are 13 inches long. In the majority of our experiments six large jars of flint glass were used, each jar having a coating of tinfoil, of about 2000 sq. centims. surface. The capacity of each jar was measured separately for us by Mr. J. R. BEATTIE, and found to vary from ·0049 to ·0060 microfarad, the capacity of the complete battery being ·033 microfarad. The jars, which had not been used for years, were found to be in bad condition. They were carefully cleaned, by soaking first in caustic soda and then in nitric acid. After coating with tinfoil to within 10 centims. from the top, the uncovered parts were heated in front of a fire to over 60° C., so as to secure complete dryness, and then, whilst hot, varnished. After this treatment the jars gave no trouble.

Great care is necessary in handling a battery of this capacity, especially as the experiments are conducted in the dark.

Fig. 7.

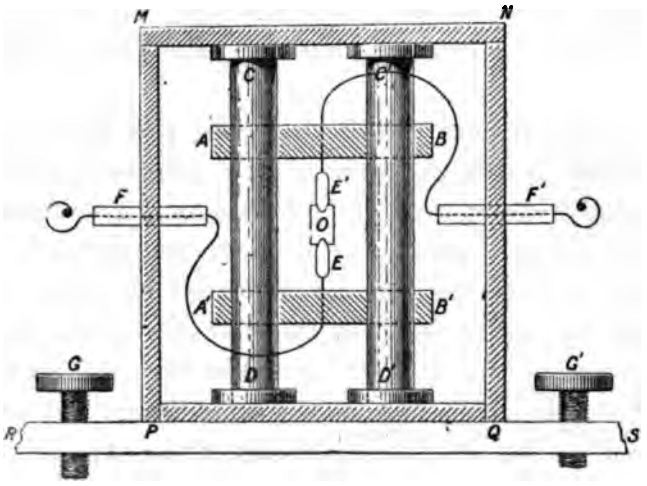
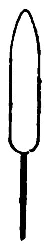


Fig. 6.



The whole battery was placed together on an insulated table, and Mr. WIMSHURST'S recommendation to keep one hand always in the pocket while handling the jars also proved useful.

The metal electrodes were fitted inside a box having an opening towards the collimator of the spectroscope. Perhaps the greatest source of trouble at first consisted in the uncertainty of the position of the points on the electrodes, from which the sparks set out. Successive sparks did not always leave the poles at the same points, so that the spark images did not always fall centrally on the slit. This was remedied to a great extent by giving careful attention to the shape of the electrodes and to their polish. The form adopted after a few trials was conical, and is shown in fig. 6. The metal having first been turned in the lathe to the required shape, was polished by means of emery paper, and finally by wash-leather. Good polish was found necessary to prevent sharp projections, which, though they may be small, cause premature discharges before the jars are sufficiently charged.

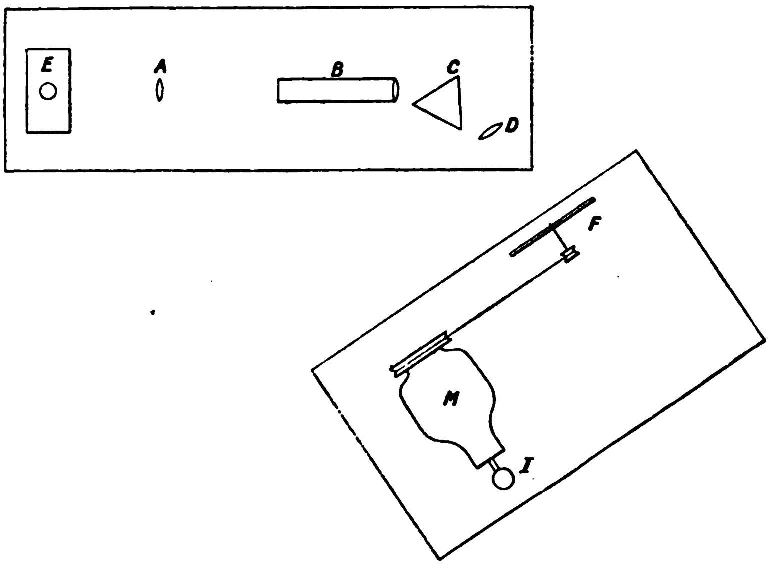
A section of the electrode box is shown in fig. 7. It is made of wood and is fixed to a board, RS, capable of being levelled so as to make the spark path as nearly as possible parallel with the slit. The electrodes are fixed to two wooden plates, AB and A′B′, which are attached to flint-glass tubing, CD and C′D′, the glass being carefully varnished. The electrodes, E, E′, are connected to the Leyden jars by means of gutta percha-covered wire (No. 18) which passed, as shown in the figure, through apertures in the plates, AB, A′B′, and finally through glass tubes, F and F′. The leads from the jars were hooked to loops formed by the wires attached to the poles, the sharp ends of all wires being protected by means of small spheres of sealing-wax. We used in the experiments a standard distance of 1 centim. between the electrodes. The electrodes having been fixed to the plates, their distance was adjusted and the plates secured to the glass rods by means of sealing-wax.

A few words should be said about the optical arrangements. It was for our purpose most important to have a good image of the spark in coincidence with the slit. As we confined our investigation principally to that part of the spectrum which, with glass prisms and lenses, is photographically most intense, the lenses were specially made for us by Messrs. ZEISS at Jena, the chromatic and spherical aberrations being corrected for the region F to H of the solar spectrum. When it is required to form a spark image on the slit, sufficient attention is not always given to the fact that lenses are generally constructed to be used with a parallel beam of light. The aberrations introduced when objects at small distances are to be focussed, are sometimes considerable, and had to be avoided in our experiments. We asked Messrs. ZEISS therefore to make a lens of 4 centims. aperture and 25 centims. focus, which when placed at a distance equal to twice its focal length from the spark, should give an image equal in size to the object within the required limits of the spectrum, free from chromatic and spherical aberrations. The lens sent to us answered all our requirements perfectly. Both collimator and camera lenses had apertures of 4 centims., and focal lengths of 46·3 and 39·9 centims. respectively. The prism used was made of old flint glass by STEINHEIL, having a refracting angle of 59° 28′ and a refractive index of 1·6227 for sodium light. The extent of the spectrum between F and H on the revolving disc was about 1·5 centim.

The general arrangement of the apparatus is shown in a diagrammatic form in fig. 8. E represents the electrode box, A the lens forming the image of the spark on the slit of the collimator B, through which the light passes before it is dispersed by the prism C and focussed by the camera lens D. The parts A, B, C, D were mounted on a slate slab secured to the wall by means of strong brackets. The lathe support which carried the motor M and spinning disc F, was placed so that the image of the spectrum should fall near the top of the disc vertically above the axle. I represents the indicator which shows the speed of the motor.

The adjustments which it is necessary to carry out with accuracy consist in the focussing of the spark image on the slit, the proper centering of the apparatus, and the focussing of the camera lens. In addition to this, it was found convenient to place the lens A as nearly as possible half-way between the spark and slit, so that the spark image on the slit should be of the same magnitude as the spark. This rendered the interpretation of the photographs more easy, as the linear dimensions measured in the direction of the spark discharge were reduced in that case by the optical arrangement in the fixed ratio of the focal lengths of the camera and collimator lenses, viz., ·86. One millimetre in length of the slit images therefore always corresponded to 1·16 millim. in length of the spark path.

Fig. 8.



The adjustment of collimator and lens A is most easily carried out if the prism C and lens D are removed, so that a clear view can be obtained through the collimator towards the spark box. The collimator was adjusted in the usual way for infinite rays. As it was separately supported on a tripod stand, it could be accurately levelled, and the centre of the slit placed, by means of a cathetometer, accurately at the same height as the centre of the spark gap. The focal length of the lens A being known, the distance between the spark and slit was adjusted by measurement to be equal to a little more than four times that focal length, and the lens A was placed at the right height, and exactly half-way between the slit and spark gap. Sparks from an induction coil were now allowed to pass between the electrodes, the slit being opened wide, and a telescope adjusted for infinite rays placed in front of the collimator, so that the observer obtained a clear view of the slit. If the adjustment is good, the electrodes illuminated by the spark are in perfect focus. By a few successive trials, in which the electrode box and lens A alone were moved, and in such a way that the latter was shifted through half the distance of the former, this could always be secured, and at the same time the lens kept half-way between spark and slit.

The lens A had now to be temporarily removed, to make sure that the axis of the collimator pointed to the centre of the spark gap. For this purpose the collimator lens was covered by a piece of cardboard with a central slit about 2 millims. wide, and the electrode box shifted sideways until the maximum amount of light passed through this slit. Replacing the lens A into its previous position, it is easy to ascertain that the adjustment of distances has not been disturbed. The adjustment being complete, the correct position of collimator was permanently secured by pouring melted paraffin wax round the tripod screws which carry it. The lens A was fixed in the same way, but a lateral motion of the electrode box must be allowed, so as to correct the small displacements which necessarily occur when one set of electrodes is replaced by another.

The prism is placed in the usual way in its proper position in front of the collimator; we worked in a position of minimum deviation for a wave-length of about $4{\cdot}3\ 10^{-5}$.

The only remaining adjustments are those of the camera lens and disc. The lens D was fixed to a brass tube which could be made to slide in a collar by means of a screw. The lens was placed so that it was completely illuminated by light passing through the collimator and prism; this is easily tested by an eye placed at the focus of the lens.

When the position of the disc had been accurately fixed, so that the spectrum was formed at the proper place, a film, which had to be sacrificed for the purpose, was placed on the disc just as during an experiment. A pointed rod was now brought into contact with the film and against some prominent spectrum lines like the blue triplet of zinc. The rod was fixed in this position, slightly touching the film, so that the disc could be rotated without appreciable friction. The disc was now removed for the final adjustment of focus, an eye-piece being put in position, so that the pointed end of the rod was in its focal plane. If the lines of the spectrum were not in focus, the lens D were moved until that was the case. The wheel being then replaced into contact with the rod, a perfect focus was secured.

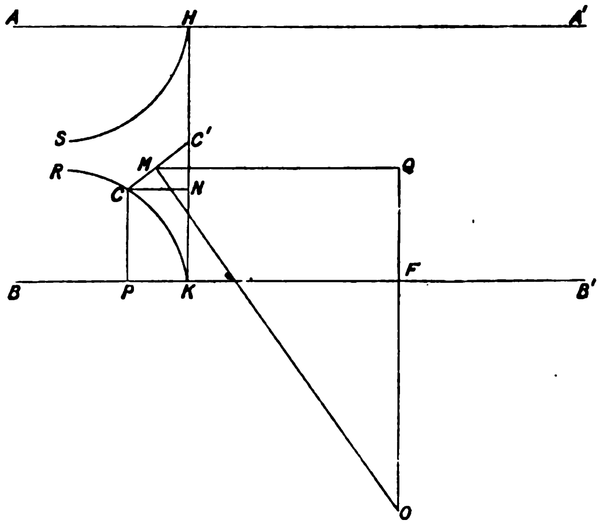
The different parts of the spectrum were not situated at equal distances from the axis. A slight correction which may in consequence be necessary in the reduction of the observations, requires the knowledge of the point of the spectrum which lies at the minimum distance from the axis, *i.e.*, if the spectrum is horizontal, we want to know the wave-length lying vertically above the axis. This may readily be ascertained by suspending a plumb-line close to the disc passing in front of its centre. The shadow of the string will then show on any photograph of the spectrum which is taken.

2. *Interpretation of Photographs.*

With the wheel which was first used, the film was placed round the rim (fig. 1

and the velocity of the film being everywhere at right angles to the image of the slit, the interpretation of the photographs was obvious. But with the film on the face of the disc, as in our final experiments, the linear velocity is not the same for different portions of the slit image, or different parts of the spectrum, nor is it everywhere perpendicular to the slit image. The following investigation shows how the velocity of the particles may be deduced from the measurement of the photographs.

Fig. 9.



Let AA′ (fig. 9) represent the upper, and BB′ the lower, edge of the spectrum, as it would be formed on a stationary film. KH represents the monochromatic image of the slit at the instant of the beginning of the discharge, and KR, HS the curved edges of a spectroscopic line as they appear on a photograph taken with a moving slit. The displacements are most easily determined by measuring the coordinates $PC = y$ and $NC = x$ parallel and at right angles to HK. If O be the centre of the disc, a point, C′, of the slit image will describe a circular arc, C′C, on the moving film, and the time taken by the moving particle to go from K to C′ is the same as that which the point C of the disc takes to describe the arc CC′, which may for our purpose be taken as equal to the length of the chord. From the middle point of CC′ draw the lines MQ perpendicular to HK, and QO parallel to HK. Then

$$C'N : NC = MQ : OQ,$$

or if $KC' = z$, $OF = a$, $FK = b$, and if the squares of y and z and their products are neglected, compared to a^2,

$$z - y = bx/a.$$

For the line HS we should similarly find $z - y = -bx/a$. Also from the figure

$$CC' : CN = OM : OQ,$$

or writing s for CC′ and r for OM,

$$s : x = r : a + \tfrac{1}{2}(y + z).$$

Again neglecting the square of small quantities,

$$s : x = r : a + y.$$

It is convenient to introduce in place of a the distance (c) between O and the *centre* of the spectrum, so that $a = c - h$, where h represents half the width of the spectrum. To the required degree of accuracy it is then found that

$$\frac{s}{x} = \frac{r}{c}\left(1 \pm \frac{h - y}{c}\right),$$

where the minus sign refers to measurements taken along HS.

The time (t) taken for the description of the arc CC′ is $s/r\omega$, ω denoting the angular velocity of the disc, and if v be the linear velocity at the centre of the spectrum

$$t = \frac{sc}{rv} = \frac{x}{v}\left(1 \pm \frac{h - y}{c}\right).$$

As y and x are obtained by measurement of photographs, and h and c have been practically constant in the experiments (though also measured in each case), the above equations allow us to calculate

$$z = y \pm \frac{bx}{c}, \qquad t = \frac{x}{v}\left(1 \pm \frac{h - y}{c}\right).$$

If x_1, y_1, and x_2, y_2 are the measured coordinates of two points lying near each other on the curved images of the spectroscopic lines, the velocity of the luminous particles at the corresponding point of the spark will be $\mathrm{K}\,\frac{z_2 - z_1}{t_2 - t_1}$, where K is the ratio of the length of the spark gap to that of its image on the film, which in our case was simply the ratio of the focal lengths of the camera and collimator lenses, *i.e.*, 1·16. If both poles are made of the same material, so that both the lines which appear at the top and bottom of spectrum can be measured and the mean taken, the correction disappears, and the velocity of the particles is given directly by $\mathrm{K}v\,\frac{y_2 - y_1}{x_2 - x_1}$. Even when a line can only be measured on one side, this may approximately be taken to be the velocity, for in our experiments c was equal to 14, the distance b was seldom greater than 1 and never greater than 2, while the ratio x/y was always less than $\frac{1}{3}$, so that z and y only differed at the maximum by 5 per cent. from each other. The uncertainty of our experiments, for reasons which will be given, was greater than that amount. Similarly may the quantity $\frac{h - y}{c}$ be neglected for the present.

3. *Method of Conducting Experiments and Measurement of Photographs.*

The method of conducting an experiment is obvious from the preceding descriptions. The apparatus being in adjustment, a photograph was taken of the spark spectrum, on the stationary disc. This is for the sake of reference, chiefly for the purpose of identifying the lines. The disc was then set spinning and when the desired speed was attained and found to be uniform the Wimshurst machine was set going by hand until about six sparks had passed. We thus obtained a number of images on each film, some of which sometimes were found accidentally to overlap. The images are not found to be equal in clearness, the spark not always passing parallel to the slit; the two best were selected for measurement. For this purpose the film, after the images had been developed and fixed, was cut up, each image being carefully marked for future reference.

The measurements were made by means of a "comparator," a very beautiful instrument, made by ZEISS, of Jena. The instrument consists essentially of two microscopes at a fixed distance from each other. A sliding table underneath serves to carry the photograph to be measured under one microscope and a scale under the other. A photograph having been fixed to the table by means of soft wax, a fine adjustment screw allows a certain relative displacement between it and the scale, so that when any desired line is in the centre of the field of view of one microscope the scale may be read with the other, and adjusted to give any desired reading. This adjustment is of great convenience in the comparison of different spectra with each other, as some air line, common to all, may always be placed so that the reading is in every case the same. The scale is divided into $\frac{1}{5}$ millims., and a micrometer eye-piece allows readings to $\frac{1}{1000}$ millim.

Speaking for convenience sake of the slit images as "vertical," the photograph must be placed so that when the sliding table is moved the trace of the centre of the field of view on the photograph is horizontal, and the measurements consist in measuring the horizontal displacements of each spectrum line at different distances from the edges of the spectrum.

Our first method of conducting the measurement was to take contact prints on glass of the selected images. On these copies horizontal equidistant lines were ruled by a fine needle point fixed to a dividing engine. The displacements could then be measured along each of these lines. The disadvantage of this method consists in the labour of taking copies and ruling lines, and also in the loss of definition, which, however small, was always noticeable in the contact print. At a later stage, therefore, the method of measurement was changed to the following:—A brass frame was made in which the film could be clamped so that it lay perfectly flat under the microscope. Underneath the photograph was placed a transparent scale, made of a portion of a film from which the sensitive layer had been removed, and on which horizontal lines at a distance of about ·55 millim. had been ruled. These lines could be seen in the instrument traversing the spectrum horizontally. They served as reference lines along which the displacements could be measured.

Accurate measurements of wave-lengths were not necessary for our purpose as long as we could identify the different lines. As the prism was not disturbed during the principal series of our experiments, the air lines were found to be sensibly at the same distance from each other in all our photographs. Making use, therefore, of a few of these air lines, which are sharp, and of the lines of zinc and cadmium, which are easily identified, a curve may be drawn from which the wave-lengths corresponding to any reading of the comparator may be determined in the usual way with quite sufficient accuracy.

4. *Preliminary Experiments.*

Before entering into the main subject of our research, we desired to become familiar with the appearance of the spark itself, and with such other phenomena connected with it as could be elucidated by photographs of the spark itself. We describe the results obtained, selecting out of a number of plates those which seem to present distinctive features of interest.

Fig. 11, Plate 9, November 10, 1897.—Image of spark taken between zinc and brass electrodes ; 5 Leyden jars were used, their total capacity was not measured, but the jars were smaller than that of the battery of six used in the later experiments, and the total capacity was estimated to be about half, *i.e.*, about ·015 microfarad. The chief feature of this photograph is a slightly curved very luminous column, containing, as was subsequently ascertained, the first discharge breaking through the air. This is surrounded by a cloudy appearance, which is due to the metallic vapours generated by the first discharge.

Our later experiments, to be described further on (p. 211), point to the conclusion that the oscillations following the first discharge pass through this cloud of metallic vapour. Some sharp linear luminous filaments may be seen in the original on one side of the main discharge. These set out from small projections on the electrodes and precede the main discharge, as may be ascertained by carefully watching the image on the focussing plate of the camera. The preliminary discharges may be avoided by giving careful attention to the polish of the electrodes.

Fig. 12, November 10, 1897.—A photograph of the spectrum of the preceding discharge, taken with 15 sparks. The image of the central column was thrown on the slit. The spectrum is that of zinc and air, with one or two lines of copper, and apparently the calcium lines H and K. A trace on the calcium line at 4227 also shows on the original.

Fig. 13, Plate 9, November 10, 1897.—The same as fig. 12, except that the cloudy portion of the discharge was focussed on the slit, which had to be widened a little in this case. Thirty sparks had to be taken to secure an impression which could be compared in intensity with the previous one. The absence of air lines, verified also by eye observations, proves that the cloudy luminosity is due to metallic vapours. The presence of metallic lines in fig. 2 does not prove the presence of metallic molecules in the main discharge, as it is impossible to obtain an image of the latter free from that of the luminous cloud lying in front and behind it.

Fig. 14, Plate 9, November 12, 1897.—Spark between iron poles, showing the luminous filaments preceding the discharge.

Figs. 15 and 16, Plate 9, November 13, 1897.—Sparks taken under similar circumstances as fig. 14. In fig. 16 a current of air is blown through the spark gap. The effect of this current of air is remarkable, as the time of luminosity of the air column was found in subsequent experiments to be less than 10^{-6} second, and the actual displacement of the air during that time must have been quite inappreciable. The most probable explanation is that the spark passes through those portions of the air which have been made conducting by preliminary invisible discharges. The air put into the sensitive state by these first partial discharges has time to move over a sensible distance before the main spark passes.

It is remarkable how the metallic vapours in this photograph seem to be drawn into the glass tube.

Fig. 17, Plate 9.—Photograph of a succession of sparks taken with a small jar and large induction coil. This photograph illustrates the fact that, in the ordinary arrangement adopted to produce metallic spectra, the air discharge is predominant, while the metallic cloud is chiefly confined to the neighbourhood of the electrodes.

5. *Measurement of Molecular Velocities.*

We may now pass to the description of the results obtained when the spectrum of a single spark is taken on a moving film. A preliminary trial with various metallic electrodes had shown us that the sharpest results were obtained with zinc, and we chose, therefore, that metal for investigation under various conditions. Fig. 18 (Plate 10) gives the spectrum as we obtained it on the stationary film. The lines of zinc which appear on it are*:—

4924·8 4912·1	α.	Zinc doublet not visible in the arc spectrum.
4810·7	β.	Zinc triplet, strong both in arc and spark.
4722·3	γ.	
4680·4	δ.	
4058·0	ϵ.	Not mentioned as seen in the spark by previous observers, but given by Kayser and Runge as a strong line in the arc.

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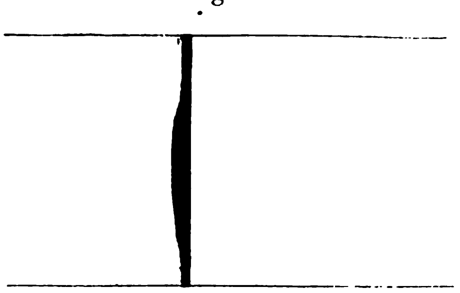
* All wave-lengths are given on Rowland's scale in air, and are taken from the most reliable available determinations. To identify the lines on the photographs we have attached Greek letters to the principal metallic lines shown in the figures, and these letters are also attached to the wave-lengths as given in the text.

The figure is enlarged in the ratio of about 4·2.

Fig. 19, Plate 10.—Disc spinning with linear velocity at centre of spectrum, $v = 99$ met./sec. The air lines are seen to run straight across the spectrum, but the metallic lines are curved and broadened. It follows that the metallic vapours remain luminous longer than the air lines, and that the metallic particles are projected with a measurable speed from the electrodes. The broadening of the nitrogen doublet at 5004, marked N_1 in the photographs, only amounts to about ·04 millim., which limits the luminosity as far as that line is concerned to $\frac{\cdot 04}{97000} = 4 \times 10^{-7}$. The air line at 3995, and marked N_2, is drawn out rather more, but is thinner near the poles than in the centre of the spark; its appearance, which in some of the photographs taken is even more marked than here, is illustrated in fig. 11. We must conclude that the air remains luminous in the centre of the spark rather longer than near the poles. This fact, which is very apparent for the luminosity of the metallic vapours, will be referred to again further on (p. 210).

Fig. 11.



The displacements on this photograph being the first that were actually measured, the best methods of drawing the reference lines (see p. 198) had not been adopted, and these lines were therefore at unequal intervals. In the tabular arrangement of our results we give in the first columns the coordinates of the curved lines which have actually been measured; x denoting the horizontal displacement at a distance y from the edge of the spectrum. If y_1, y_2; x_1, x_2 are coordinates of two points which are near together, and v is the linear velocity in metres per second, the molecular speed is $V = Kv \frac{y_2 - y_1}{x_2 - x_1}$ (p. 197), at a distance $h = K \frac{y_1 + y_2}{2}$ from the pole, where K is the factor correcting the optical reduction, which is 1·16 in all our photographs. The displacements being difficult to measure, a small error, in x_1 and x_2 near the pole, may produce very large differences in the result. For the comparison of different metals with each other we therefore calculate also the quantity $V' = Kv \frac{y}{x}$ which is the average speed between the pole and a point at a distance $h' = Ky$ from the pole;

the quantity x being larger a small error does not affect the result so largely as one in the difference $x_2 - x_1$.

TABLE I.—Zinc, $\lambda = 4810·7$ (β).

Date, Feb. 15, 1898. No. XV. $v = 102$. Distance between poles: 1 centim.

x.	y.	V.	h.	V'.	h'.
millims.	millims.				
		Lower pole.			
·07	·56	896	·33	896	·66
·18	1·12	631	·98	740	1·30
·38	1·66	316	1·61	530	1·93
·59	2·20	300	2·24	438	2·55
		Upper pole.			
·09	·44	548	·26	548	·51
·21	·88	472	·77	506	1·02
·33	1·52	598	1·39	541	1·76
·64	2·16	241	2·13	395	2·50

$\lambda = 4924·8$ (α).

x.	y.	V.	h.	V'.	h'.
millims.	millims.				
		Lower pole.			
·01	·56	6632	·33	6632	·66
·23	1·12	297	·98	569	1·30
·42	1·66	347	1·61	471	1·93
·68	2·20	241	2·24	381	2·55
		Upper pole.			
·10	·44	505	·26	504	·51
·24	·88	371	·77	427	1·02
·50	1·52	289	1·39	356	1·76
·75	2·16	306	2·13	340	2·50

The above experiment was only considered to be of a preliminary nature, and the photographs obtained were used chiefly to find the best method of reducing the observations, but it gives a good idea of the general nature of the results. It is seen that the velocity near the pole is a very uncertain quantity, owing to the sources of error which will be discussed (p. 210), but the velocity gradually diminishes, and about

2 millims. from the poles it reduces to about 400 metres / second, which is not very different from the velocity of sound in air at the ordinary temperature. It is well to realise this, for it shows that the sound-wave which is produced by the spark has moved outwards through a distance of a few millimetres only by the time the metallic molecules have reached the centre of the spark.

In order to see whether the capacity of the jars and distance between the poles have an appreciable effect on the result, the displacements of the zinc lines were measured for sparking distances of approximately ·5, 1·0, and 1·5 centims., and in each case the capacity was altered by taking the discharge from 2, 4, and 6 jars. Table II. embodies the result; the velocity (V') given being, as above explained, the average speed between the pole and a point at a distance of 2 millims. from it.

TABLE II.—Average Velocity (V') in metres / second of Zinc Molecules.

Sparking distance.	Wave-length.	Number of jars.		
		2	4	6
centims. ·51	4925 (α)	814	556	416
	4811 (β)	1014	668	529
1·03	4925 (α)	400	499	415
	4811 (β)	501	548	545
1·54	4925 (α)	723	1061	435?
	4811 (β)	1210	1526	492?

The first striking fact shown by the table is the uniformly slower speeds derived from the doublet at 4925 as compared with that of the least refrangible line of the adjoining triplet, and we have assured ourselves that there is no difference in displacement between the two first components of the triplet. The third component is too weak and too near an air line to admit of satisfactory measurement. It was one of the main objects of our investigation to discover, if possible, such differences in the velocities as might indicate the presence of different kinds of molecules, but we hesitate to ascribe the differences found to such a cause. The line 4925 is the least refrangible component of a double line, which is wider and much stronger at the base than in the centre of the spark. In order to measure the displacement, the cross wire of the reading microscope has to be set on the edge of the displaced line, first near the pole, and then on a corresponding point nearer to the centre of the spark gap. Where the line is strong, the edge of the line would be the least refrangible edge of the *least* refrangible component, but when the line is weak, the strongest part of the line would be that part where the two components begin to overlap, *i.e.*, the edge of the *most* refrangible component, and there would be a tendency to setting the microscope on the strongest part of the line rather than on the edge of it. It is easily seen that an error would be liable to arise, giving too great a displacement or too small a velocity. We are not at all certain that if this source of error were eliminated the double line would not show higher rather than lower values for the velocity, and hope to decide this question by using greater dispersion.

Comparing different capacities with each other, we find that for the spark gap of 5 millims. the velocities are greater for small than for large capacities; we offer at present no explanation of this unexpected result, which requires confirmation and further investigation. When the sparking distance is increased to 1·5 centim., the course of the sparks becomes so erratic that not much importance can be attached to the figures. A query has been attached to those that are specially doubtful. On the whole, there seems a tendency towards greater velocities in this case.

Our normal spark gap of 1 centim. does not show any decided difference due to capacity, and with our normal capacity of six jars the spark gap does not seem to affect the result. Hence we are justified in thinking that under the conditions named a reliable comparison may be made between the velocities obtained for different metals.

We have collected in Table III. the velocities V and V′ as measured on our photographs. The results are nearly always the mean of two sets of measurements from different photographs. In some cases, such as magnesium, iron, manganese and silver, no satisfactory measurements could be made. The spectrum, as it appears on our films, does not always give the same distribution of intensity among the lines as is shown by the spectra of sparks taken with an induction coil and jars of small capacity, and it differs also, of course, from the arc spectrum. A discussion of the peculiarities of our spectra lies outside the range of this paper, and must be reserved for another occasion, but in the following account some of the chief features of our spectra are shortly pointed out. The metals are arranged in order of atomic weights.

TABLE III.

Metal.	Wave-length.	V′ (metres per second).	V (metres per second).
Aluminium	4512 (α) 4478 4446 3613 β 3602 3585	1890?	1300?
Zinc	4925 (α) 4912	415	406
	4811 (β) 4722 (γ)	545	524
Cadmium	5379 α 5339	435	472
	5086 β 4800 γ 4416 δ 3613 ε	559	515
Bismuth	5209 α 4561 β 3696 γ	1420	1480
	4302 δ 4260 ε	533	488
	3793 ζ	394	270
Mercury, from Cadmium Amalgam	5461 α 4359 β	988	550
	3663 γ	590	406
„ from Zinc Amalgam	4359 β 3663 γ	481 383	

Magnesium.—In photographs taken on a stationary film we note the complete absence of the triple line in the green, which forms a prominent feature of the magnesium spectrum under ordinary conditions. The whole energy of the spectrum seems to be concentrated into the line at 4481 (α), which is exceptionally strong, and into the triplet beginning with 3838·4 (β), which appears as a doublet in our films, the two last lines probably not being resolved. A strong line seen both in arc and ordinary sparks at 4703 is also absent. When the spectrum is photographed on the rotating film, the lines resolve into remarkable clouds (fig. 20, Plate 10), which do not admit of measurement. The appearance of the lines shows, however, that the velocities of the magnesium particles is great, and approximately the same as that of aluminium.

Aluminium.—The prominent lines are the two triplets beginning with 4512 (α) and 3613 (β) respectively, and the doublet which lies between H and K. The former do not appear in the arc spectrum, which shows the violet doublet strongly. The velocities we have obtained are the largest we have measured, but the displacements (fig. 21, Plate 10) are very small, and the lines are measurable only near the poles, so that the numbers are doubtful. The violet triplet seems displaced through a greater distance than the blue, but the measurements are so uncertain that we have taken the average displacement without distinguishing between them. The violet doublet almost disappears in the rotating film, leaving only a faint cloudy formation similar to that shown in the magnesium spectrum, but possessing peculiarities which require further investigation.

Manganese.—Many lines of manganese appear on our stationary photographs, but on the rotating film they completely disappear under the usual conditions of our experiments. When the slit is widened and the speed reduced, the displacements could be observed, but were not measured.

Iron.--The iron lines were not well marked, and completely disappeared in the rotating films.

Copper did not show any well-marked lines, the displacements of which could be measured.

Zinc.—This metal has already been discussed in detail, and it only remains to mention that the line at 4058 (ϵ) disappears on the rotating film.

Silver showed no prominent lines when our battery was discharged through it, but the calcium lines are always seen (see p. 212).

Cadmium.—A number of cadmium lines appear on our photographs, which agree in position with the lines so frequently measured; but there is some difference in the relative intensities as noted by different observers. The double line 5379 and 5339 (α) shows a greater displacement than that of the other lines, and the distance between the components is so great that the explanation given for the corresponding zinc lines does not seem to hold here. Fig. 22 shows the behaviour of the cadmium lines.

Mercury.—Sparks were taken from a surface of liquid mercury, and though the displacements can be readily seen (fig. 23), and the edges of the lines are fairly sharp at some distance from the electrode, they are so diffuse in close proximity to it that no satisfactory measurements could be made at points corresponding to those taken in the case of other metals. Better results were obtained when the electrodes were either amalgamised zinc or cadmium; but the measurements in these cases, though they are given in the table, are difficult to make and cannot be trusted. When cadmium amalgam is used (fig. 24), the photographs show clearly that the mercury line 3663 (Hg, γ) is parallel to the cadmium line 3613 (Cd, ϵ), and the measurements are fairly consistent, but when the pole was zinc amalgam, the displacements of the mercury lines seemed decidedly greater than that of the zinc lines. This would indicate a higher velocity for zinc than for cadmium molecules, while our previous results had given almost identical values for the displacements of the zinc and cadmium lines. There is here some contradiction which requires clearing up by further experiments.

Bismuth (fig. 25, Plate 11) gave us most interesting results, as it possesses some lines, such as γ, which are curved so little that the velocities of the molecules giving these lines is found to be larger than that observed in any other case except aluminium. On the other hand, the line at 3793 (ζ) indicates the smallest velocity measured. There are some special difficulties standing in the way of the measurement of the bismuth lines, which made us hesitate some time before definitely asserting the different curvature of the lines, but we think that our best photographs, one of which is reproduced in fig. 25, leave no doubt on the question. One of the difficulties lies in the fact that bismuth and mercury are the only metals in which the lines are repeated, owing to the oscillatory discharges. When six jars are in circuit, the appearance is that of fig. 26, and the lines become mixed and difficult to measure. Another difficulty lies in the great difference in the sharpness of the lines even on the stationary film, the lines which show a small curvature being much sharper. There was a possibility of an illusion due to this cause, similar to that explained in the case of the double zinc lines, but we cannot believe that the difference in curvature between the lines marked (γ) and (ζ) in fig. 5 can be due to this cause.

[April 26.] We possess very few investigations on the spectrum of bismuth· LECOQ DE BOISBAUDRAN, who is acknowledged to have purified his substances with extreme care, gives the line 5209 (α) as one of the characteristic lines of bismuth, and he also gives 4302 (δ) and 4260 (ϵ) as belonging to bismuth. The relative intensities of the lines as given by him cannot be expected to coincide with our own, as he used very different spark conditions. Our spectrum agrees, on the other hand, perfectly with that given by HARTLEY and ADENEY,* who traced no coincidence between the lines we made use of in this research with those of other metals.

As the great difference in the molecular velocities suggested the possibility that bismuth was a mixture of elements, we obtained, through the kindness of Messrs. JOHNSON and MATTHEY, samples of bismuth prepared from three different sources. The visible portion of the spectrum was examined with great care, but no difference in the relative intensities of the lines could be detected.

6. *Discussion of Results.*

When we compare together the results obtained for different metals, the first question that arises refers to the connexion between the velocities and atomic weight, or rather vapour densities. Dr. FEDDERSEN was led by his researches to conclude that the spark through air volatilised the metal, which afterwards, according to him, took no further part in the discharge. If that is the case the process of molecular diffusion should, at equal temperatures, be inversely proportional to atomic weight. There is no doubt that the first luminosity of the discharge is entirely due to the spark breaking through the air. The air lines are so little widened in our photographs that it is only close to the pole that any metal vapour can be present while the air is luminous. During the interval between the initial current and the first return, the metal vapour will diffuse chiefly, if not entirely, by reason of the molecular velocities. The initial discharge starts a sound-wave which must leave for a short time the air between the electrodes in a state of rarefaction, and it is perhaps right to consider that the mass of metallic vapour suddenly formed is driven by its own pressure into the partial vacuum formed by the heated air. It would seem more correct, therefore, to compare the process with that of a gas under pressure flowing into a vacuum than to that of pure thermal diffusion. There is not much difference between these views, and we may take it that in our experiments we have approximately measured the velocity of sound in the metallic vapour. This gives a relation between its temperature and density. Neglecting a possible difference in the ratio of specific heats, the relation

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* 'Phil. Trans.,' vol. 175, p. 130, 1884.

$$V = 80 \sqrt{T/\rho}$$

should hold approximately, T being the absolute temperature, V the velocity of sound, and ρ the vapour density referred to hydrogen. Thus, for cadmium, the average molecular velocity found was 560, and substituting $\rho = 56$, we obtain $T = 2700$, which seems a possible but rather low value. We must conclude that the molecule of cadmium in the spark cannot have a mass which is much smaller than that determined directly near the boiling point of the metal.

If we compare different metals with each other we are struck with the almost identical numbers obtained for zinc and cadmium. It is possible, of course, that zinc vapour may be diatomic, but it seems more probable that as the spectrum of zinc and cadmium show homologous lines, the molecular constitution of the two vapours is the same. Aluminium, with a small atomic weight, has a high velocity, and so has magnesium, but the ratio of the velocity of aluminium to cadmium is roughly as 3 : 1, while the ratio of the square roots of the atomic weights is only as 2 : 1.

The uncertainty of our numbers is so great that we only wish at present to draw the general conclusion that the two metals having the lowest atomic weights, which have been examined by us, are also those showing the highest velocities. For the same reason we forbear discussing the question as to the different behaviour of different lines, especially with regard to other peculiarities possibly existing in the behaviour of lines which show abnormal velocities. The data are at present too scanty and uncertain to allow us to attach any value to the coincidence of such peculiarities.

An interesting question arises as to whether, in the case of an alloy, the different components affect each other or not. The evidence, so far, goes to show that they do; the zinc lines, for instance, are less curved when the zinc is amalgamated, and similarly when electrodes of bismuth are moistened with calcium chloride the displacement of the bismuth lines is reduced.

7. *Experiments without Prismatic Decomposition.*

Dr. FEDDERSEN took photographs of the entire spark by means of the revolving mirror, and the appearances he obtained were very irregular, though his experiments allowed him to draw some general important conclusions. We considered it to be of interest to take some photographs after removal of the prism, retaining the slit; fig. 29 (Plate 12) shows the appearance with zinc poles under the normal condition of our experiment, *i.e.*, with the six jars and a pole distance of 1 centim. The straight luminous initial discharge which passes the air gap is followed by a number of curved lines, which represent the oscillatory discharges. Fig. 30, in which the oscillations are spread out by the interposition of self-induction, gives a better representation of the phenomenon. We notice, in the first instance, the alternation at each pole between strong and feeble discharges: the strong discharge at one pole being opposite a weak discharge at the other. This peculiarity was pointed out already by Dr. FEDDERSEN. Our experiments do not allow us to decide at which pole the discharge is most luminous, and it would be important to find this out, as it would allow us to decide whether the discharge through the metal vapour resembles more that of a vacuum tube or that of the voltaic arc. Fig. 30 shows the initial discharge to be followed by a second straight luminous band before the curved lines begin. The curvature is irregular, and the two poles do not behave exactly alike.

The distance between successive discharges is the same for all metals we have tried, which shows that the resistance of the metal vapour cannot be a dominating portion of the whole resistance, for doubtless different metal vapours will differ in resistance, especially as they must be present in very varying quantities, owing to differences in volatility. As we can count about ten discharges, we may take it that the damping is small and therefore the time of an oscillation $2\pi\sqrt{LC}$, if L is the self-induction. From this we calculate that our self-induction was about 3000; for steady currents a rough estimate of our circuit gave a self-induction of 1000, and we know that for rapid oscillations it must be greater. If we adopt the higher number, we may further conclude that the total resistance of our circuit must have been small compared to 20 ohms, while, taking the lower estimate for the coefficient of self-induction, the resistance was small compared to 12 ohms, a further proof, if one were needed, of the fact that once the insulating property of air is broken down, its conductivity may be large. We do not wish to enter further into some interesting questions connected with this matter, as we are only concerned with the bearing of our experiments on the main subject of our research.

The curved lines of the oscillatory discharges may serve as a basis for the calculation of the molecular velocities, and if this is done, higher values are obtained than those we have derived from our experiments with the complete apparatus. Aluminium and magnesium gave again, however, the highest velocities near the pole, but that of the zinc molecules exceeded considerably the velocities found for cadmium.

There is here, however, a considerable difficulty in the interpretation of the result, as without the prism it is impossible to separate the different lines, and, what is perhaps more important, we do not get the effect of the first discharge at all, as that is hidden behind the dense luminous column of the straight air discharge.

If a photograph be taken of the spark on the rotating film, the image is always drawn out most near the middle of the spark, and we obtain images such as fig. 31. It is not quite easy to see why the metallic particles should remain luminous in the centre of the spark longer than near the pole, unless there is some inflow of cold air from the poles inwards. Such an inflow might be produced through the effects of rarefaction produced by the initial heating of the air by the spark.

8. *Sources of Error.*

Different photographs, obtained exactly in the same way, sometimes differ considerably in the value they give for the molecular velocity, and there is little doubt that the chief cause of error lies in the fact that the discharge is not a straight line parallel to the slit, but takes place in irregular curves, and, as appears already on Dr. FEDDERSEN'S photographs, the successive oscillations of the same discharge take sometimes very different courses. When we first began to work we used smaller capacities, and our results were more irregular, because the course of the discharge was more erratic. The large capacity acts in the direction of making the path of the discharge straighter, unless the sparking gap becomes too great. If the molecular stream is a straight line, our calculations are based on the further assumption that the image of the point of divergence falls exactly on the slit. If that point is at a distance h, measured perpendicularly to the slit, the molecules would have to traverse a distance $\sqrt{y^2 + h^2}$, if the projection of the distance along the slit is y. Calling V the velocity of the molecular stream, v that of the photographic film, it follows from $x = \mathrm{V}t$ and $vt = \sqrt{y^2 + h^2}$ that

$$\mathrm{V}^2x^2 - v^2y^2 = h^2v^2.$$

x and y are proportional to the coordinates of the spectroscopic lines as measured on the film, and the curve represented by the equation is a hyperbola.

The error introduced is such that a constant velocity would appear as one infinitely large close to the pole and gradually diminishing. If h is small, the hyperbola would soon coincide with its asymptote, and in that case the error in the molecular velocities as deduced from points which are not in close proximity to the poles, would not be large. Some of our photographs give evidence of being affected by this error.

Other inaccuracies which cannot readily be evaluated are introduced by the curvature of the spark. If we imagine the molecular streams to follow the course of the spark, the appearance on our photographs might be considerably modified. In particular, if the curvature lies in a plane passing through the slit, the molecules near the pole will move towards the slit or away from it, and the components of the velocity resolved along its direction will have a smaller value than near the centre of the spark, where its direction is nearly always parallel to the slit. The error introduced is in a direction opposite to that found in the previous case, the molecular velocities near the poles now appearing too small.

One further point remains to be noted. Photographs like those illustrated by fig. 29 and fig. 30 cannot be easily explained, unless we take it that the metallic molecules actually carry the electric current. If that is the case, and if the process of discharge is similar to that advocated by one of us in 1882* and now generally accepted, the molecular stream will carry with it the ionic charge, at any rate in the positive part of the discharge. Now, considering successive oscillations, electric forces must continue to act on the metallic molecules, first accelerating, then retarding, and then again alternately accelerating and retarding. Under the conditions of our experiments, and with the velocities measured by us, the molecules would only have got to a distance of about 1 millim. from the pole before the second acceleration takes place. If that is a correct view, we should expect a sinuous curve for the molecular path, and there seems indeed a tendency towards such a form in some photographs, as for instance that given in fig. 22. The only way to overcome this complication will be to increase the period, so that we may be able to measure the velocity of the stream separately for each oscillation. The effect of magnetic forces on the velocity of the stream may also supply useful information.

9. *Effect of Self-Induction in the Spark Circuit.*

In the course of our investigation we were led to insert a coil of wire into the spark circuit in order to separate the oscillatory discharges. We discovered in this way a curious effect on the appearance of the spectrum, the air lines completely disappearing when the coil was inserted, provided the self-induction was sufficient. The most plausible explanation seems to be that the air lines are produced entirely by the first initial discharge, when the spark gap contains no metallic vapour. The subsequent oscillations pass, on the contrary, through the metal vapour, which in the meantime has had time to diffuse away from the electrodes. By inserting the coil, the initial discharge takes place more slowly and apparently does not heat up the air sufficiently to yield the line spectrum. The whole duration of the spark is considerably lengthened, and the discharge may pass through the metal vapour, which after a few millionths of a second fills the whole spark gap. The spectrum is modified by the change in the mode of sparking, and Mr. HEMSALECH is at present investigating this change, but our joint photographs are sufficient to show that the spectrum is *not* reduced to that of the arc discharge, the double zinc line at 4925, even with large self-inductions, remaining strong, though it thins out a little. We consider the method of investigating spectra a most useful one, and it allows us to distinguish at once what is an air line and what is not. The obnoxious noise of the spark is also much diminished.

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* 'Bakerian Lectures,' 1884 and 1890; 'Roy. Soc. Proc.,' vol. 37, p. 317, 1884, and vol. 47, p. 526, 1890.

[April 26, 1899.] Professor KAYSER has kindly drawn our attention to a paper by Mr. AUER v. WELSBACH ('Wiener Sitzungsberichte,' 88, II., 1883), in which an interesting device is described by which strong spark spectra may be obtained without the usual induction coil. It is essentially an arrangement by which the spark at the break of the primary current is used in place of a gap in the secondary circuit. The air lines also disappear under these conditions. In how far the spark is of a similar nature to that used by us is difficult to say without direct comparison of the spectra obtained.

When the above method is applied to different metals, it is found that the calcium lines H and K of the solar spectrum appear in most, possibly in all, cases. There is perhaps nothing surprising in this, owing to the presence of calcium in the dust of the air, and the probability of its occurring as an impurity in many metals. But what is surprising is the great intensity of these calcium lines, as compared with those of the metal proper when the poles used are silver. We deposited some of the metal electrolytically, and Mr. J. CROWTHER fused the deposit into poles on a block of willow charcoal, which contains only a very small amount of ash, but we must admit the possibility of some traces of calcium having been taken up by the silver in the process.

The silver so prepared gave not only the H and K lines, but also the line at 4226, with undiminished vigour. We do not offer any explanation as to the possible sources of calcium, either in the metal or in the dust of the air, but the fact is rendered remarkable by the appearance of the lines in question when the spectrum is taken on the rotating wheel. Fig. 27 is a photograph taken on the stationary film without self-induction, and the H and K lines are seen strongest near the centre of the spark. With self-induction, the strongest lines of the spectrum are those of calcium. In fig. 28 the disc was spinning, and there was no self-induction. The lines H and K present a comet-like appearance, K being much the strongest. The head of the curve corresponding to the K line is displaced towards the violet, and if our interpretation of these curves is correct, the photograph seems to prove that the H and K vibrations start not at the pole, but in the centre of the spark, and luminosity begins not when the main initial discharge strikes through the air, but, roughly speaking, about the 200,000th part of a second afterwards, when the luminosity of air as shown in our photographs has completely ceased. Further experiments are in progress, and the matter is only mentioned here because any one repeating our experiments on the influence of self-induction will be struck by the intensity of the H and K lines when the spark is taken from silver poles. The extraordinary persistency with which the slightest trace of calcium is known to give the lines in question renders any investigation on the actual source of the calcium present very difficult. The effect of polishing, which was necessary to obtain good sparks, undoubtedly results in calcium contamination, as we were able to ascertain by experiment, but silver poles carefully prepared and not brought into contact with any foreign material, but polished by friction against each other, still gave the H and K lines very prominently.

10. *Conclusion.*

We point out in conclusion the principal results arrived at in the preceding investigation.

We do not consider that the numbers arrived at for the molecular velocities can lay claim to great accuracy, owing to the irregularities in the results, some of which have not quite been traced yet. But we think that we are able to say generally that metals of light atomic weight, like aluminium and magnesium, are projected from the poles with greater velocities than the heavier ones we have tried, such as zinc, cadmium and mercury. Different curvatures of different lines are marked in the case of bismuth, and are not easily explained, except by assuming the presence of different kinds of molecules having different masses, the lighter ones diffusing more quickly. We have thus established a method which is likely to prove of extreme value in separating the effects of different molecules.

Our experiments also allow us to draw another important conclusion. When two lines of a metal are of unequal intensity, it is not always due to the fact that at any period of the discharge the vibrations which appear the strongest are really the most intense. Our eye or the photographic film only perceives the total energy sent out, and the time of luminosity is in many cases very different for different lines. A vibration which is weak but persists may appear stronger than one of greater intensity which only appears for a very short time.

We have been led in addition to a new method of taking spark spectra with an induction coil, by the discovery that self-induction in the spark circuit leads to the disappearance of the air lines, which are often very troublesome in the investigation of spark spectra.

Finally, the appearance of the calcium lines in the photograph of the silver spectrum has shown the existence of a new type of spectroscopic lines, namely, one that starts in the centre of the spark and is propagated towards the poles.

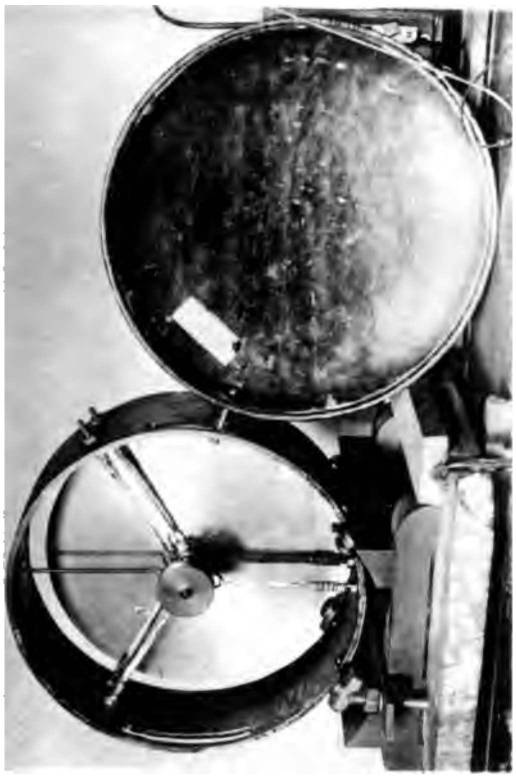


Fig. 3

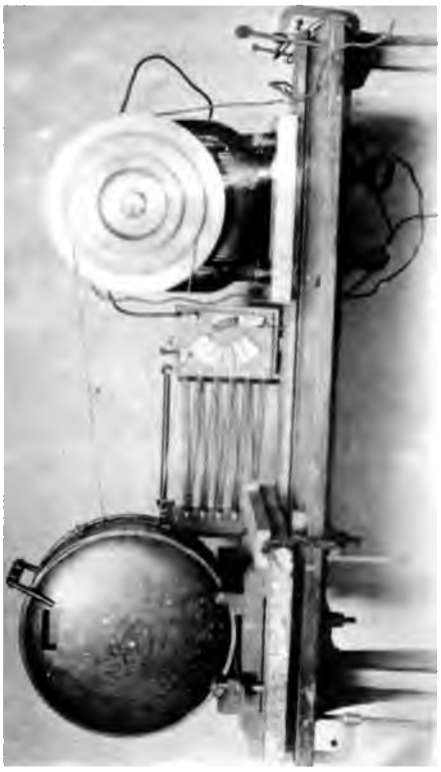


Fig. 5

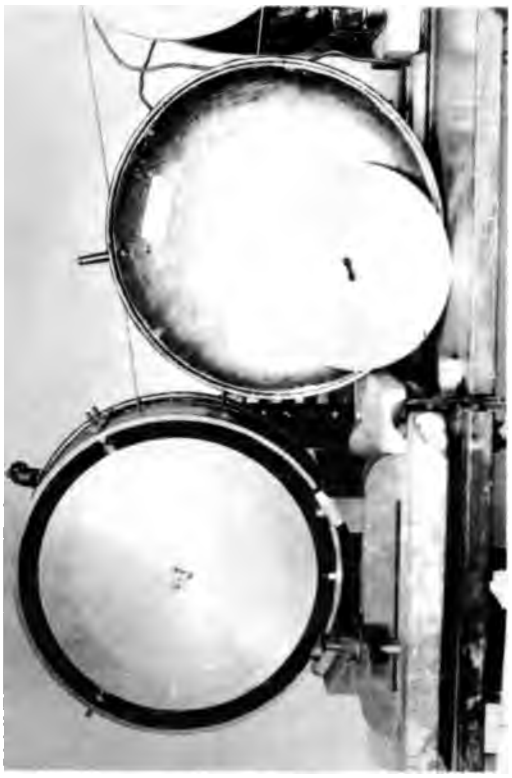
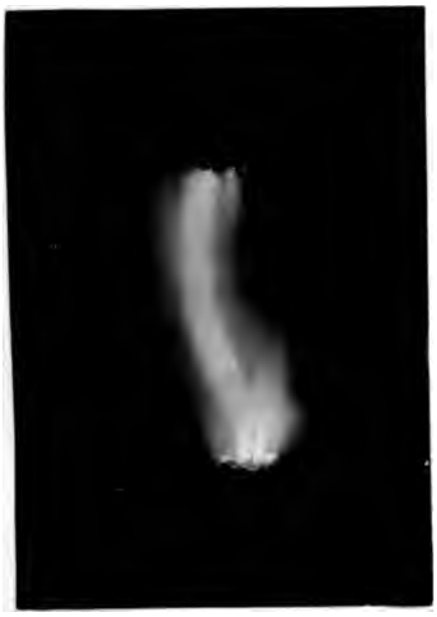


Fig. 4



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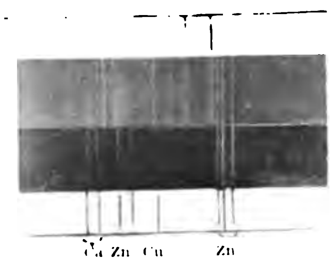

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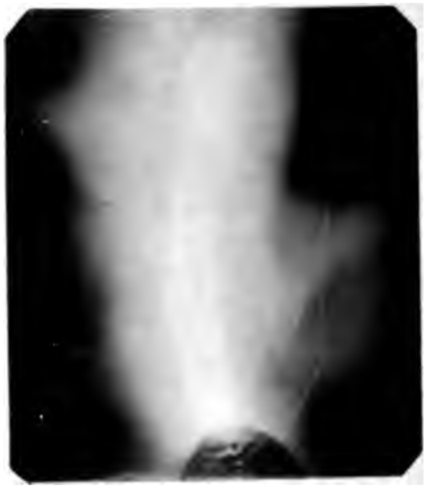

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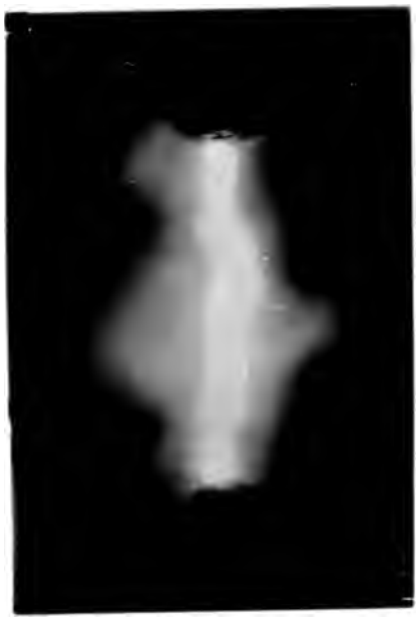

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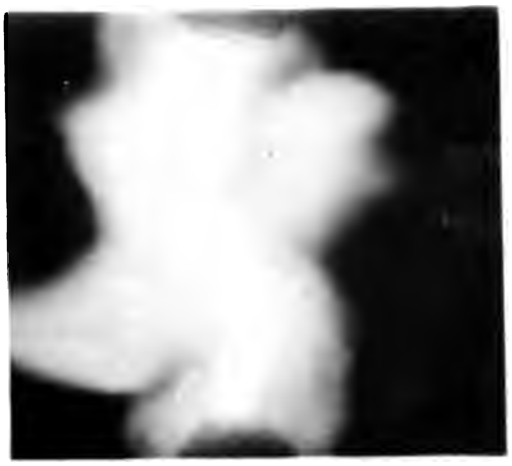

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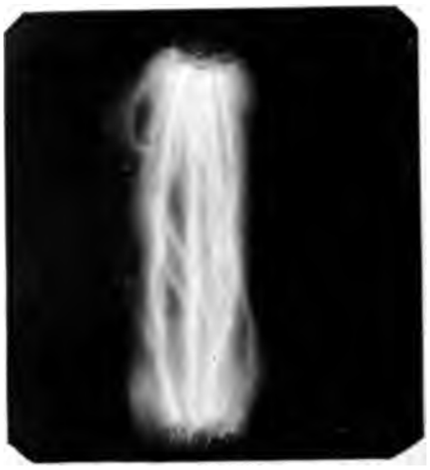

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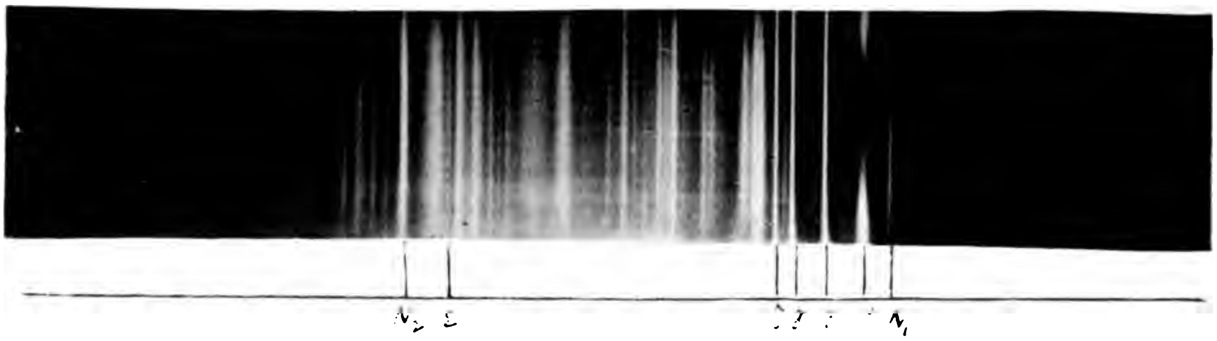

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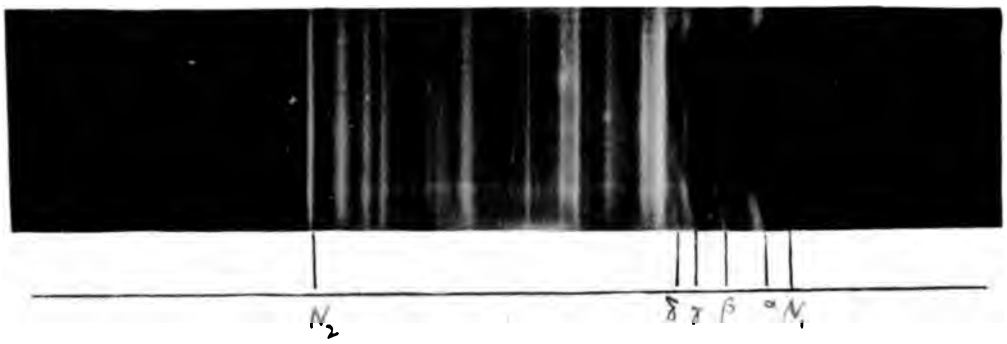

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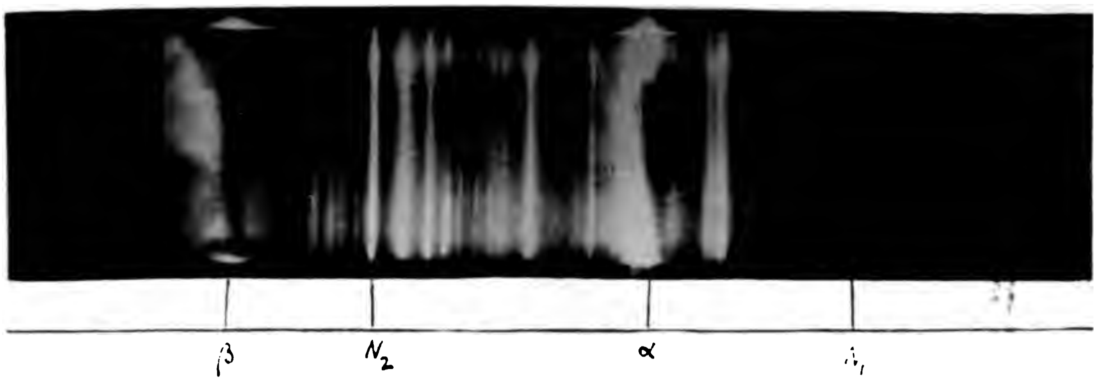
MAGNESIUM.



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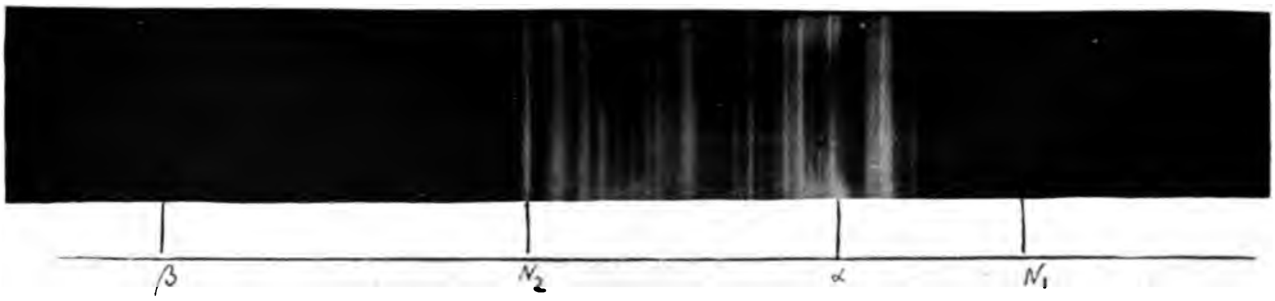
ALUMINIUM.



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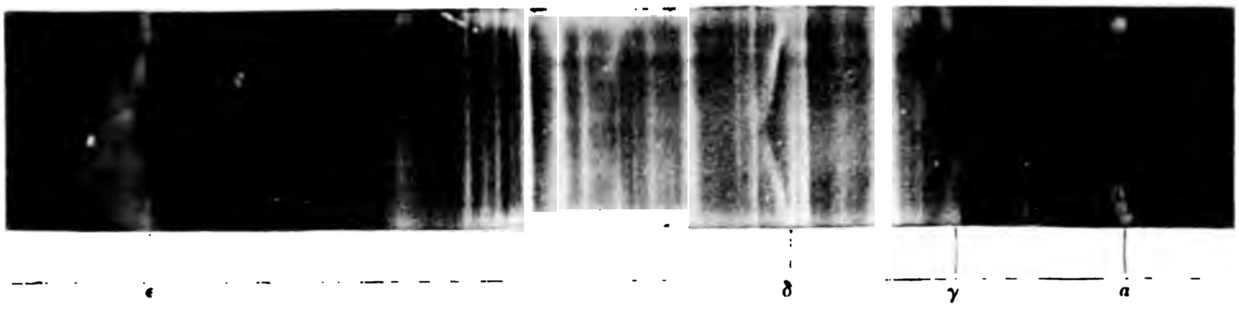

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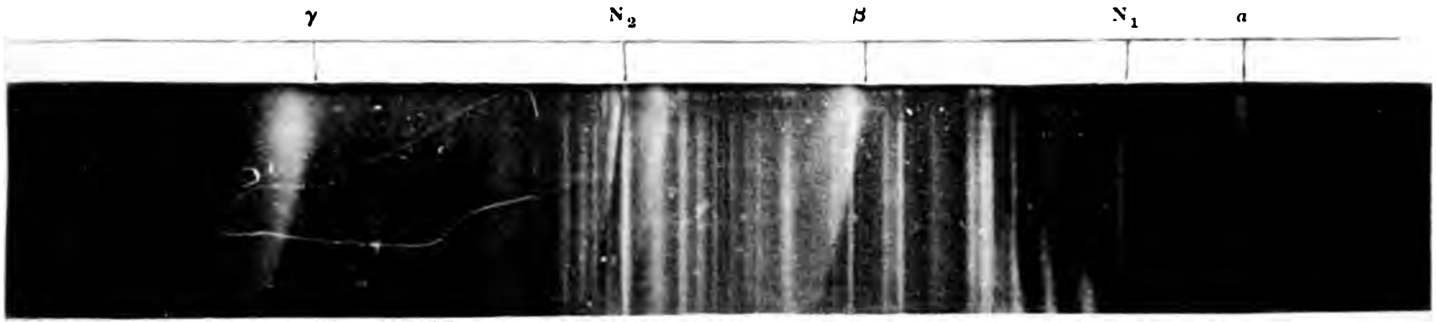

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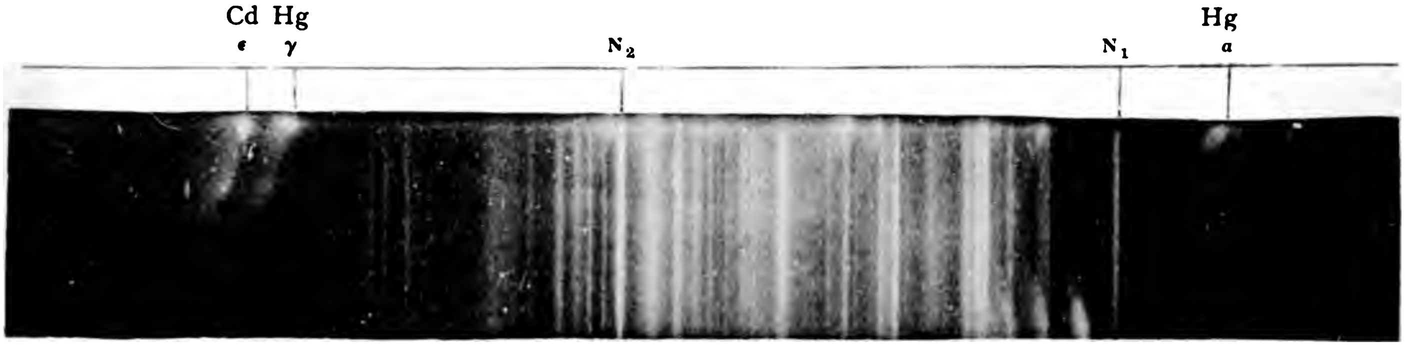
Upper Pole Mercury, Lower Pole Zinc.



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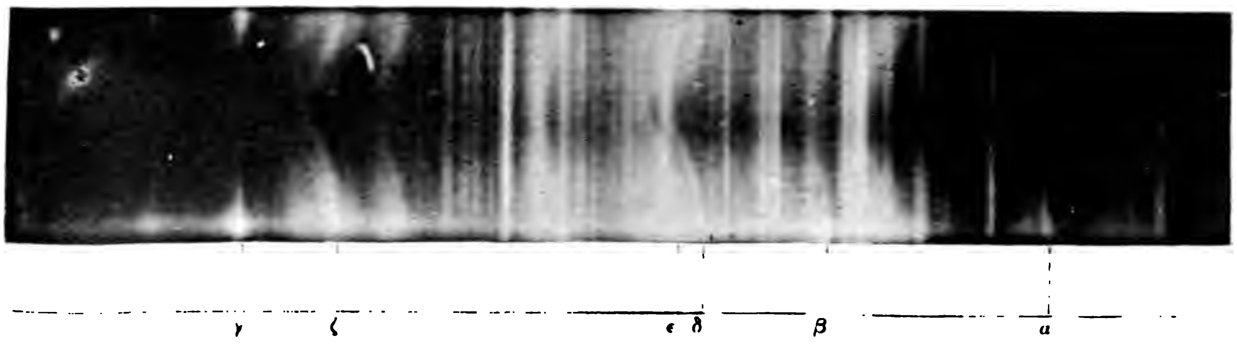

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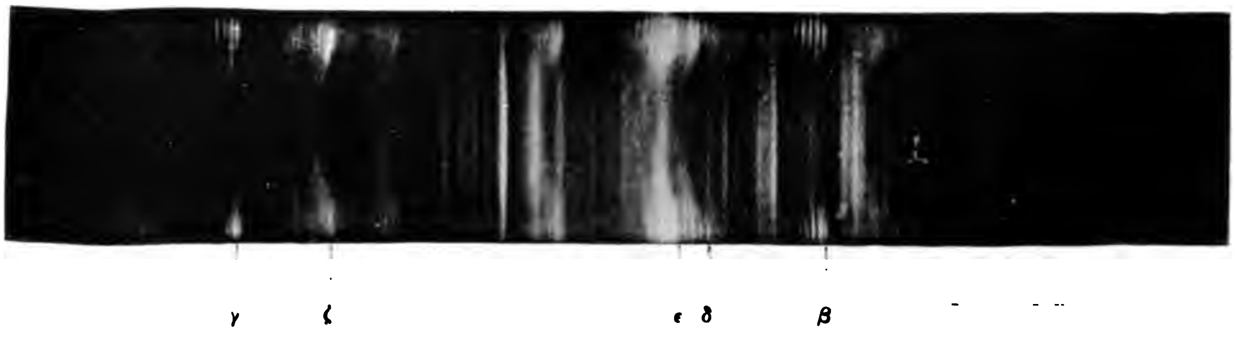

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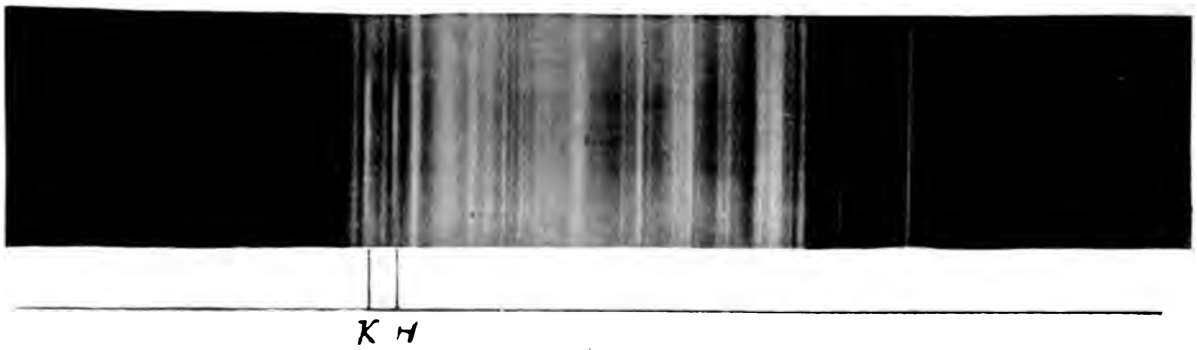

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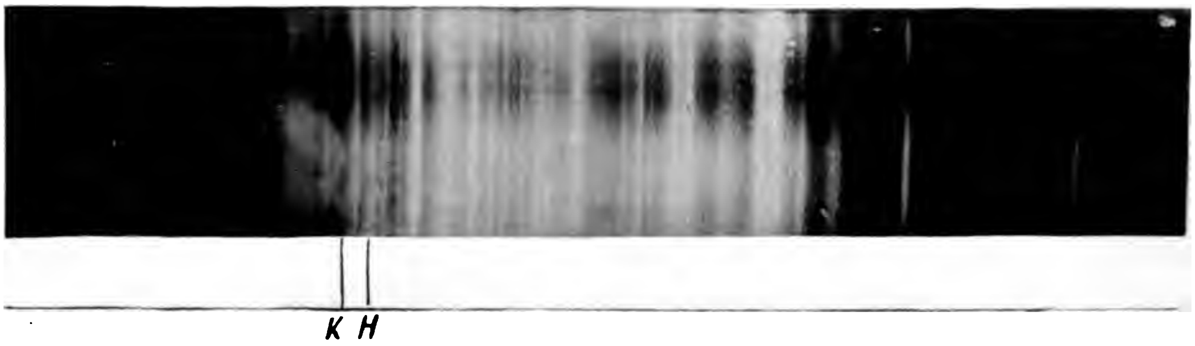

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VII. *On a Quartz Thread Gravity Balance.*

By RICHARD THRELFALL, *lately Professor of Physics in the University of Sydney, and* JAMES ARTHUR POLLOCK, *Demonstrator of Physics in the University of Sydney.*

Communicated by Professor J. J. THOMSON, *F.R.S.*

Received, April 11,—Read, April 27, 1899.

[PLATES 13, 14.]

THE problem of replacing dynamical by statical methods of studying the variations of the earth's gravitational force has long occupied the attention of astronomers and physicists, and a good many attempts have been made to construct an instrument which should enable relative measurements of gravitational force to be carried out with a smaller expenditure of time and trouble than is incidental to the observation of pendulums.

The only kind of force which is practicable as a means of opposing gravitation in the construction of such an instrument as is here contemplated, is that derived from the elastic properties of matter.

Accordingly the problem is reduced to that of constructing either a spring balance of sufficient accuracy, or of making use of the elastic properties of a gas.

In both cases, when we approach the limits of accuracy obtainable by pendulum observations, we have to face very great difficulties arising from the necessity of ascertaining the temperature of the apparatus within very narrow limits, and this obviously implies the even greater difficulty of insuring that the apparatus shall have the same temperature at every point.

If we add to this the consideration that the apparatus must be reasonably portable, and of such construction that it is not possible to disturb its mechanism by the shaking inseparable from transport, it is evident that we have to face a mechanical and physical problem of considerable difficulty.

Until Mr. BOYS discovered the unique properties of fused quartz, in 1887,* no material having elastic properties of the requisite simplicity and constancy was available, and from this cause, if from no other, all attempts at constructing a statical instrument of reasonable accuracy must necessarily have failed—as they all did. But even setting this aside, we are satisfied that all the designs—some of them of great ingenuity—which have hitherto been submitted, must necessarily have failed from the difficulty of securing in them the requisite uniformity of temperature.

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* 'Phil. Mag.,' June, 1887, p. 489.

With regard to forms of apparatus in which advantage is taken of the elastic properties of gases, it is proper to observe that, though it is possible to arrange a compensation effective to the first order of the variations due to change of temperature, yet the apparatus must almost necessarily be too large to satisfy the condition as to the temperature being sufficiently uniform throughout.

Very closely associated with the problem above discussed is that of making an instrument by which small secular variations in the gravitational force at any one place may be ascertained, though in this case the problem is very considerably simplified, because the question of portability does not arise, and arrangements may be made for minimising fluctuations of temperature. In this case, however, the observations should be of a higher order of accuracy than is necessary during a gravity survey.

As soon as we became familiar with the properties of fused quartz, it became obvious to us, as well as to others, that by taking advantage of these qualities it might be possible to construct a balance having sufficiently permanent elastic properties to give a practical solution of the problem of constructing a gravity meter.

A committee of the British Association, which in 1886 had invited designs for a gravity meter, reported in 1889 that work had been suspended pending a trial of fused quartz. Our own attempts to construct a gravity balance began in September, 1889, and have continued uninterruptedly ever since.

When we began to work at the matter we formed an impression that the problem of observing small variations in the intensity of gravity at any one place would prove simpler than that of constructing a portable instrument, and consequently we first turned our attention in this direction. We worked at instruments of what may be called the non-portable balance class for two years, at the end of which time we had satisfied ourselves that we were not likely to attain to sufficient sensitiveness, and accordingly we turned our attention to the construction of a portable instrument. We experimented in this direction for some time, with the result that in October, 1892, we began to construct what we hoped would prove to be a final form of instrument, but it was not until September, 1893, that we had got it forward enough to commence systematic observing.

With regard to portability we may say that we have travelled with the present balance over 6,000 miles. Of seven quartz threads which we have had in actual use since the balance was completed in September, 1893, only one has broken. During the first journey in February, 1894, and through the negligence of a person who had undertaken to look after it, the balance was knocked off its stand and practically destroyed; the only part which was not broken was the thread. The present thread was mounted on the 10th of September, 1896. It has travelled over 4,600 miles by cart, railway, and steamer. Observations have been taken with this thread at four stations in New South Wales, at Melbourne in Victoria, and at Hobart and Launceston in Tasmania. On November 23rd, when hurrying to catch the train at Springwood, N.S.W., one of the handles of the box containing the balance broke, and one end of the box fell through a distance of about two feet on to the asphalt platform. The reading was found to be altered when we got to Sydney, but the thread was undamaged. As the final result of the present investigation we have determined the value of "gravity" at Hornsby, a station 21 miles from Sydney and 472 feet above the laboratory, relatively to that at Sydney, in three journeys, with a maximum difference less than one part in 500,000. For the purpose of a survey the evidence shows that a single observation with the balance will enable *g* to be determined relatively to a standard value to within one part in 100,000. There is a great probability, however, that the error would be less than one part in 200,000.

Before proceeding to the discussion of the subject of this paper we desire to perform the pleasant duty of thanking those who have assisted us in one way or another, and first amongst these is Mr. JAMES COOK, F.R.A.S., our mechanical assistant. Mr. COOK made the whole of the instrument, except the thermometric appliances, including his own working drawings from our sketch designs. It is not too much to say that had we not been so fortunate as to have commanded his great mechanical skill and accuracy we should in all probability have failed in our undertaking.

To Mr. J. J. E. DURACK, Deas Thomson Scholar of the University of Sydney, and to Miss FLORENCE MARTIN, we owe our thanks for much observational assistance. Through the enlightened liberality of the Commissioners of the Railways of New South Wales we were provided with free passes over the government railways. To them, and to the Secretary for Railways, Mr. HUGH MACLACHLAN, we desire to tender our thanks, not only for our free passes, but for the unfailing kindness and courtesy which they showered upon us.

Amongst those who have assisted us with advice upon special points we desire especially to thank our colleague, Professor GURNEY, Mr. G. H. KNIBBS, Mr. E. F. J. LOVE, Mr. G. F. FLEURI, M. GUILLAUME, Mr. GRIFFITHS, and Mr. CHARD, L.S.

During our journeys we were provided with facilities for making our observations by Professor LYLE, of the University of Melbourne; Professor MCAULAY, of the University of Tasmania; Mr. ALEXANDER MORTON, Curator of the Tasmanian Museum at Hobart; Mr. W. J. BAIN, of Launceston; Sir PHILIP FYSH, (late) Premier of Tasmania, and Dr. WIGAN, Acting Mayor of Armidale in 1897.

The Surveyor-General of New South Wales also deserves our thanks for having placed a disused theodolite at our disposal, the circle of which was employed for a long time as part of our balance.

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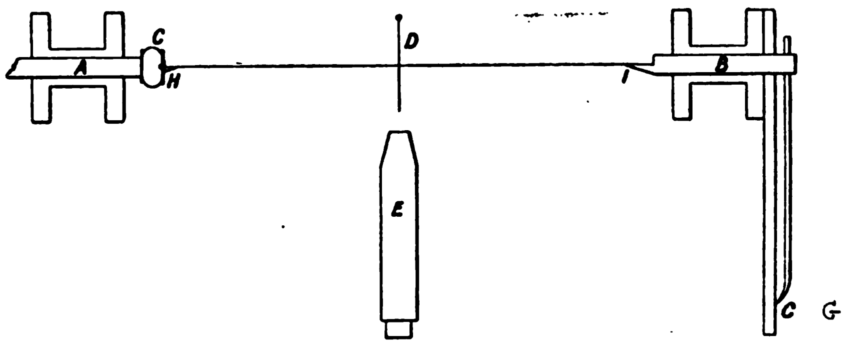
General Description of the Instrument and Preliminary Remarks.

A and B (fig. 1) are two metallic rods capable of adjustment by sliding along their common axis, but having no freedom to move transversely; in our earlier experiments we made use of the head stock and back centre of a watchmaker's lathe.

Fig. 1



C is a coach spring attached to A, and carrying a point H, to which the end of the quartz thread is fastened.

B is a brass axle capable of turning about the axial line HI, which also represents the quartz thread; it is attached rigidly to an arm carrying a vernier G. The vernier moves over a divided circular arc—a sextant being in fact employed.

The quartz thread has as nearly as possible a diameter of ·0015 inch = ·0038 centim., and is very uniform; it is soldered up to the points prepared for it at H and I.

The distance of A from B is regulated so that the thread is stretched tight by the spring at C.

The length HI is 30·5 centims.

D is a piece of gilded brass wire 5·3 centims. long, and its mass is 0·018 gram; it is attached to the thread by soldering, the thread lying in a little kink in the wire.

It is adjusted so that its centre of gravity lies a little to one side of the thread.

E is a microscope attached to the frame of the machine so as to occupy an invariable position with respect to the supports A and B.

The microscope is provided with a riding level showing 2″ of arc per division of 2·5 millims. at ordinary temperatures.

The whole apparatus is provided with levelling screws, so that the axis of collimation of the microscope can be brought into an invariable position by means of the riding level, which invariable position is very nearly or quite horizontal. Also the thread itself can be adjusted to lie in a horizontal plane by means of a subsidiary level, which is permanently attached to the frame work, and is at right angles to the riding level.

The apparatus is thermally insulated so far as possible, and a platinum wire thermometer wound upon very thin glass lies alongside the thread.

There is also an arrangement for arresting the end of the lever for security during transport.

The gravitational moment of the lever about the thread requires to be adjusted by the addition of small drops of fusible metal solder. The moment is so adjusted that about three whole turns of each end of the thread are required to keep the lever horizontal. This adjustment is made once for all; during the process of observing the thread is twisted from one end only, viz., the end attached to the sextant arm. It will be shown directly (in the section dealing with the theory of the instrument) that the equilibrium of the lever becomes unstable when its centre of gravity rises a few degrees above the horizontal plane passing through the thread.

The microscope is so arranged with respect to the lever that, when the riding level is horizontal, the image of the end of the lever lies on the cross wire; and the lever itself is almost, but not quite, in the position of instability.

By turning the vernier arm it is possible to increase or diminish the twist of the thread from I up to the lever. When the thread is increasingly twisted, the side of the lever on which is the centre of gravity rises until the position of instability is reached and the lever upsets; it is caught, however, by the arrester, and is not allowed to fall right over. When the intensity of gravitational force increases, the centre of gravity of the lever falls, and the thread has to be twisted by the vernier arm in order to bring the lever back to its former position.

The immediate subject of observation is the amount of twisting or untwisting necessary to bring the lever to its sighted position.

In order to calibrate the instrument, it is necessary to know the total twist of the thread, that the thread is uniform, or at all events of a definite shape, the position of the centre of gravity of the lever, and the exact position occupied by the lever with respect to the ends of the thread. As these quantities are not exactly ascertainable, we prefer to calibrate the instrument by observing the change of reading which occurs when it is taken from Sydney to Melbourne—the gravitational data of both of these places being sufficiently well established.

It appears from the theory below that this information, together with the vernier reading at a third station and the corresponding temperature readings at each station, give us all the data necessary for calculating the value of gravity at the third station.

The necessity of observing the temperature arises from the fact that the rigidity of fused quartz threads increases with increase of temperature. The effect is, of course, complicated by the changes which take place in the dimensions of the threads. The nett result is that as the temperature rises the "stiffness" of the threads increases.*

The dimensions of the lever also change, and in such a manner as to partially compensate the changes of "stiffness" taking place in the thread. In the actual instrument the resilience as well as the figure of the coach spring vary with temperature, the line of collimation of the microscope may also vary relatively to the level, and the metal framework may also twist and vary in shape in any manner. All these effects are found to be capable of collection into a single temperature coefficient—which, so far, we have found to be sufficiently well expressed by a single term—the temperature scale adopted being the platinum scale of CALLENDER.

Many attempts were made to annul the effect of variation of temperature by constructing levers composed of two bars of metals of different expansibilities, an account of which will be found in the appendix, but all such attempts came to nothing.

The sensitiveness of the instrument is at present such that to compensate for a change of one part in 100,000 in the value of g it would be necessary to move the vernier arm through 2·12 sextant (doubled) minutes = 1·060 minutes of arc. Also a change of temperature of one-tenth of a degree alters the circle reading by 3·15 sextant minutes, gravity being invariable.

Although quartz is immeasurably superior to any other known material as regards the constancy of its elastic properties, it must not be supposed that it is entirely free from elastic defects. As a matter of fact, it is only just good enough for the present purpose. One of the great difficulties we have had to overcome has arisen from the fact that it is only by the most judicious choice of dimensions that it is possible to reduce the viscous yielding of the thread to within practicable limits, and even as it is, although the thread of the present instrument has been twisted for more than two years, and indeed purposely overtwisted for part of that time, we are still obliged to apply an important though practically constant correction on this account. In other words, the reading of the sextant arc is still slowly decreasing, and though the rate of decrease is now constant for all practical purposes, it has to be taken strictly into account in interpreting the indications of the instrument, even when consecutive observations are separated by an interval of time as short as a single day.

It will, no doubt, occur to the reader that we ought to use a finer thread, in order to get rid of this source of inconvenience. This, however, experience has shown us, we are not at liberty to do, for a finer thread means either a finer lever or one soldered to the thread at a point nearer to its own centre of gravity. The former solution is inconsistent with the necessity of having the lever strong enough to bear arresting without change of form, and the latter has the drawback that it tends to magnify the effect of such variations as do actually occur in the shape of the lever. In short, the design of a gravity balance becomes a matter of compromise just like any other engineering undertaking. In calculating the moment of the gravitational forces brought to bear upon the thread by the weight of the lever, it must not be forgotten that the effective weight is reduced in consequence of the flotation of the lever by the air surrounding it. At an accuracy of one part in a hundred thousand in the estimation of g the effect of variations of barometric pressure or humidity would become sensible, and it is therefore necessary to protect the lever against variations of air density. This is done by enclosing the whole apparatus in an air-tight casing within which the air is kept at a constant density. In practice the density selected corresponds to a pressure slightly below the minimum external pressure of the air. The apparatus must also clearly be filled with dry air to avoid the deposit of dew. As a consequence, the twisting of the thread necessary to afford a reading of the instrument, has to be carried out by means of a shaft working through a stuffing box; and the requisite accuracy in the estimation of the twist given to the thread makes it necessary that the rod shall work almost without friction in the stuffing box.

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* THRELFALL, 'Phil. Mag.,' July, 1890.

We have devised a sort of mercury and tallow joint which has got over the difficulty fairly well; and which is also applied to the stuffing box through which the rod of the arrester works.

An aneroid in connection with the internal space enables any leakage to be detected.

The constructional difficulties above indicated were further increased by our determination to have no iron or steel work about the machine: this was due to fear of magnetic interference; with which, however, we have never been troubled.

The degree of portability attained will be understood from the following statement of the sizes and weights of the various appliances requisite for an observation of gravity, as, for instance, during a survey when everything must be provided.

	Length.	Breadth.	Depth.	Weight.
	ft. ins.	ins.	ins.	lbs.
Box containing balance	2 9½	19	15¼	106
Box of resistance coils	2 4½	10	10	23
Box of accessories	2 1½	17½	11½	82
Legs of tripod stand	...	...	...	15
		Total weight		226

The above weights and dimensions refer to a balance of gun-metal and copper, and accessories taken just as they came to hand from the laboratory. Both weights and dimensions might be greatly reduced if so desired. The weights certainly might be easily halved.

It is necessary to observe either upon a stone, cement, earthen or asphalte floor. Boards, however strong, do not form a sufficiently inelastic support.

INSTRUMENTAL DETAILS.

The drawings on Plate 13 are to scale, and ought to be self explanatory. Notes, however, have been added on the page facing the drawings in order to make it easier to follow the latter. It will be seen that the general principle of construction is as follows:—All the essential parts are rigidly held together by a system of bars forming a complete mechanism. This is then thrust into a tube of copper, which it fits precisely, and the tube of copper is further surrounded by a packing of paper to insure some degree of thermal insulation. The paper in its turn is surrounded by an outer tube of brass, and this is held down to the base by brass clips. The copper tube is closed at one end by the brass cover carrying the vernier arm and mercury stuffing box, which constitute the circle end of the machine. At the other end there is a smaller brass cover, through which projects the end of the arrester shaft and the end of the thermometer; or, rather, the thick terminals and the end of the ebonite shank on which the glass tube is mounted. The aneroid tube also passes to the interior of the apparatus through this end.

The copper tube has two openings at opposite ends of a horizontal diameter, and into these openings are fastened, at one side a window, at the other the microscope focussed upon the end of the lever.

The under frame is provided with levelling screws so spaced as to fit the grooves of the top of a Kew magnetometer tripod. The instrument having been reconstructed several times does not now occupy the best position on the under frame, and this has necessitated the addition of a heavy lead counterpoise. Without this counterpoise the weight carried by the back levelling screw is too great to permit of the screw being turned with sufficient ease for exact levelling. The counterpoise is not shown in the drawing, but appears in the photograph (see Plate 14).

Attachment of the inner mechanism to the copper tube.—The bars connecting the bearing (which supports the "spring" end of the thread) to the bearing which carries the vernier arm are of gun-metal, and the bearing which carries the spring fits with considerable accuracy into the copper tube, carefully bored for the purpose. During our earlier observations the spring bearing was quite free to slide up or down the copper tube according as the temperature rose or fell. After one of our journeys, however, we thought that there was some evidence of the spring end of the thread having moved, and it appeared possible to connect this motion with the freedom of the spring bearing. Accordingly at present the bearing is wedged tight to the copper tube.

Stuffing Boxes.

The idea was to render the brass work incapable of being amalgamated by plating it with nickel, which for ordinary purposes may be regarded as incapable of amalgamation. The fit of the shafts both of the vernier arm and of the arrester is made as good as possible without being at all stiff. The joints between the shafts and the cylindrical holes in which they work are further filled in with tallow. Outside a cup-shaped depression is arranged, and this is filled with mercury. As there is a partial vacuum inside the apparatus, the tendency is for the mercury to drive the tallow through the joint and then to follow it. The joint is very fine, however, and the surface tension of the mercury tends to oppose this motion. The practical result is that we obtain a smoothly working motion and a sufficient air tightness so far as inwardly directed air pressures are concerned.

Microscope.

The microscope is supported on a flange soldered to the thick copper tube. There is also a flange on the tube within which the microscope is fixed, and the adjustments of the microscope are partly made by sliding one flange over the other, the clamping screws which press the opposing surfaces together being passed purposely through holes much larger than the screw diameters. The flanges are ground to fit, and are put together with a little tallow for the sake of securing air tightness.

The microscope itself is an ordinary Zeiss microscope tube furnished with the "A" objective, giving a magnification of about 100 diameters. The adjustments are made as follows. It is required to adjust the microscope so that the image of the end of the lever will be sharply in focus, and bisect the cross wire when it is a small definite amount below its position of instability. For the purpose of arriving at this adjustment the object-glass is first soldered by its screw mounting to a length of brass tube which fits inside the stronger tube very well. The outer tube is attached by soldering and screwing to the flange, and carries the eye-piece itself mounted in a short length of tube in which the cross wires are fixed. The eye-piece is focussed on the cross wires as usual, the object-glass is placed in position, and the outer tube is levelled by means of the riding level. The focussing is then accomplished by sliding the object-glass tube in or out, and when the desired adjustment has been obtained, the flange is unclamped and the object-glass tube sweated in position with tinman's solder, by which means it becomes rigidly attached to the outer tube. The outer lenses of the object glass are also fixed in their mountings by means of wax, but the workmanship of the cell is so good that the wax acts merely as a means of preventing the leakage of air. No doubt some change in the position of the axis of collimation occurs as the temperature rises. This will do no harm if the position of the axis of collimation can be regarded as a function of the temperature, but the chance of irregularities must be put up with so long as brass mounts are used: with platinum a greater degree of certainty is to be expected. It is fair to say, however, that we have no reason to suspect the axis of collimation of movement rather than any other part of the apparatus. When the preliminary focussing has been accomplished, and the soldering satisfactorily finished, the microscope is again mounted and levelled, and the final adjustment made so that the lever upsets when its pointed end rises above the cross wire by one diameter of the point. It generally happens that the focus is not quite correct, but this can now be set right by means of the eye-piece. Finally the eye piece (which is waxed and otherwise fastened to a tube several inches long sliding very tightly inside the outer tube) is finally waxed air tight and firm. All the parts of the microscope are now firmly fixed to the outer tube, and the whole is correctly focussed upon the end of the lever when the latter is in position. It remains to explain how the outer tube of the microscope is prepared. It must be remembered that the outer tube carries the riding level, and that the whole theory rests upon the assumption that the axis of collimation can be brought into an invariable position with respect to the horizontal. In order to avoid spending too much time over levelling the instrument when it is in actual process of observation, it is of some assistance to have the tube of the microscope and the V-grooves of the riding level so perfect that, in ordinary phraseology, the level will reverse. The microscope tube was ground and lapped by us on one of BROWN and SHARPE'S cutter grinders, which are not recommended by the makers for producing a cylindrical surface. However, by applying care and attention and rotating the tube between centres we have succeeded in making a tube so cylindrical that we are unable by any of our appliances to detect any defect of form. We have to thank the mechanical assistant of the laboratory, Mr. JAMES COOK, for making the V-grooves so perfect that the sensitive level we employ does in fact reverse when mounted on the microscope tube.

The riding level is a very substantial affair, the frame is of brass cut away as much as possible for the sake of lightness. A cross level is attached to the frame by adjusting screws. The main level is mounted in a copper tube (see p. 232).

Quartz Thread and Lever and Attachments.

This being the essential part of the instrument, requires considerable care both in its design and its manufacture. The general methods have been described by BOYS,* also THRELFALL;† but it will suffice for the present purpose to refer the reader to a book on 'Laboratory Arts,'‡ Sections 80 to 91.

We have already referred to the fact that it is necessary to attend to the dimensions of the thread employed, and will not repeat ourselves except to point out that to arrive at the best relative dimensions as the result of a compromise necessarily implies an immense amount of experimenting. At one time the quartz behaved so badly that we actually took the trouble to assure ourselves by trial of its superiority over other materials. We mounted a very fine steel wire, kindly sent to us by Mr. ELLERY of the Melbourne Observatory, on another balance, and observed that its rate of viscous subsidence was about a hundred times greater than we were accustomed to in the case of quartz. We repeated the experiment with a very fine platinum wire with a similar result.

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* 'Phil. Mag.,' June, 1887, p. 489. 'Journal of the Society of Arts,' 1889. 'Journal of the Physical Society,' 1894. 'Phil. Mag.,' 1894, vol. 37, p. 463.

† 'Phil. Mag.,' July, 1890.

‡ THRELFALL, Macmillan, 1898.

It must not be supposed that all fused quartz, as derived from clear rock crystal, has the same properties. Almost every crystal examined by us contains both sodium and lithium—the latter in large spectroscopic quantity—indeed, we first noticed it from the colour it gave to the blowpipe flame. The observation of the almost universal presence of lithium in quartz was first made by TEGETMEIER,* a fact of which we were ignorant when we made the observation. There also appears to us to be a distinct though small difference in the viscosity of various samples of fused quartz, and this independently of the sodium or lithium they may contain. It has been our practice to select the most infusible quartz independently of the amount of lithium it contains, for lithium seems to be burned out by continued heating in the oxy-gas flame: whether this is really the case or whether the lithium forms a compound which does not give the flame re-action, we have not attempted to inquire. We have aimed at securing the greatest possible uniformity in the thread and a mean diameter of about ·0038 centim. We do not pass a thread unless the diameter is uniform from end to end within the limits of observation (exceeding those of measurement), looking at samples taken from each end of the thread through a microscope magnifying 100 diameters. We have also spent a great deal of time in trying to make certain that there were no drawn-out air bubbles in the thread, and though we have succeeded in arriving at the satisfaction of both these conditions simultaneously, still our present practice is to ignore very small bubbles, and to direct our attention principally to obtaining uniformity. Since, however, we now adjust the torsion of one end of the thread only, we are inclined to think that for the future we would put up with a slight taper in the thread, provided of course that the twisting for adjustment was done from the thinner end.

The bow-and-arrow method of Mr. BOYS gives better threads for our purpose than the catapult method ('Laboratory Arts'), which does not lend itself to the production of threads of great uniformity. There may also be some difference of tempering, owing to the different rates at which the threads are pulled out and cooled in the two methods considered. A great deal of time may be spent in preparing the thread, for at present we have no method of predetermining its diameter. All that we can do is to be guided by experience, and shoot threads till one is got satisfying the conditions. The process is so uncertain that we have on occasion got a thread within a few days, and on others we have spent a fortnight over it. The thread which forms part of the instrument to which the observations refer took a fortnight's continuous shooting. The best way of carrying out the examination is to draw the thread, comparing it all along its length with a bit broken from one end laid beside it under the cover slip of a microscope slide, the space between the slide and cover slip being filled with stained glycerine.

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* 'Wied. Ann.,' 41, p. 19, 1890.

When a sufficiently perfect thread is obtained, it is silvered all over, as described in 'Laboratory Arts,' p. 222, and the ends are then coppered and soldered up to the supports. The centre of the thread is then coppered for about a centimetre of its length. The lever is adjusted in position, being held tight by a temporary clamp mounted from the back girder. It is then soldered up to the thread by tinman's solder. The excess of silver is removed with nitric acid, and the thread is well washed. It is surprising how difficult it is to do this thoroughly; we use a brush made of glass, but we are not sure that it is the best way.

Lever.

We have made levers of many metals, but at present use one of gilded brass wire of the smallest diameter we could get with our draw-plates, say ·005 inch. The conditions to be satisfied are (1) magnetic indifference, (2) undeformability by the arrester, (3) lightness. At first we used to make levers of aluminium foil, shaped like a cross; we then tried soft annealed copper, in the hope of doing away with secular changes of shape, but it was not stiff enough to resist the arresting.

Adjustments of the Lever.—Having decided to use three turns in each half of the thread, it is necessary to adjust the lever so that the line joining the centre of gravity of the lever to the thread is nearly horizontal when the thread is twisted with this amount of twist.

It is shown in the part of this paper dealing with the theory of the instrument that with this twist in the thread the lever becomes unstable when the line joining its centre of gravity to the thread rises about three degrees above the horizontal. We have adjusted the lever in the present case so that it upsets when the thread is twisted by about three whole turns. The adjustment is made by weighting the lever with a small drop of fusible metal, rather larger than a pin's head.

In making a new machine we think it would be worth while to make the lever out of a thickish needle of fused quartz. We have occasionally thought that the thread was slipping in its silvering, but though there is a distinct danger of this happening, we have not had a really clear case. Some attempts to guard against the possibility of this, by grinding the thread flat on one side, failed on account of the way in which the thread was weakened.

The Spring and its Adjustments.

It is almost if not quite essential that one end of the quartz fibre shall be carried by a spring, otherwise the difficulty of manipulation becomes intolerable. It is also clear that the spring must satisfy some peculiar conditions. The thread must be stretched by a constant, or nearly constant force, and the spring, while freely allowing this to occur, must be incapable of moving so that the point of attachment of the thread shall become displaced transversely. The spring must also be subject to so much damping that it will not vibrate during transport, otherwise the rate of subsidence of the thread may be affected and errors introduced. We have attempted to meet these conditions in the following manner. The spring, as shown in the drawings (Plate 13), consists of a pair of elliptical springs crossing each other at right angles, the point of support for the thread being at the centre of the crossing. The point to which the thread is attached is very light and might be lighter with advantage, so as to increase the effect of the damping.

There are four bars of spring steel (watch main-spring, in fact), which are attached to the plate carrying the coach springs so as to stand up perpendicular to the plate. The free ends of these springs are attached by wire links to the thread support. The object of this disposition is to damp down the free transverse vibrations of the support. For a long time this formed the complete apparatus, which will be referred to hereafter as the "rosette" spring.

In October, 1898, however, we had reason to believe that some small residual irregularities were traceable to a transverse movement of the thread attachment. We therefore put on three stays, made of very fine glass hairs, so as to hold the support to the three girders of the main framework. The stays were cemented in position by paraffin, and the wire linkages were also cemented at their points of contact by means of paraffin. This has been completely successful, as the observations will show. We are, however, of opinion that we could improve this part of the apparatus still further, for paraffin is not an ideal cement. In fastening the thread up to the support, and afterwards in stretching the thread, it is very convenient to have a slow adjustment. This is supplied by fastening the spring system on the end of a bar; the bar is passed through the end bearing and is adjusted by means of a nut working on a thread cut on the bar. The proper tension of the thread having been arrived at, the nut and bar are soldered up to the end bearing. In view of the great difficulty of preparing a good thread, it is well worth while to provide every possible convenience for adjustment, and for minimising the risk of breakage.

Thermometry.

We believe that we wasted some time in endeavouring to obtain consistent results with mercury-in-glass thermometers. We were very unwilling to adopt platinum thermometry on account of the loss of portability which is its inevitable concomitant. Through the kindness of Mr. GRIFFITHS we obtained three of TONNELOT'S thermometers, which were studied very carefully by M. GUILLAUME before they were sent to Mr. GRIFFITHS. They were again studied by Mr. GRIFFITHS before they were sent out to us, with the result that he wrote to say he believed they were the three finest thermometers ever made. We mention this to show that we really went to some trouble in the matter, and did not take to platinum thermometry until it was actually forced upon us. We desire to thank M. GUILLAUME and Mr. GRIFFITHS for the trouble they took in the matter. The reason why mercury-in-glass thermometers fail for our purpose is that they are too slow when of the necessary sensitiveness, that they do not give the mean temperature of the thread, and that the zero corrections present difficulties in field work. We are glad to acknowledge that it was M. GUILLAUME'S opinion from the first that platinum thermometry would be the most suitable for us. The instrument we at present employ is made simply according to the instructions of Mr. CALLENDER,* but we have made one or two small modifications in the resistance box which was constructed to work it. The platinum wire, of the diameter recommended by Mr. CALLENDER, is wound in a double spiral on a very thin glass tube, 1 centim. in diameter. The tube is mounted on an ebonite plug through which the leads pass, and which serves to screw it in to the inner tube of the balance. The tube is unsupported, except by the plug, and is arranged to lie parallel to the thread, and at a distance of 2 centims. more or less below it, *i.e.*, 2 centims. from thread to centre of tube.

When we first adopted the platinum thermometer we were of the opinion that the temperature of the thermometer would lag behind that of the thread, but experience has shown that the opposite is the case. When the temperature is rising to a maximum the maximum is always reached and passed some few minutes before the lever reaches its highest point.

The apparent slowness of the thread in taking up the temperature of the surrounding space may be due to the following circumstance. Though we are not in possession of the complete mathematical theory of a stretched, weighted, and twisted thread, we have found by experience that an increase of tension of the thread, as by tightening the rosette spring, acts in the same way as an increase in the twist of the thread. Consequently, if the spring takes some time to reach the surrounding temperature, its effect will be the same as if the temperature of the thread itself were lagging, at all events so far as the observer at the microscope is concerned. This appears as follows. The action of a rise of temperature on the tension of the spring is complex. The expansion of the bars increases the tension on the thread; the decrease in the elastic forces of the spring produces an opposite effect. On the whole the former is probably the larger effect and therefore predominates. Consequently, the rise of temperature will stretch the thread; this will act like an increase of twist, hence less twist will be required to keep the lever in its sighted position, but this is exactly the effect produced by the temperature increase in the rigidity of the thread.

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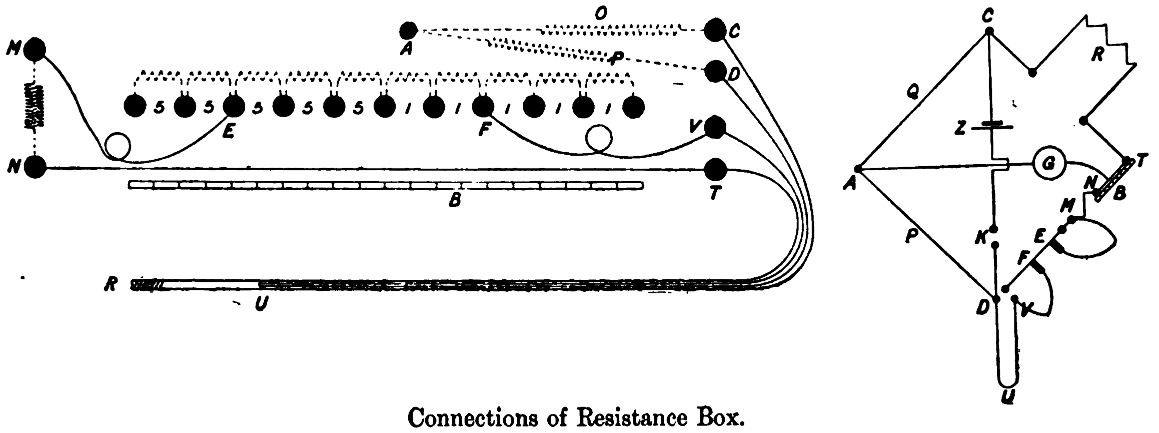
* 'Phil. Mag.,' July, 1891.

Resistance Box.

The resistance standards are made of manganin wire according to St. Lindeck's indications.* After the box had been in use for a year it was recalibrated and found to have changed slightly, sufficiently to affect estimates of temperature supposed accurate to ·002 centim. degrees. The arrangement will be seen from the diagram (fig. 2). We have five coils each of one "box unit," in our case 0·1166725 "legal ohm." We have also five coils five times as great, *i.e.*, equal to five box units of resistance each. The box unit is a resistance equal to the one-hundredth of the change of resistance of the platinum thermometer, when its temperature is changed from 0° C. to 100° C. The manganin bridge wire lies over a glass scale of equal parts which is graduated over a length corresponding to a resistance of one box unit. This

Fig. 2.



Connections of Resistance Box.

A. The point of junction of the two equal arms P and Q.
B. The contact of the galvanometer connection to the bridge wire.
C. The junction of the resistance, Q, to the flexible lead of the thermometer.
D. The junction of the resistance, P, to the flexible "dummy" lead for compensating the resistance of the thermometer leads.
E. The connection of the short flexible lead to one end of the variable resistance.
F. The connection of the other short flexible lead to the other end of the variable resistance.
G. The galvanometer.
K. The battery key.
M. One end of the coil balancing the thermometer resistance at 5° C.
N. The other end of the coil.
P. One of the equal arms.
Q. The other equal arm.
R. The platinum wire resistance, constituting the thermometer.
T. The junction of one of the flexible leads of the thermometer and the divided bridge wire.
U. The end of the loop of the "dummy" lead.
V. Junction of the "dummy" lead and the short flexible lead.
Z. The battery.

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* 'B. A. Reports,' 1892, p. 139.

length is divided into two hundred parts, a movement of the slide over one part thus corresponding to a change of temperature of one-hundredth of a degree. Each division is 1·73 millim. long. In order to avoid the difficulty which arises from the alteration of resistance of plug contacts, we have made a slight innovation. We have placed all the coils in series, and between each of them we have brought a stout clamping terminal to the top of the box. Connection is made to the coils which it is desired to use by means of stout terminals attached to flexible leads. The diagram shows how the resistance of these leads is compensated. The "dummy" leads used to compensate the thermometer leads are made too short by the length of the flexible leads, which are, of course, made of the same wire as the dummy leads. Assuming that the temperatures of the dummy leads and of the flexible leads are the same, this gives a perfect compensation.

The advantage of choosing the particular value of the box unit above mentioned lies in the resulting fact that, setting aside corrections, the box is direct reading; the disadvantage is that with any other thermometers the box would lead to inconvenient arithmetic. The apparatus is only intended to measure temperatures between 5° C. and 35° C., consequently the large coil of the box has the same resistance as the thermometer at 5° C. The contact on the bridge wire is also of manganin, to reduce thermoelectric action, and there is no difficulty in subdividing the bridge divisions to tenths by estimation, *i.e.*, in reading to one-thousandth of a degree. We use one of AYRTON and MATHER's D'Arsonval galvanometers and find it very sensitive and portable.

The following constants will end our discussion of this matter :—

	Legal ohms.
Resistance of platinum thermometer at 100 C.	45·99173
" " " 0° C.	34·32448
Difference	11·66725
Resistance of equal arms	10·027

Testing Resistances.

(1.) The resistance of each of the box units was ascertained in terms of the resistance of one division of the bridge wire, the wire having been previously tested for uniformity in the usual way.

(2.) The same process was gone through with the "fives."

(3.) The five box units in series were tested against each of the "fives" in succession.

(4.) The value of the large coil was ascertained in terms of the platinum thermometer at 0° C. through the intermediary of another box.

We do not desire to dwell on this part of our work, as it is foreign to the main object of this paper, but we must state that the comparisons were made with care and repeated six times. From the comparisons a table was prepared showing the corrections to be applied at each point of the scale. The arrangement of coils adopted allows us to measure each temperature in two ways, and at every five degrees in three ways; this provides a valuable check against large errors or mistakes.

It may possibly be considered that as we have to refer to a table of corrections in any case, we wasted time in adjusting the coils with the nicety we employed. This is not the case, however, for in the process of observing it is necessary to make many approximate estimates of temperature as well as the final maximum, and the uncorrected direct readings are very useful for this purpose. We must acknowledge the assistance we received from a paper by Mr. GRIFFITHS in 'Nature,'* in which the construction of a box of coils and a standard thermometer is described.

We usually observe by closing the galvanometer circuit before we close the battery circuit. A special experiment showed that this was legitimate with our inductionless coils and thermometer, and thus we are not troubled by residual thermoelectric effects; rendered small in any case by the design of the contact maker.

The Arrester.

The arrangement made for arresting the lever is in reality more simple than the description would seem to indicate. We have a framework adjustable to the bars of the main supports by means of clamp screws. This framework carries two jaws worked up and down by means of a pinion with teeth cut by an involute cutter. The rack teeth are therefore triangular, the tops being cut off; it is necessary to have a good smooth motion. The lower jaw of the arrester is convex so that the lever is held against a convex surface. The upper jaw is wedge-shaped, and presses the lever against the convex surface of the lower jaw. This arrangement was adopted with a view to preventing the lever getting bent by the arrester. The active surfaces of the jaws are connected to the racks through the intermediary of spiral springs, and the upper jaw has a stronger spring than the lower jaw. By a simple mechanical arrangement the lever is always held by the same pressural force, and this depends on the elasticity of the springs, and not on the exact angle through which the arrester shaft may have been turned. The arrester frame is adjusted until the lever is practically arrested in its observing position and is not displaced out of the field of view of the microscope. What happens is this. As the jaws close the upper jaw reaches a fixed position. On further turning the arrester handle the spring of the jaw gets compressed but the jaw remains fixed. Then the lower jaw closes on the upper one, and on further turning its spring gets compressed. An essential part of this construction is that the spring of the lower jaw must be weaker than that of the upper one. The arrester shaft of course works through a Hook's joint. Outside the thermometer end of the machine a milled head is placed, and this works a key which passes through a stuffing box and fits on the end of the arrester shaft. The milled head may be clamped by a clamp screw on the end of the balance case, and thus we can make sure that the arrester does not work loose during transport. The thread being quartz in a perfectly dry atmosphere will of course insulate perfectly. We have a check on the possible electrification of the lever because we can set it swinging so as to touch the open jaws, and we can then observe whether any change in the reading occurs. We have often suspected that we were troubled by electrostatic effects but we have no conclusive evidence that this was really the case.

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* November 14, 1895.

The whole of the inside of the balance is blacked so as to increase the surface conductivity, and so enable the temperature to equalise itself with the greatest possible facility.

In fitting the instrument together the surfaces are coated with soft wax before being screwed up. This makes all the connections air-tight. The last operation after putting the machine together is to exhaust it down to about 24 inches of mercury pressure, and then admit air dried by sulphuric acid and phosphorus pentoxide. The air is also passed over potash and through a filter of cotton wool, the object being to have dry dust-free air in the balance case.

The process of exhaustion and readmission of air is repeated many times; in making a new instrument we would have two openings into the balance and draw dry air through it instead of exhausting and readmitting air. Finally, the balance is left with air at a pressure of about 25 inches of mercury, but, of course, the actual pressure selected depends on the temperature of the balance at the time; a temperature of about 25° C. would be suitable for the pressure named. It is, of course, our object to have such a degree of exhaustion that there will always be an inward pressure wherever the balance is taken, and yet not to have a greater pressure difference than is necessary. A survey in mountainous regions would naturally require a higher degree of exhaustion.

Mounting the Riding Level.

The mounting of a sensitive bubble tube, such as is employed in our riding level, is a matter calling for attention. We have availed ourselves of the experience of our friend, Mr. G. Knibbs, and, acting on his advice, have mounted the glass tube in a stout tube of copper, and packed it in position by means of glass wool. The packing is tight enough to put any displacement of the bubble tube out of the question, and yet the tubes are free to follow their own tendencies in the matter of expansion or contraction. We have proved by experience that a level mounted on the tube of the balance case is not sufficiently to be relied upon, and this at least suggests that some of the irregularities we have observed may be due to warping of the metal work under the influence of changes of temperature. In the event of such warping exceeding the elastic limits of the material, we should have permanent sets which would be most noticeable after extreme temperature changes. When we come to discuss the observations we shall see that there appears to have been some effect of the kind.

Thermal Insulation.

It has already been mentioned that the copper tube, which forms the inner case of the balance, is surrounded by a packing of paper, about half-an-inch thick, over which the outer brass tube fits. It is clear that heat is able to enter or leave through the metallic ends, the sextant part being especially difficult to insulate. We, therefore, put the whole affair into a box of thin copper sheet and packed the interspace with cotton wool. It is, however, obvious that the sextant arm and the microscope must project, and, consequently, it is quite a question whether it is really worth while to insulate the other part so carefully as we have tried to do.

Packing and Transport.

The balance in its copper box is lifted by its movable handles on to a tray provided with mild steel handles, extending upwards above the top of the copper box. The tray carrying the balance is then lifted into a pine box which it just fits. Rigid connection between the machine and the box is secured by means of two hard wood strips which slip and dovetail into notches at the top of the box. When the copper box is in position the wooden strips lie exactly above it, and there are brass screws which pass from the strips into the frame of the balance inside the thin copper box, which is not strong enough in itself to form a proper connection. In this way all relative motion of the instrument and the box is avoided. The riding level is taken off the balance and screwed to the box by special screws. The pine box containing the balance is supported on a set of sofa springs, which are attached to a false bottom, and some side support is given to the box by certain iron rods which are attached to the false bottom and extend upwards, so surrounding the box by a kind of iron framework. Connection between the box and the framework is secured by means of rubber buffers, and in this way the box is prevented from swinging about on its spring bottom.

The balance is handled by means of two handles screwing into the framework through two holes in the copper lid of the outer box.

THEORY OF THE BALANCE.

The tension of the spring which stretches the thread is so large in comparison with the weight of the lever that the thread is very nearly straight; we will suppose that it is exactly so.

Let HI (fig. 1, p. 218) represent the plan of a thread lying in a horizontal plane with a lever having mass, but of linear dimensions, fixed at the middle point of the thread. Let the thread be considered uniform in diameter, and homogeneous. Let the centre of gravity of the lever be at D, and at first consider the lever as hanging vertically, so that D is vertically below the thread. Under these circumstances, let there be no twist in either half of the thread.

Let θ be the angle through which the circle end of the thread, I, is rotated from its initial position, in other words, let θ be the twist in the circle end of the thread when the lever is kept vertical in its initial position.

Let ϕ be the corresponding quantity referring to the spring end of the thread.

Let ψ be the angle which the lever makes at any moment with the vertical plane drawn downwards through the thread.

Let l be the distance of the centre of gravity of the lever from the thread.

Let m be the mass of the lever.

Let g be the value of the earth's acceleration at the point considered.

Let τ be the moment of the forces exerted on the lever by either half of the thread when twisted through unit angle.

The equation of equilibrium, neglecting for the present the effect of variations of temperature, is

$$mgl \sin \psi = \tau (\theta - \psi) + \tau (\phi - \psi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

If the system is in equilibrium and we increase θ, the lever will take up a new equilibrium position, provided that $\frac{d\psi}{d\theta}$ is finite. If $\frac{d\psi}{d\theta}$ is infinite any increase of θ will make the lever upset. Now, from Equation (1) above

$$\frac{d\psi}{d\theta} = \frac{\tau}{mgl \cos \psi + 2\tau},$$

so $\frac{d\psi}{d\theta}$ will be infinite when

$$mgl \cos \psi + 2\tau = 0,$$

i.e., when

$$\cos \psi = - \frac{2\tau}{mgl}.$$

Also from the original equation

$$mgl = \frac{\tau (\theta - \psi) + \tau (\phi - \psi)}{\sin \psi},$$

so $\frac{d\psi}{d\theta}$ will be infinite when

$$\cot \psi = - \frac{2}{(\theta - \psi) + (\phi - \psi)}.$$

In the case of the instrument as constructed, we know that $(\theta - \psi)$ and $(\phi - \psi)$ are both approximately equal to 6π, so that the upsetting position of the lever is approximately given by $\cot \psi = - \frac{1}{6\pi}$, or the lever is above the horizontal plane through the thread by about three degrees when upsetting takes place. This is in exact accordance with our observations in so far as they are able to test it, and indicates that the simplified theory is fairly applicable to the actual instrument.

The accuracy with which a setting of the microscope may be made upon the mark at the end of the lever depends upon the value of $d\psi/d\theta$, and is, therefore, greater the nearer the position of the lever at the moment of observation is to the upsetting position. When the observing position of the microscope has been chosen, ϕ and ψ become constant. In the present instrument it is found possible to observe so close to the upsetting position of the lever that the microscope can be set much closer than the circle can be read. It is because of this fact that we have adopted the plan of increasing the effective circle sensitiveness by giving the thread three whole turns. The circle has a radius of 7 inches, and with our present experience, we are inclined to think that it would be better for the future to use a larger circle and reduce the twist of the thread.

The sensitiveness of the instrument to gravitational changes is given by the value of $d\theta/dg$, and from (1) this is

$$\frac{d\theta}{dg} = \frac{ml \sin \psi}{\tau}.$$

This is greater the nearer the position of the lever chosen for observation is to the horizontal plane, in so far in accordance with the last result.

Effect of Variation of Temperature.

We find that the relation between θ and the temperature as given by our platinum thermometer scale, for the small changes in the latter factor which we encounter in practice, is a linear one, within the limits of accuracy of our observations. We may therefore include all the effects of a change of temperature in a single coefficient. Equation (1) may be written

$$\theta = \frac{mgl \sin \psi}{\tau} - (\phi - 2\psi).$$

As ϕ and ψ are independent of the temperature, and as θ decreases as the temperature rises, to make this a working formula we must re-write it thus

$$\theta = \frac{mgl \sin \psi}{\tau}(1 - \alpha t) - (\phi - 2\psi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where α is the temperature coefficient and t is the platinum temperature. In consequence of the fact that the constants in this equation can only be approximately determined, we are limited to the consideration of relative values of g, so we may write the equation

$$\theta = \mathrm{K}g(1 + \alpha t) - \mathrm{C},$$

where K and C are constants.

Let θ_s be the reading corresponding to g_s the value of g at Sydney.
,, θ_m ,, ,, g_m ,, g ,, Melbourne.
,, θ_p ,, ,, g_p ,, g ,, any other place.

We have then

$$\theta_s = \mathrm{K}g_s(1 - \alpha t) - \mathrm{C}$$

with corresponding values for θ_m and θ_p.

For any one value of t we have

$$\frac{\theta_s - \theta_p}{\theta_s - \theta_m} = \frac{g_s - g_p}{g_s - g_m},$$

or

$$g_s - g_p = (g_s - g_m)\frac{\theta_s - \theta_p}{\theta_s - \theta_m}.$$

From the reading at any station at any temperature we may therefore determine the difference of g between that station and a standard station, say Sydney; provided we know the reading at Sydney at that temperature, the difference of reading between Sydney and Melbourne at that temperature and the difference in the values of g at Sydney and Melbourne.

To get an idea of the numbers involved in the use of our present instrument, we may take the Sydney reading at 21° on October 6, 1898, as 83°. We have then

Reading at Sydney 83°, corresponding to $g = 979{\cdot}639$, as given by Mr. LOVE.
,, Melbourne 82°, ,, $g = 979{\cdot}916$, ,, ,, .
,, the Pole 70°·5, ,, $g = 983{\cdot}11$, EVERETT's 'Units and Physical Constants.'
,, the Equator 88°·6, corresponding to $g = 978{\cdot}1$, EVERETT's 'Units and Physical Constants.'

The readings are those which would be given by our sextant arc, the temperature being 21° C.

From Equation (2) we get

$$\frac{\delta\theta}{\delta t} = -\frac{mla \sin\psi}{\tau}g.$$

If δt is taken equal to 1° C. on our thermometric scale, we have by observation

$$\frac{\delta\theta}{\delta t} \text{ at Sydney} = -31{\cdot}50 \text{ sextant minutes.}$$

We can at once deduce that

$$\frac{\delta\theta}{\delta t} \text{ at Melbourne would be } -31{\cdot}51.$$

and

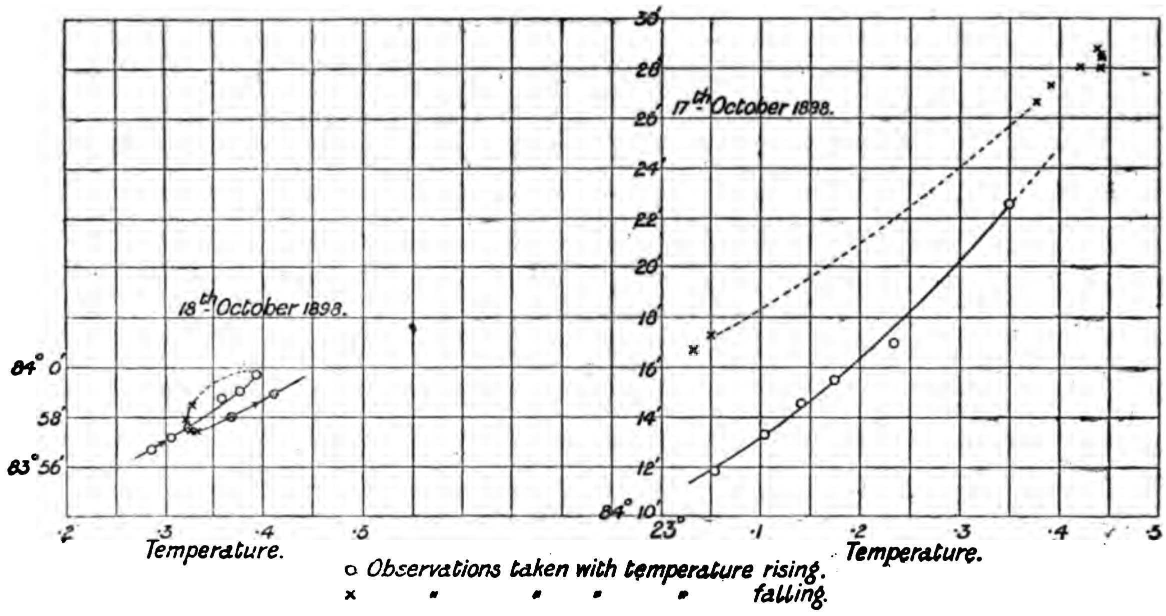
,, ,, the Pole ,, $-31{\cdot}61$.
,, ,, ,, Equator ,, $-31{\cdot}41$.

These values being so nearly the same, it is obvious that an approximate value of g at any station is all that is necessary to enable all observations to be reduced to a common temperature, which, if desired, may then be used to obtain a second approximation; this, however, is not necessary with our present accuracy of observation.

OBSERVATIONS.

Supposing that the line of collimation of the microscope occupies a fixed position, the variables which have to be observed at any one place are the temperature of the interior of the instrument as given by the platinum thermometer, and the amount of twist in the thread necessary at any moment to bring the image of the end of the lever coincident with a cross wire in the eye-piece of the microscope. This latter factor is given by the reading of the position of the vernier arm on the sextant arc. We have found that the relation between these variables depends very greatly not only on whether the temperature is rising or falling, and on the rate of the change, but also, theoretically at all events, on all the previous variations of temperature since the last steady state. In our instrument, the platinum wire, after a change of temperature, assumes its new resistance appropriate to the final temperature more quickly than the lever takes up its new position.

Plot 1.



In Plot 1 are given observations showing the relation between the temperature and the position of the vernier arm when the temperature is first rising and afterwards falling. The change of temperature on October 17th was due to a "southerly burster" coming up after a hot westerly wind. On the 18th the change is the natural change which takes place every evening, the second rise of temperature being due to the room being heated by the lights used for observation. These observations gave us what we call a natural minimum. These examples are sufficient to explain the reason of our ordinary procedure which is described in the next paragraph.

Method of Observing.

In the afternoon, when the temperature of the room just commences to fall, a watch is kept by one observer on the temperature of the box. When the rise of temperature of the box becomes very slow, the other observer prepares to take observations of the lever end. The preparation consists in carefully levelling the instrument along a line parallel to the thread, placing the long striding level on the microscope, and levelling cross ways until the sum of the readings of the two ends of the bubble of this level is within a few tenths of a division of the sum found when the level was last reversed. When the observer watching the temperature considers that the temperature of the box will reach its maximum value within the next few minutes, the lever is unarrested, a note being generally made of the time at which this is done.

A careful watch being kept on the levels, the image of the end of the lever is taken slightly above the cross wire by screwing the tangent screw of the vernier arm, the motion of the screw is then reversed, and the edge of the image brought down to coincidence with the cross wire. (There is a difference in the reading of a coincidence, depending on whether the image of the lever end is brought *up* to the wire or *down* to it, of from 30″ to 40″ (sextant), so that settings are habitually made downwards.) An observation of temperature is taken when the coincidence is exact. Immediately a setting is made, the positions of the ends of the bubble of the microscope level are noted, and the time, temperature and level readings entered. A reading of the position of the vernier arm is then made at leisure. A complete observation takes from three to four minutes. On levelling again, if necessary, it is seen that the image of the end of the lever has moved upwards, so the former procedure is repeated, until the end of the lever remains steadily in coincidence with the cross wire. An independent entry is made in the note-book of the time at which the temperature commences to fall. The image of the lever end is watched, levelling at intervals, until it begins to come down, to ensure that the maximum reading has been obtained. The lever is now arrested, and the vernier put at the constant reading of 85° (as it happens), so that the amount of twist in the thread may be kept constant with the exception of the time, never longer than an hour, occupied in an observation. The aneroid and air thermometer readings are entered, the level reversed and then removed from the microscope, and the observation is complete.

Specimen Observation.

NATURAL Maximum. December 20th, 1898.

Time. P.M.	Temperature of instrument.	Readings of end of level.	Circle reading.	Remarks.
h. m. s.	°C.		° ′ ″	
1 1 0	21·509	...	...	Air temperature 22°·1
2 12 0	21·779	...	...	„ „ 22°·4
2 42 0	21·890	...	...	„ „ 22°·5
3 53 20	22·033	6·1 41·1	81 26 50	
3 57 30	22·033	6·2 41·1	81 26 50	
4 2 30	22·033	6·2 41·1	81 27 0	Lever continues to rise, but temperature is steady
4 21 0	22·033	6·2 41·1	81 27 20	
4 24 0	Begins to fall			
4 26 0	22·31	6·2 41·1	...	Lever turned, air temperature 21°·6

Aneroid 28·2. Lever clamped at 85°
Level reversed. B end to eye-piece. Readings of bubble 6·2 41·1
„ „ A „ „ „ „ 6·2 41·1
„ „ B „ „ „ „ 6·1 41·1

The time between the first indication of a fall of temperature and a fall of the image of the lever end may be anything between 0 and 12 minutes, depending on the rate at which the temperature is changing. The temperature when the image of the lever end first gave signs of coming down, has been 0·02° below its maximum value, though it is generally much less. We assume that the maximum value of the temperature corresponds to the maximum reading of the vernier arm—an assumption which can only be rigidly justified by the accordance of the results which the procedure gives. To distinguish the observation of a maximum temperature and a maximum reading when the temperature alters naturally, from readings taken in other ways, we call the former a natural maximum observation. As the hour at which the temperature rate reverses varies greatly from day to day, the preliminary watching of the temperature is rather tedious.

A minimum reading may be obtained in a similar way by waiting until both the temperature of the air and of the box are falling, the temperature of the air being lower than that of the box. This occurs, of course, every evening after sunset. It is easy now to increase the temperature of the air of the room, by using lamps or small stoves; the lights necessary for observation are generally sufficient. The temperature of the box, and the position of the vernier arm, then go through minimum values which are observed as before. A reading taken under these circumstances we call a natural minimum. The actual observing for a natural minimum does not take half-an-hour.

NATURAL Minimum. Sydney, December 18th, 1898.

Time. P.M.	Temperature of instrument.	Readings of end of level.	Circle reading.	Remarks.
h. m. s.	°C.		° ′ ″	
6 38 0	23·110			
7 56 0	22·950	...	...	Air temperature 22°·3, temperature steady
8 9 0	22·950	...	...	Temperature beginning to turn
8 12 0	22·954	6·5 41·0	81 54 30	
8 16 0	22·960	6·4 41·0	81 54 40	Independent settings (2)
8 21 0	22·969	6·3 40·9	81 54 40	„ „
8 24 30	22·975	6·4 41·0	...	Lever has turned

Aneroid 28·00. Air temperature 23°·7. Lever clamped at 85°
Level reversed. B end to eye-piece. Readings of bubble 6·7 41·1
„ „ A „ „ „ „ 6·6 41·1
„ „ B „ „ „ „ 6·3 40·9

By far the greater number of our observations have been made by artificially heating the lower part of the case by placing a batswing gas burner, or a lamp, on the floor directly under the centre of the Kew magnetometer tripod, with a view of getting the temperature of the box between 2° and 3° above that of the air of the room. The source of heat is taken away after about three-quarters of an hour, and a maximum reading is obtained within the next hour. It is surprising that such barbarous treatment as heating the instrument from *below only*, with a naked flame, should have given results worth recording, but from the plots of the observations taken in this way it will be seen that they are most accordant. They will be referred to as artificial maxima. The success of this method of observing depends entirely on the perfect freedom of expansion of every part of the instrument. For reasons, which we give afterwards, we now only observe either natural maxima or minima.

ARTIFICIAL Maximum. Hornsby, December 15th, 1898.

Time. P.M.	Temperature of instrument.	Readings of end of level.	Circle reading.	Remarks.
h. m. s.	°C.		° ′ ″	
9 23 30	26·958			
9 30 0	27·030			
9 32 0	27·030	...	...	Temperature commencing to fall
9 34 0	27·040	6·5 40·9	84 14 0	
9 39 10	27·059	6·6 40·9	84 14 40	
9 46 30	...	...	...	Lever well down

Aneroid 28·20. Air temperature 24°·5. Clamped at 80°
Level reversed. B end to eye-piece. Readings of bubble 6·9 41·0
„ „ A „ „ „ „ 6·6 40·8
„ „ B „ „ „ „ 6·9 41·0

The adequacy of the temperature observation, and of the instrumental adjustments, will now be discussed.

Temperature Corrections.—Any given temperature measurement is clearly affected with the sums of the errors of the coils employed. The accuracy with which the coils are compared with each other and the bridge wire is chiefly a matter of galvanometry. We made our comparisons with the galvanometer used in the temperature observations, and, consequently, cannot hope to attain the highest possible accuracy of which the method is capable. A long series of comparisons shows that the working accuracy, or rather consistency, attained in the temperature measurements is within 0°·01. If we suppose that any observation is in error by this amount, the resulting uncertainty may be stated as 0·3 minute (sextant). This would lead to an error in the estimation of the value of g of one part in 700,000 very nearly.

Accuracy of Setting on the Lever and Reading the Circle.—The position of the vernier arm can be read to 10 seconds of the graduation of the arc, or to 5 seconds of arc. The magnification of the microscope is such that if we move the vernier arm through 10 seconds (sextant), the end of the lever is displaced on the cross wires by a comparatively large amount—an amount two or three times larger than the least we could see. We may say then, that the accuracy of setting is such that we are not affected with any errors on this score comparable with those which occur in reading the vernier. With regard to the latter we will suppose that, taking the sensitiveness as before, an error of 10 seconds in reading the vernier arm would lead to an error in the estimation of g of one part in 1,300,000. This is about half the temperature error.

Errors of Levelling.—The indications of the instrument depend on the assumption that the line of collimation of the microscope can be brought to the same position with respect to the horizontal at each observation. We have already discussed the precautions taken to insure this being the case; but, as there pointed out, there is a certain outstanding theoretical uncertainty on this score, and, moreover, there is no way of applying a check. Setting this aside, we consider that the axis of the microscope tube occupies the same relative position with respect to the horizontal plane whenever the level reverses. The readings of the position of the ends of the bubble can be got with certainty to within 0·2 of a division. It is not always possible to adjust the level during an observation so that the bubble occupies its reversing position. In practice our observations have been made with the maximum error of 0·6 in the position of the bubble. By trial, we have found that if we displace the bubble by 9·0 divisions we alter the circle reading by 150 sextant minutes. It follows that an error of levelling of one division of the level scale would introduce an error of 0·3 sextant minute, or one part in 700,000 in the value of g.

If all these three maximum errors conspire, we shall obtain a value of g in error by one part in about 300,000. It must not be forgotten, however, that with the exception of the temperature error, we have taken as possible errors quantities which could only be realised by bad observing and by omitting to apply a level correction, which, hitherto, we have not found requisite.

Reduction of Observations.

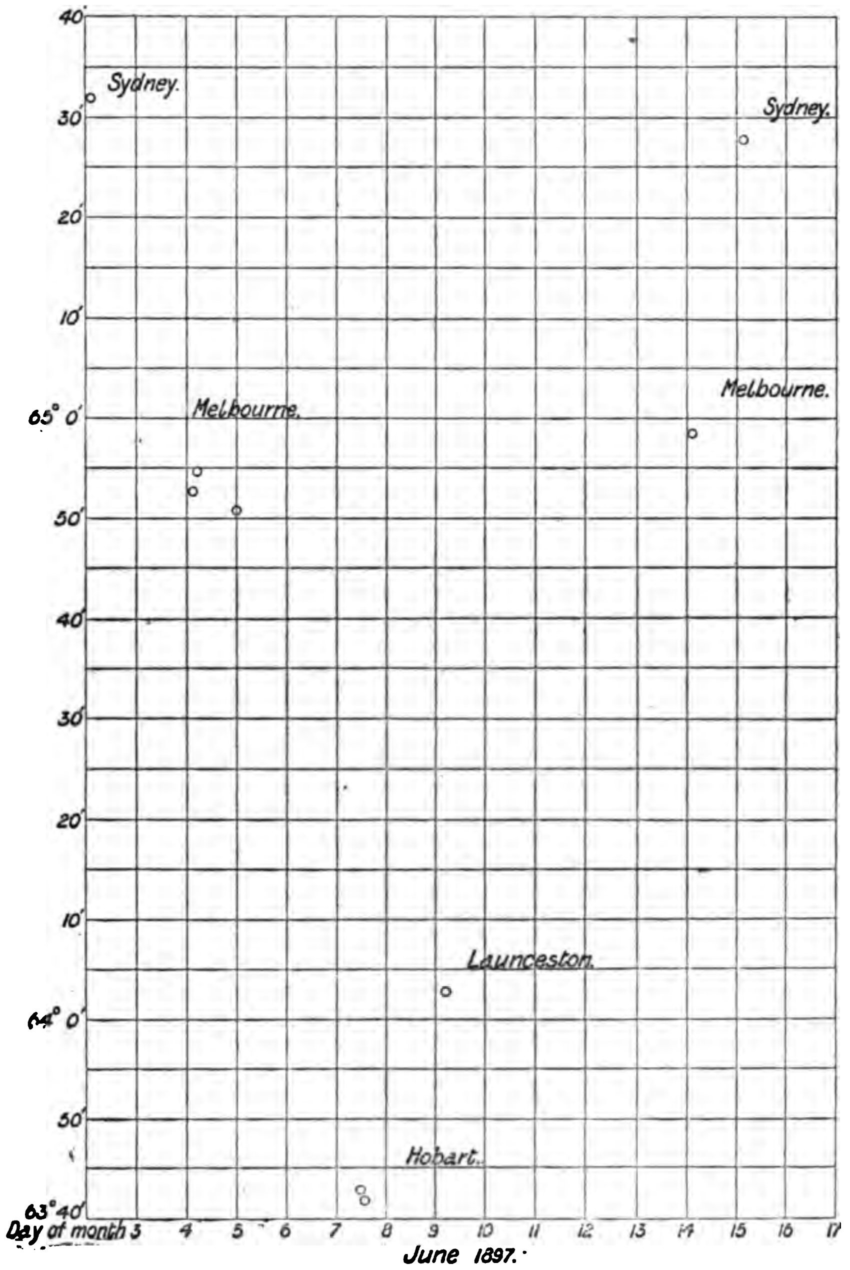
The readings of the instrument at any one place differ from day to day from two causes—from change of temperature and from the slow elastic after-working of the thread and its supports. When a new thread is set up, or any alteration made in the lever or supports, the first thing to be determined is the effect of temperature on the readings. When a considerable number of daily observations have been collected, in the first place, the maximum temperatures of each day's observation, as entered, are corrected according to a scheme drawn up from the comparison of the coils of the resistance box. The maximum readings of the position of the vernier arm are then all reduced to one temperature by an assumed temperature coefficient, on the assumption that the relation between the temperature and readings is a linear one. A plot is then made with time as abscissa and readings as ordinates. If there appear to be systematic errors connected with the temperature, a new coefficient is taken, and the observations again reduced. This procedure is repeated until the systematic errors cease to be apparent—sometimes a lengthy process. If there was no elastic after-working, the plot of the observations from day to day at any one place should lie on a line parallel to the axis of time. If elastic after-working exists the line joining the plots of the observations will be more or less sloped to the same axis.

DISCUSSION OF RESULTS.

There are two conditions which a balance of this kind must fulfil for it to be a working instrument—firstly, it must give accordant readings at any one place from day to day; and secondly, the readings must not be affected by the vibration inseparable from transport. We shall adopt the historical method of treatment, discussing the deviations from the rigorous fulfilment of the two conditions in the order in which they were observed.

The first journey made with the instrument (other than preliminary ones) was commenced in June, 1897. The instrument was taken from Sydney to Melbourne by train, and set up in a cellar of the Physical Laboratory of the University. It was then taken to Hobart by steamer, observed in a cellar of the Museum and in the University Physical Laboratory, then to Launceston by train and observed in the strong room of the Custom House, then to Melbourne by steamer, and to Sydney again by train. On this trip simultaneous observations of temperature and twist were taken over long periods with the temperature rising or falling more or less rapidly, as we had not at this time discovered that it was essential only to observe maximum or minimum readings. From the numerous observations taken we have selected those obtained under conditions most nearly approaching those which we now know to be required. They are given in Plot 2. In the long series of observations taken discontinuities appeared, amounting in one instance to 7 sextant minutes. At this time the two axles to which the thread was attached were connected by a rectangular frame, intended to make both ends of the thread twist by

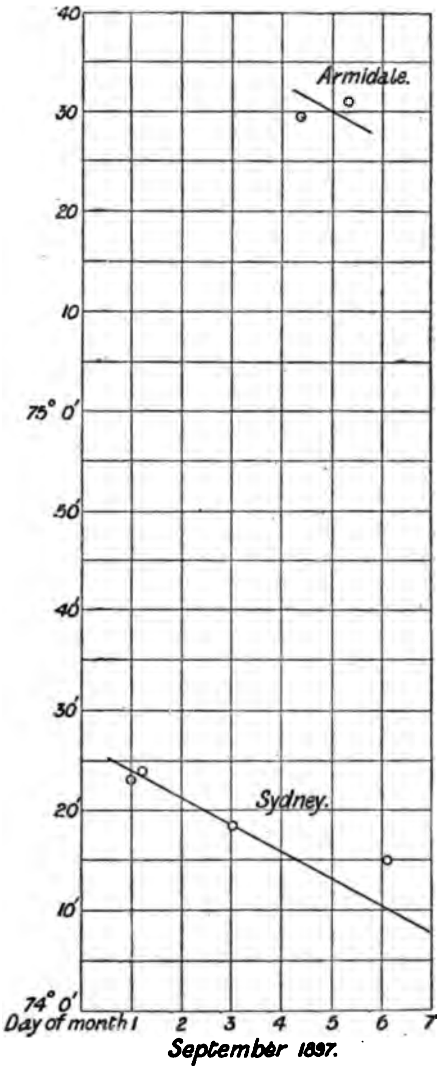
Plot 2.



Readings reduced to 16° C.

the same amount. This frame was put in as the result of an after-thought, after the instrument had been completed, and there was not then room for an adequately designed girder. We attributed the discontinuities observed to the bending of this frame, and it was taken out immediately on our return to Sydney, with the result that there has not been any appreciable discontinuity in a series of readings since that date. The difference between the reading at Sydney on June 2nd, and that at Melbourne on the 4th and 5th, may be taken at 39 (sextant minutes), and that between the observation at Melbourne on June 14th, and that at Sydney on the following day, as 29 (sextant minutes). The mean of these, 34 (sextant minutes), confirms, so far as it can be said to confirm anything, a result which we obtained in the following year, as will be seen later. We had quite concluded before we returned

Plot 3.

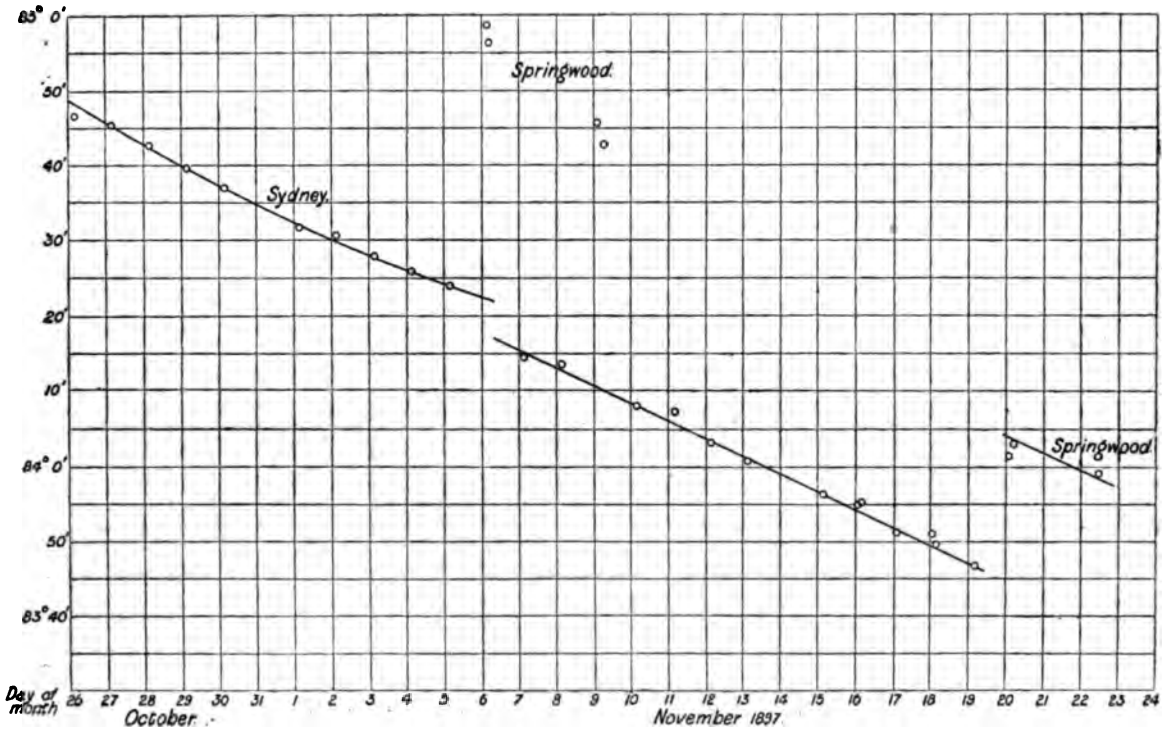


Readings reduced to 14° C.

to Sydney that the observations would not be of high value, but they afforded us some evidence that the instrument could be transported under actual conditions without serious derangement.

The removal of the frame connecting the two axles leaves one-half of the thread with a constant twist, while the other half has during each observation a slightly variable amount of twist depending on the temperature and the intensity of the gravitational force. Any change of gravitational force has now to be compensated by a change in the twist of one-half of the thread, consequently the vernier arm has now to be moved through twice the angle necessary with the former arrangement; the reading-sensitiveness of the instrument was, in fact, doubled by this change.

Plot 4.



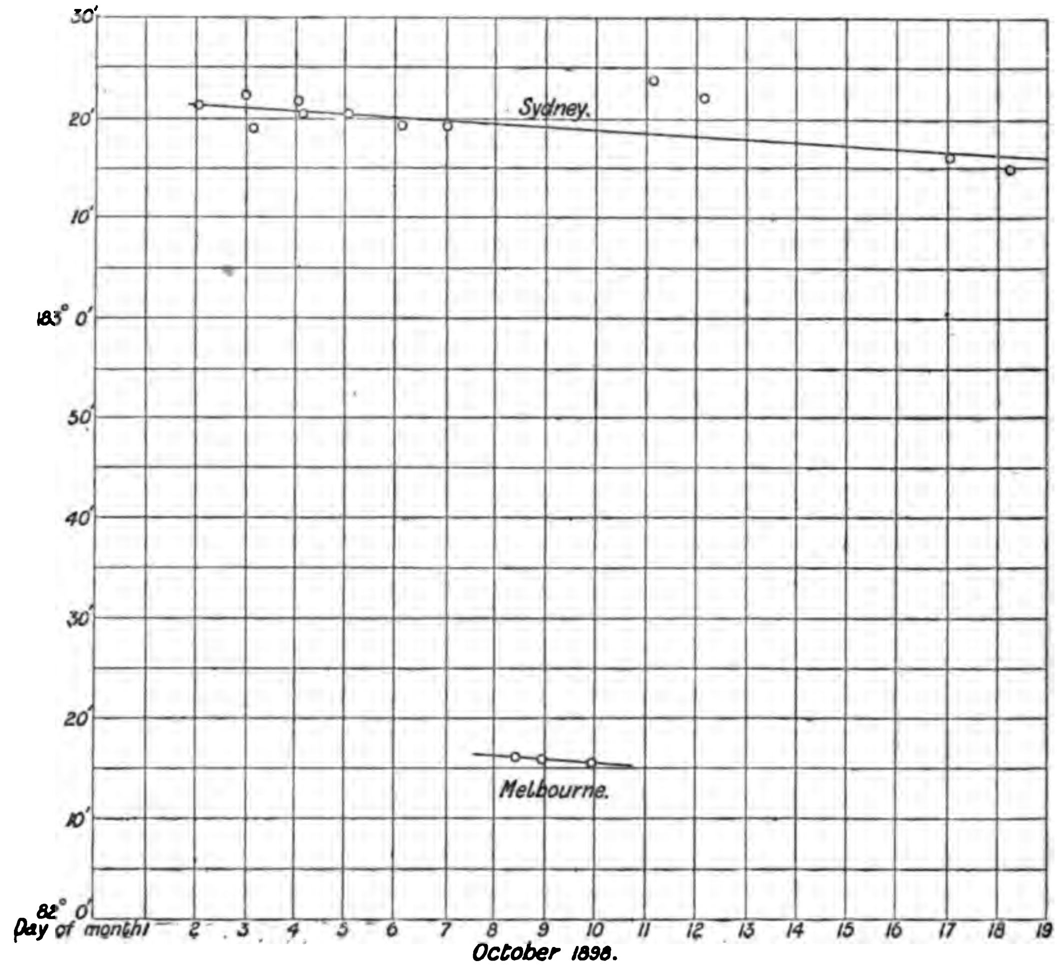
Readings reduced to 25° C.

A gravitational survey, connecting the towns of the eastern Queensland seaboard with Sydney, was now projected, the first connecting station chosen being Armidale, a town on the table-land in the northern part of New South Wales. The observations given in Plot 3 show an immense improvement in the behaviour of the balance, though they give evidence of a break of 5 sextant minutes in the reading on returning to Sydney—due to the shaking which the instrument had experienced in transport. We had, by this time, found out the necessity of observing only when the temperature was at a maximum or a minimum.

On the return from Armidale it was found that some air had leaked into the instrument. In trying to discover the leak, through carelessness, the delicate soldering of the lever to the thread was severely strained. The accident had the effect of permanently increasing the reading by 10°.

On November 6th the instrument was taken by train to Springwood, a station 48 miles west of Sydney, 1216 feet above sea level. It was here observed in the cellar of the Oriental Hotel, and was brought back to Sydney the same night. Plot 4 shows that a break of 5·0 sextant minutes had occurred in the Sydney reading, and that the daily rate due to elastic after-working (about 2·5 sextant minutes per day) had slightly increased. The balance was again taken to Springwood. On its return the next day there was no definite evidence of a further break. But a third journey on November 20th, returning to Sydney on the 23rd, strengthened the evidence as to a permanent change taking place due to the travelling. When putting the instrument into the train at Springwood on the 23rd, one of the handles of the box broke, and one end of the box fell about two feet on the asphalte. The thread was not broken, but the observation of the 23rd at Sydney differed by 1° (sextant) from

Plot 5.



Readings reduced to 21° C.

its reading on the 19th. It was considered that the breaks in the readings when the instrument was "travelled" were due to a looseness in the joint fixing the lever to the thread, which might have been caused by the accident after coming from Armidale, and observations were discontinued until the joint could be re-made.

In February, 1898, the bars carrying the thread were taken out of the outer case and the lever was re-soldered to the thread. This necessitated the little lump of fusible metal at the end of the lever being altered in order to keep the twist in the thread the same as before. In putting the instrument together again the further end of the internal framework was wedged to the copper tube. Previously permanency of relative position between the microscope and the thread depended on the attachment of the inner framework to the copper tube by the screws at the circle end only.

Other work did not permit observations to be commenced until the end of September. The temperature coefficient was now 26·7 sextant minutes per degree centigrade. Plot 5 shows that the daily rate due to elastic after-working had decreased from 2·5 minutes in September, 1897, to 0·3 minute in September, 1898. (The present thread was mounted in September, 1896.) Satisfactory observations were made in Melbourne on October 8th, 9th, and 10th. The observations on our return to Sydney show a difference of 5·5 minutes compared to the ones taken before going to Melbourne, but the change, instead of being permanent as it was in 1897, completely came out by the 17th.

The journey to Melbourne was undertaken with a view to finding the sensitiveness of the instrument. The rate, as given by the observations at Melbourne, which were completely satisfactory, is the same as that at Sydney. We may suppose it likely then that the change in the Sydney reading is due to something which occurred on the return journey. The difference between the readings at Sydney and at Melbourne, taken from the plot, is 63 sextant minutes.

If three whole turns in the thread, or 2160 (sextant degrees), are required to keep the lever horizontal against a g of 980, assuming that the relation between the necessary twist and g is a linear one, in spite of the accumulated after-working, 72 sextant minutes would be required to compensate for a change of g of 0·277, which is the difference between g at Melbourne and Sydney, as given by Mr. LOVE.*

We have then for the difference of reading between Sydney and Melbourne for the instrument, as at present constructed, calculated from the number of twists in the thread, known only approximately, 72 sextant minutes.

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* Mr. E. F. J. LOVE, who has given attention to the determination of gravity at Sydney and Melbourne, both experimentally and by discussing the results by other observers, was good enough to give us the following values of g as the most probable:—

Values of gravity at Melbourne Observatory	979·916
Values of gravity at Sydney Observatory	979·639
Difference	0·277

From the unsatisfactory comparison in June, 1897, remembering that the sensitiveness is doubled since that date, 68 sextant minutes.

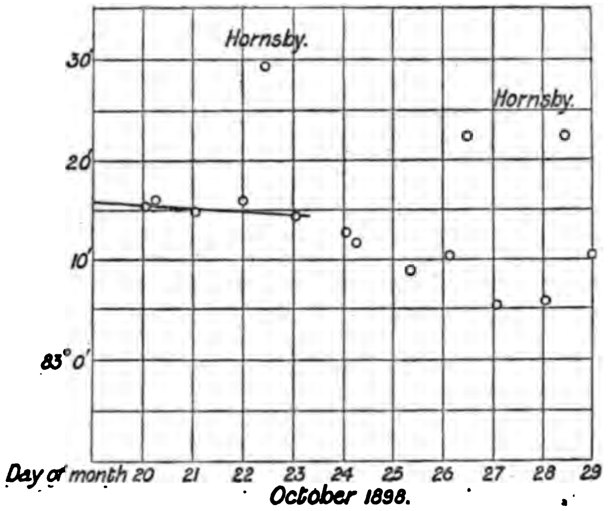
From the comparison in October, 1898, 63 sextant minutes.

Until the difference has been more accurately determined, we may take say 60 sextant minutes as the difference of readings at Sydney and at Melbourne.

Mr. LOVE considers that the value of g at Melbourne is 979·916, and that the Sydney-Melbourne difference is 0·277. If the change in the reading of our instrument between Sydney and Melbourne is 60 sextant minutes, then a change in the value of g of 1 part in 100,000 would be represented by a change of reading of 2·12 sextant minutes.

There is now ample evidence before the reader in the plots of the observations taken in November, 1897, and from the 2nd to the 18th October, 1898, to show that, apart from discontinuities due to travelling, it is extremely unlikely that any single observation would differ from the mean reading at any one station by 1 sextant minute.

Plot 6.



Readings reduced to 21° C.

There is only one observation, that on the afternoon of October 3rd, 1898, which differs from the mean Sydney reading by 2 sextant minutes. If, therefore, the instrument can be carried from place to place without altering its behaviour, we may say that the value of g at any station may be determined relatively to that at some standard station by a single observation, with extreme probability to 1 part in 200,000, and with certainty to 1 part in 100,000.

We had, therefore, at this time to determine the cause of the effect produced by travelling the instrument. We determined to take the balance to Hornsby Junction, a station 21 miles north of Sydney, 592 feet above sea level.

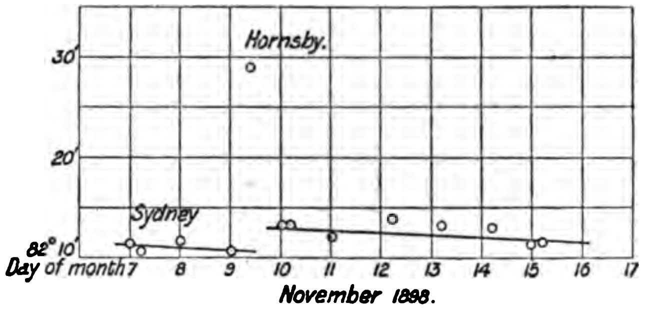
The observations at Hornsby were made at night in the lamp room of the station, and the balance was brought back to the laboratory in the morning. The result of the observations is shown in Plot 6. Without correcting for daily rate, the Sydney-Hornsby differences of reading, as determined from these observations, are 13·5, 15·0, 12·0, 16·9, 16·4, 12·0 sextant minutes, and confirm the existence of a serious change produced by travelling. We now know that the difference is 18·2, so that from one of these observations the difference might be in error by 6·2 sextant minutes.

The only points of the instrument which we could think of as being affected by the travelling, were: the end of the rosette spring to which one end of the thread is attached, and the lever itself, the after-end only of which is clamped, leaving the forward-end not incapable of vibration. Although the end of the rosette spring was stayed by wires to four upright pieces of watch spring, it was still capable of some vibration in a direction at right angles to the length of the thread.

On October 29th, 1898, the inner framework was taken out. The arrester springs were strengthened, so that the lever was held more firmly than before. The junctions of the little wire stays of the rosette spring were paraffined at both ends to prevent the slightest play in the links, and three fine glass hairs were attached with paraffin, one end in the centre of the rosette spring and the other on one of the bars of the framework, the three glass hairs lying in a plane at right angles to the line of the thread. The point of the rosette spring may now be considered to have no freedom of movement in a direction at right angles to the length of the thread.

The instrument was re-mounted, and observations commenced on November 7th. It was found that the temperature coefficient had increased, an increase being what one would expect if the resilience of the spring system had increased, as it must have done owing to the additional constraint imposed by the three glass hairs.

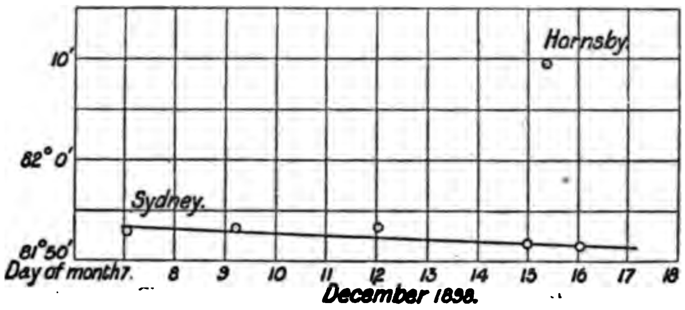
Plot 7.



Readings reduced to 23° C.

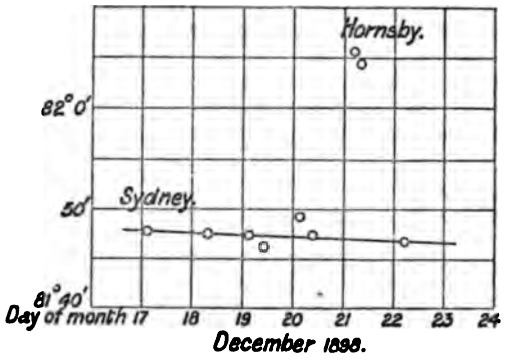
It was very soon seen that the changes due to travelling had been greatly diminished, the change in the Sydney reading, after an observation at Hornsby on November 9th, being only 2·6 sextant minutes. It was found, however, that by daily observation at Sydney, from November 10th to December 7th (those up to November 15th being shown in Plot 7), that discrepancies amounting to 4 sextant minutes on each side of the mean line now appeared in a most erratic way. This we traced to the method of observing by heating the instrument from below with a naked flame, as before described. When we observed only natural maxima and minima, the readings at Sydney regained their old regularity, with the exception of one observation on the afternoon of December 20th, which is 2 sextant minutes from the mean line. In our present instrument there is a fault in design, inasmuch as the microscope is fixed to a copper tube, while the thread is carried by a gun-metal framework which we now try to fix firmly at both ends to the copper tube. It is only to be expected that, owing to the different expansibilities of the two metals, the instrument may be twisted in a most erratic way during changes of temperature, and this effect will be exaggerated if the changes of temperature are rapid. Moreover, by the addition of glass tie bars the character of the spring system has been entirely changed.

Plot 8.



Readings reduced to 23° C.

Plot 9.



Readings reduced to 23° C.

Two more journeys were made to Hornsby, and the observations are shown in Plots 8 and 9. It is seen from these plots that the readings are not now affected by travelling, for, without correcting for daily rate, we have the following differences between an observation at Hornsby and the observation before and after at Sydney :—

November 9th	18·3	sextant minutes.	
„ 10th	15·9	„ „	
December 15th	17·9	„ „	
„ 16th	18·3	„ „	
„ 20th	17·7	„ „	Taking the mean Hornsby reading.
„ 22nd	18·3	„ „	

We may reject the observation of November 10th. Discrepancies twice as great as this from the mean were discovered afterwards to be due to observing immediately after too rapid temperature changes.

When we correct for daily rate and take the mean reading at Sydney, the Sydney-Hornsby difference comes out—

November 9th	18·5	sextant minutes.
December 15th	18·1	„ „
„ 21st	18·1	„ „

We have, therefore, determined the value of g at Hornsby relatively to that at Sydney in three journeys, with a maximum difference of 0·4 sextant minute, or to less than 1 part in 500,000 in the value of g.

This sensitiveness refers, of course, only to the Sydney-Hornsby difference. For the purposes of a survey the mean reading at the standard station may be determined with extreme accuracy—with an error at any rate negligible with respect to the probable error of a single observation at any one station. Travelling does not now affect the instrument, so that the accuracy of a determination of g, from a single observation at any station, depends on the possible deviation of a single observation from the mean.

APPENDIX B.

NOTES ON EXPERIMENTS MADE WITH VARIOUS FORMS OF GRAVITY BALANCES.

In September, 1888, we made calculations as to the sensitiveness of a balance with a horizontal thread twisted several times. We then constructed several trial instruments, mounting our threads and appliances on an old watchmaker's lathe-bed. One end of the thread was fastened to a rod working in a barrel against a spiral spring, and the other end was attached to the centre of a bar provided with a large roughly-divided circle.

A large number of trials were made as to the right thickness of the thread, &c., and much trouble was experienced in preventing the thread from breaking after having been exposed to twist and tension during several days. The thread and levers were massive compared with those we now employ. One of the earliest methods of cementing employed by us was by means of clean fusible metal. Attempts were also made to grind the thread slightly flat so as to give a sort of key to the cement.

We then began to attempt to compensate the natural increase in stiffness which occurs as the temperature of a quartz thread rises. It was clear that compensation to the requisite extent could only be attained by the use of some form of compound lever. Professor GURNEY suggested to us to try a lever consisting of three bars. Two of these were to be horizontal and situated at a short distance one above the other in the same vertical plane, they were to be of metals of as dissimilar expansibilities as possible, the less expansible one being uppermost. One end of each lever was to be attached rigidly to a cross piece to which the thread could be fastened. At the free ends of the levers the following disposition was to be made : a bar was to be pivoted from the end of the lever of the less expansible metal, and was to bear against the wedge-shaped end of the lever made from the more expansible metal. When the system of two bars was horizontal, the pivoted bar was vertical. As the temperature rose the more expansible bar would deflect the pivoted bar, and so increase the moment round the thread. Many levers were made on this principle, for which purpose we instructed ourselves in the art of the watchmaker, and finally managed to make very small levers, using aluminium and platinum-iridium as the two dissimilar metals. We finally made a compound lever weighing less than a decigram, and having all its dimensions correct to produce the compensation required. After some experimenting with this lever, it was abandoned on account of the inevitable want of permanency of form to which its hinged joint gave rise.

During the experiments with compound levers an iron box was made which would contain the whole appliance, and which was air-tight and could be exhausted. Most of our earlier trials were made in this box; it lent itself especially to the investigation of the flotation of the levers by air of different densities. We had hoped to be able to deduce the sensitiveness from the flotation effects, but the cement of uncertain density occupied so large a relative volume in comparison with the lever that these attempts failed.

By May, 1891, we had sufficient experience to hope to detect the lunar disturbance of gravity. At this time we used a lever of aluminium shaped like a cross, the thread being cemented across the shorter arm by shellac. A mirror was mounted on the lever so as to be vertical when the lever was horizontal; the mirror was close to the thread, being carried by the head of the cross. The balance, in the air-tight box, was placed in a cellar and supported on a very heavy stand, made by filling a large iron cylinder (a mine case for 500 lbs. of guncotton) with sand and stones. The temperature was taken by means of a very fine mercury in glass thermometer, the bulb of which was placed inside the iron box. The temperature of the cellar was very uniform, but was disturbed considerably by the presence of the observer. The sensitiveness of the instrument was ascertained by weighting the lever with shreds of fine wire; the calculation involved a knowledge of the distance of the shred of wire from the thread, a quantity which could not be got with any great accuracy.

The highest sensitiveness attained was that we got an observable deflection corresponding to an increase of gravity by 7×10^{-6} of itself. This involved reading to one-tenth of a millimetre on a scale distant 3 metres from the mirror $\equiv$ 3 seconds of arc. The observations, of course, included a study of the daily rate of the system, as well as of its temperature coefficient. We varied the form of the lever, the dimensions and twist of the thread, &c. Exceedingly good mirrors were made and mounted during this work, and we found that the best cementing material for thin mirrors is a mixture of equal parts of white and red lead made into a paste with a little boiled linseed oil. The mirrors were fastened on with a trace of this cement; they were left for about a day, and then stoved at 100° C. till hard. Every kind of cement deforms thin mirrors, but this paint deforms them less than anything else we tried. A variation was attempted on the following lines. Two threads were mounted, one many times the diameter of the other, the thicker one supports the working lever, the other acting as a torsionless axle. From the end of the working lever, distant, perhaps, 2 centims. from the thread, the finest possible thread also of quartz was stretched to join the second lever mounted on the thin thread, and was cemented to it quite close up to the thread. The second lever carried the mirror, and any motion of the first lever was magnified in the ratio of the distances of attachment of the connecting thread from the two main threads. We only got one-tenth the sensitiveness of our best single system by this arrangement, which was besides very cumbersome and difficult to set up. The finest quartz thread is too stiff to act properly as a flexible connection and acts more like a rigid bar.

During the latter part of 1891 we made an adjustable system of MICHELSON'S arrangement of interference mirrors, and tried to increase the sensitiveness of our angular measurements by observing the motion of the interference fringes. The general result arrived at after much patient work was that the method presented no advantages in practice; and this even when we replaced our single mirror by two small mirrors separated as far as the mechanical conditions permitted.

By March, 1892, we became convinced that it was hopeless to attempt to disentangle the lunar effect from the instrumental irregularities, even if we could bring the sensitiveness up to the necessary point; of which there seemed to be no hope. The research was therefore abandoned.

Investigations with the view of constructing our present form of portable instrument were begun in May, 1892. For this purpose we mounted a thread on the spiral head of our milling machine, and supported the fixed end of the thread on the back centre. The twisting of the thread was observed by a mirror and scale, the mirror being connected to the rod carrying the end of the thread to be twisted; and a lever and mirror were mounted at the centre of the thread as before. The machine was provided with a special slow motion of rotation for the spiral head. The position of the lever was observed by a telescope, using the method of reflected images. These trials led to the present form of balance, the construction of which was commenced in August, 1892. The instrument was not ready till July, 1893, and the first thread was not mounted till September, 1893. This was a catapult thread, and was twisted with $2\frac{3}{4}$ turns. The aluminium lever carried a mirror, and this was observed by reflexion of cross wires as before. At first the thread was intended to be twisted from one end only to compensate changes of gravity—exactly as in our present form of instrument—but as irregularities occurred we were led to put in an extra set of bars so as to twist both ends of the thread at once. We had also an idea that the theory would be sufficiently simplified to enable us to avoid the necessity for any calibration, the total twist being capable of exact estimation in consequence of the presence of a theodolite circle, which at that time occupied the place since taken by the sextant arc. We also hoped that an exactly symmetrical distribution of twist in the thread would reduce the viscous yielding, and so more than compensate for the loss of sensitiveness due to twisting two ends of the thread instead of one.

The lever was adjusted by drops of paraffin, which are easier to regulate than drops of solder. The whole structure of the instrument was mounted on a turntable so as to allow us to take observations in any azimuth, for the purpose of eliminating magnetic effects if such should appear. In October, 1893, this machine was taken to Armidale, with the result mentioned at the commencement of the paper. The temperature was taken at this time by a mercury-in-glass thermometer.

During the time that the instrument was undergoing repair and alteration, in accordance with the experience obtained, we constructed a fresh instrument in which the whole of the working parts were immersed in mineral sperm oil. The machine was only intended as a trial instrument, and was put together out of brass tube, &c., but it was well made. We observed it in the cellar with the appliances formerly used in the attempt to discover the lunar disturbance of gravity. It was found that the readings went through a regular daily cycle which was ultimately traced down to the action of minute convection currents. The regularity of these minute currents was one of the most surprising things we have ever met with. In order to get an idea as to whether it would be possible to observe at sea, we mounted the machine on a swing, and satisfied ourselves that no amount of damping would enable accurate observations to be made under such circumstances. A subsidiary set of experiments were made in connection with this matter in order to find the resisting properties of different cements when immersed in oils, *i.e.*, to find out whether they were gradually softened by the oils. These experiments lasted for two years, and showed us that the resistance of shellac is surprisingly great. Mineral oils did not appear to have any influence at all, and turpenes only a very slight effect, the other cements tried were not nearly so resisting.

Experiments of one sort or another with the oil balance were continued till January, 1894.

Meanwhile the portable instrument had been restored, and in November, 1893, the catapult thread was replaced by a shot thread, this was under observation during the beginning of 1894. We now had a balance much worse than the one that had been broken: the thread appearing to show a viscous yielding in the wrong direction. We attributed this to the shellac, so in January, 1894, an experimental thread was mounted on yet another balance. This thread was drawn down so as to be thick in the middle and at both ends, with the view of reducing the stress intensities in the shellac. The results of this experiment were such as to lead us to think the abnormal behaviour of the portable balance was to be traced to the thread or lever, and not to the shellac. However, by February the balance seemed to have settled down, and it was again taken to Armidale. On this occasion, however, the results were quite disappointing, and the cross wire images were so difficult to observe that it was clear that a new mirror or method of observing was necessary. The viscous subsidence from whatever cause arising was also unsatisfactory, so that we decided to mount a fresh thread, choosing now a much finer one. For two months of incessant work we struggled with fine threads, finally mounting what we thought a very good one in April, and this was then observed till July, when we have the following note: "This thread, the very finest we have had with a very small lever, has been most unreliable. It seems from its behaviour (especially when the lever becomes unstable) that the centre of gravity of the lever has been moving relatively to the thread, perhaps the fine thread has been moving through the shellac."* A defect in the arrester rendered this possible.

In consequence of this observation a separate experiment on another balance was made from May till September, using a soldered thread. We were led by this experiment to solder up a fine thread to the portable balance and to mount on it a soft copper lever. On August 21st, after continued observation, we were forced to the following conclusion:—"From the general appearance of the observations it is clear that the readings are becoming less with lapse of time—or the thread requires more twist to bring the lever to its sighted position—the effect to be anticipated . . . The readings are, however, irregular . . . The thread is not to be compared with others we have had whose variations were within a degree, whereas this thread has altered its reading by ten degrees. The thread is so fine that the centre of gravity of the lever must be extremely close to it, so that the very smallest change in the lever or attached mirror will greatly affect the reading. The very fine threads have not been a success." As a result of this experience we abandoned the lever and mirror in favour of a microscope, and also brought the arrester to its present form. During September and October, 1894, several threads were mounted, and these all broke before we could get a series of observations. These persistent failures led us to examine the threads more closely than we had done hitherto, both as to uniformity and freedom from air bubbles. Threads of the following diameters all broke: ·00126, ·0015, and

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* We are now inclined to think that a good deal of the difficulty which we attributed to the imperfect elasticity of the threads was in reality due to bad means of attachment, imperfect spring, and bad temperature estimations.

·00154 inch. Also the twist was far less than our previous experiments had indicated as allowable. Thus, we thought that we ought to have been allowed to put in a twist of one turn per centimetre, and we only put in a twist of ·204 turn per centimetre. This showed that either we had got hold of a bad sample of quartz, in spite of all efforts to the contrary, or that the conditions of continuous stress obtaining in a balance affect the elastic properties in some hitherto unrecognised way.

Now the thin thread which did so badly from June till August, 1894, had a diameter of ·0004 inch or ·00102 centim. The length of all the threads was 12 inches. We concluded that the diameter of the thread should lie between ·0035 and ·001 centim., using levers of the mass hitherto employed, so as to avoid breakage on the one hand or irregularity on the other. The cause of breakage was also seen to be related to the sag produced by the use of too heavy a lever. It was at this time that we made the comparative examination of different samples of quartz referred to in the text. We did not terminate these investigations till the end of 1894, and it was not till March, 1895, that we succeeded in obtaining a thread to satisfy us. The diameter of this thread lay between ·0014 and ·0015 inch, and was made of our most infusible quartz and mounted with a very light straight wire lever. We observed this thread till September, 1895, finding, amongst other things, that the flotation correction was very small, as had been anticipated. The viscous yielding was large, and we suspected that the silvering had not been removed sufficiently close to the coppering (from fear of the nitric acid getting in and gradually loosening the thread); this was tested by a reapplication of nitric acid, and the reading promptly changed by 40° of twist.

Finding the great weight of the instrument a drawback, we re-made the base and replaced the theodolite circle at the twisting end by a sextant arc. By November we had traced such irregularity as still persisted, to the sticking of a barrel spring, which caused the tension of the thread to vary irregularly whenever the temperature changed. We therefore designed and made the "rosette" spring referred to in the text, but were so unfortunate as to break the thread in dismounting it; we spent no less than two months in getting a fresh thread to satisfy the conditions. On January 22nd, 1896, this thread was mounted, with about three whole turns at each end. It was observed continually until July 9th, when it was pleased to break. The daily rate of subsidence of this thread was 6 sextant minutes in March, and fell to 2·5 in the latter part of June. There appeared to be some effect depending on the time which elapsed between the releasing of the lever and the time at which the reading was taken. We did not find the reason of this at the time, but we now know that it was due to the heating up of the balance by the observer. However, we wasted some time over it.

A new thread was got in September, 1896, after some weeks' shooting. This thread is still in use and is the best we ever got. We had a great deal of trouble in stopping leaks in the apparatus, which was beginning to show signs of wear in the bearings. In April, 1897, we detected a mechanical irregularity due to this wear, the axle of the sextant arm was working with too much friction ; this was remedied. During May the instrument was considered to be fit to travel, and we took it to Bowenfels, a place in the Blue Mountains, some 2000 feet above Sydney, and here we made some promising observations. We have to thank our friend, Mr. FLINT, for allowing us to use his house as an observing station, and for helping us in every way. When we came to discuss these observations, we found that it was now necessary to improve the thermometry, and we accordingly made and tested the platinum thermometer. By dint of some very continuous work we were able to get away to Melbourne in June. An account of our work from this date is contained in the paper.

Description of Plates.

PLATE 1.

N.B.—The drawing does not show the long inner tube carrying the eye-piece inside the outer microscope tube, otherwise it is a working drawing.

Index to Lettering on the Drawing. Horizontal Section.

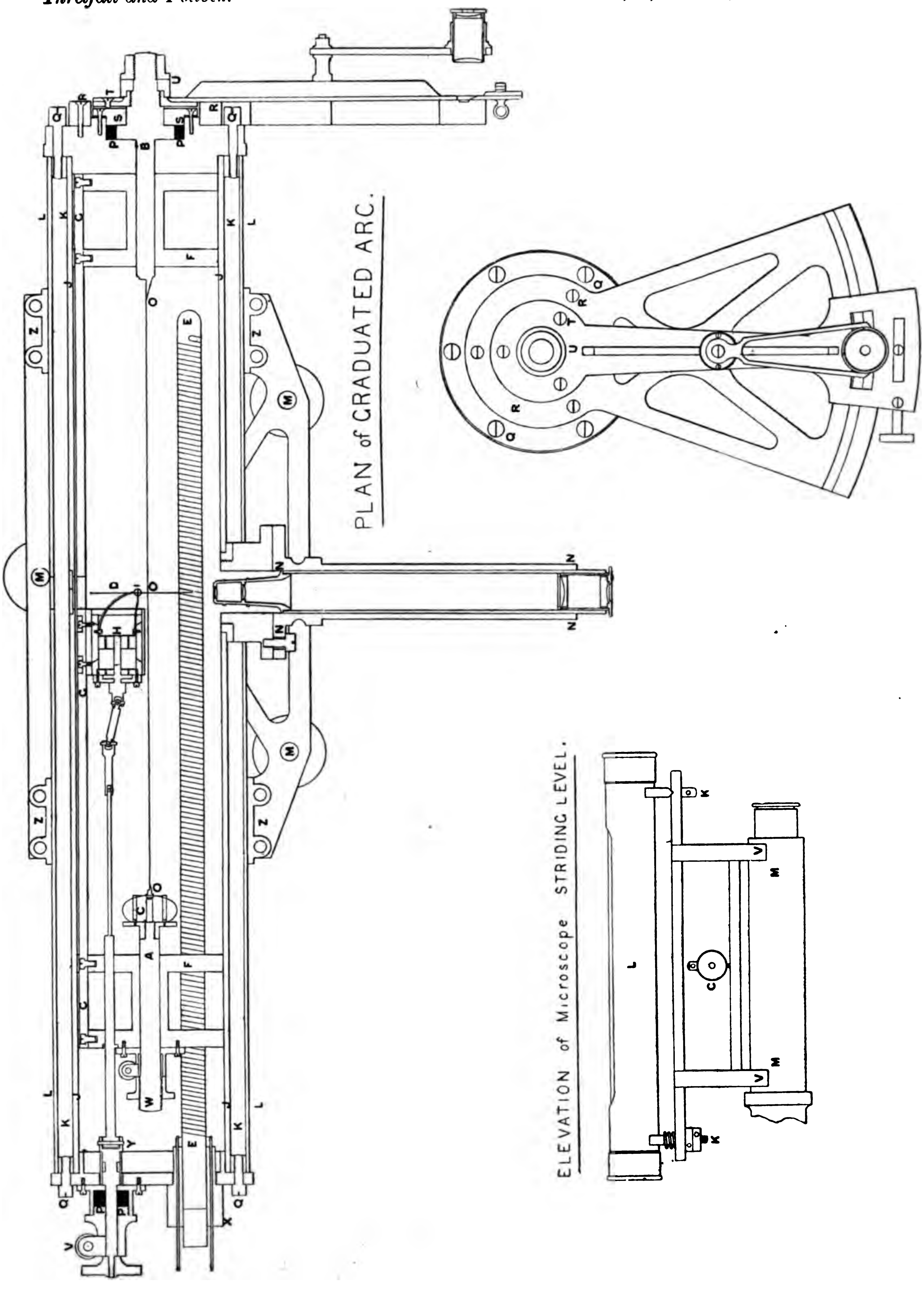
A. Axle carrying rosette spring. B. Axle carrying vernier arm. C. Rosette spring. D. Wire lever.
EE. Glass tube carrying platinum wire thermometer wound in a double spiral.
FF. Main bearings. GGG. One of the three bars holding the bearings together.
H. Pinion of arrester working into two racks carrying the arrester jaws. I. Jaws of the arrester.
JJJJ. Copper tube bored to fit the main bearings and rigid with respect to the inner mechanism.
KKKK. Paper insulation round the copper tube. LLLL. Outer tube of polished brass.
MMM. Levelling screws forming the supports of the balance on the tripod stand.
NNNN. Ground and lapped tube carrying the microscope.
OOO. Points at which connection is made to the thread by soldering.
PPPP. Mercury stuffing boxes on the vernier and arrester shafts.
QQQQ. (Also in the plan of the graduated arc.) Screws attaching the inner mechanism to the copper tube.
RR. (Also in the plan of the graduated arc.) Part of the framework of the webbing of the sextant arc.
SS. Cap of mercury stuffing box on vernier axle.
T. (Also in plan.) One of the screws fastening the vernier arm to a strengthening piece screwing to the vernier axle.
U. (Also on the plan of the graduated arc.) Clamping washer and lock nut with left-handed thread to insure rigid connection between the vernier arm and the axle.
V. Clamp screw of arrester shaft.
W. Slow-motion screw used in the preliminary adjustment of the tension of the thread, afterwards clamped by the clamp screw and soldered up to an invariable position with respect to the frame.
Y. Box key on the arrester shaft. This key can be drawn out by unscrewing the stuffing box from the end of the balance case. The drawing shows the box key too large to pass through the hole, this is wrong.
ZZZZ. Screw clamps or clips holding the balance proper to the under frame.

Index to Lettering of the Elevation of the Microscope and Striding Level.

L. Level. KK. Adjustments for the level tube. C. Cross level.
VV. The sides of the V grooves of the level carriage. MM. Worked tube carrying the microscope.

PLATE 2.

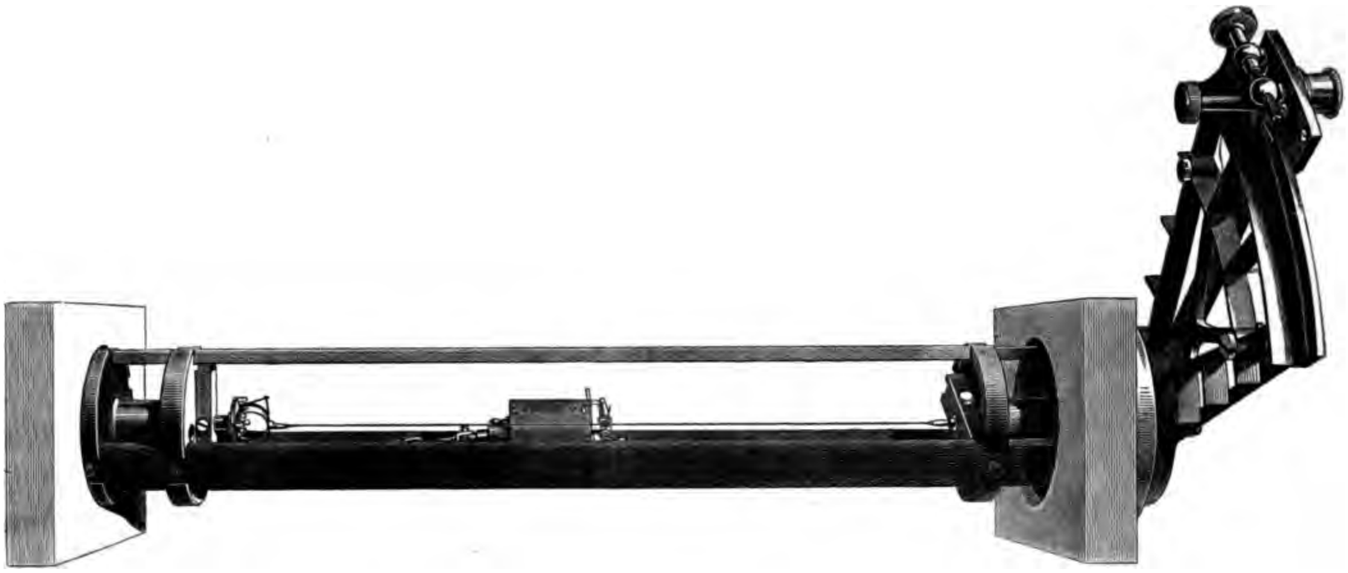
Reproduction of photographs of the apparatus.

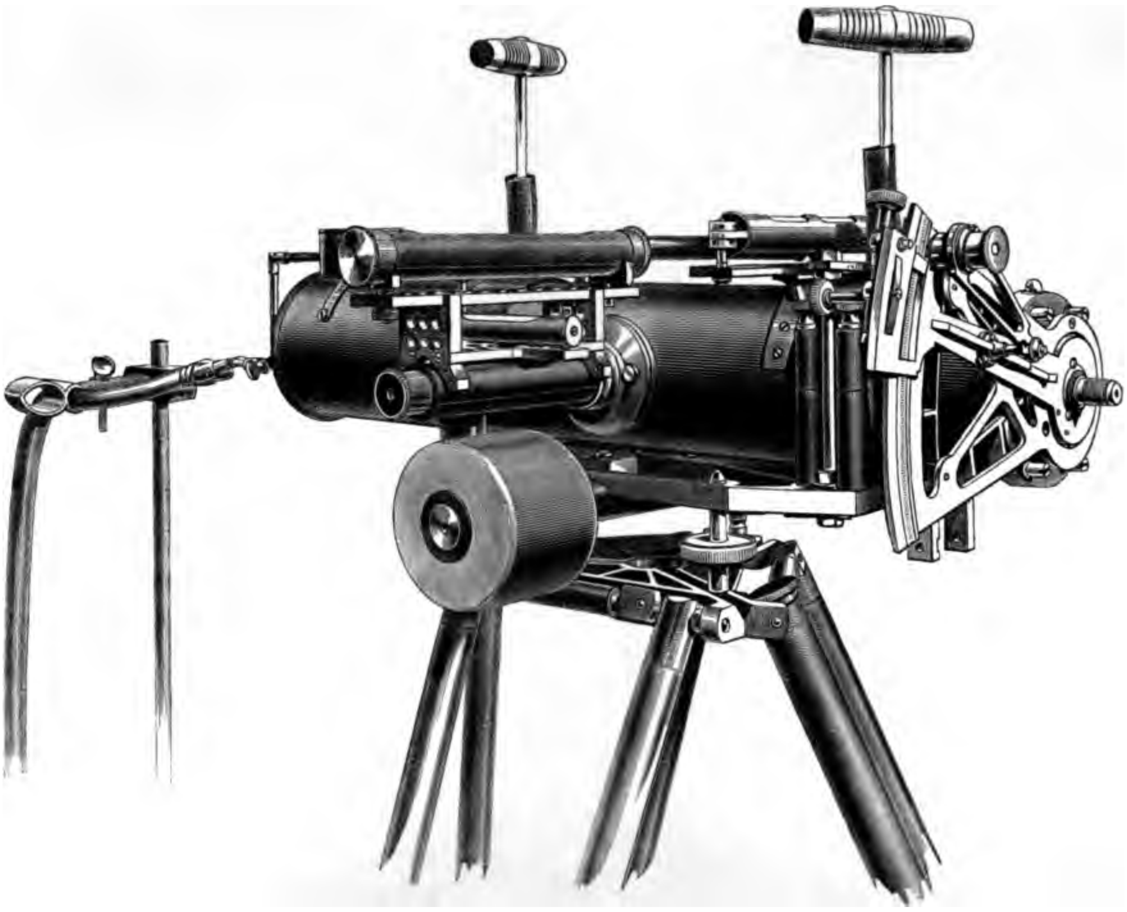


HORIZONTAL SECTION of GRAVITY BALANCE.— $2/5^{ths}$ SIZE.

PLAN of GRADUATED ARC.

ELEVATION of Microscope STRIDING LEVEL.





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VIII. *The Colour Sensations in Terms of Luminosity.*

By Captain W. DE W. ABNEY, *C.B.*, *D.C.L.*, *F.R.S.*

Received February 23,—Read June 15, 1899.

(I.) *Introductory.*

MY attention has been directed recently to the theoretical considerations involved in the production of photographs in approximately the colours of Nature, by combining together the images from three positives backed by appropriate colour screens, the colours chosen being those which should best represent the three Colour Sensations of the Young Theory.

During my investigations into the matter I found it necessary to ascertain what these colours were, for although serious objections may be raised to the Young Theory when considering it in detail, yet when expressed in a general form it adequately explains the phenomena which arise when colours fall upon the centre of the retina. The sensation curves have been given by KŒNIG, but it appeared that a redetermination by a luminosity method might well be undertaken, for they did not altogether agree with the results of some preliminary measures that I had made in order to trace them. In my work on 'Colour Vision' I have given a rough diagram as to what the sensation curves might be when they are shown as luminosities which together make up the total luminosity of the spectrum of the crater of the arc light, but it was only intended to be an approximation to the correct diagram. The method, however, by which the problem could be attacked and by which a rigid determination could be made was indicated. It is by this method that the results given in the following pages have been obtained.

(II.) *A Preliminary Survey.*

The red sensation can be perceived in purity at one end of the spectrum. From the darkest red to a point near the C line, a little above the red lithium line, the colour is the same, though, of course, the brightness varies, but the brighter red colour can be reduced to form an exact match with the dark red, and no mixture of any colours will give a red of the description we find at the end of the spectrum.

At the violet end of the spectrum we also find that the colour is the same throughout, from the extreme visible limit to a point not far removed from G, but it is not for this reason to be accepted that the colour is due to only one sensation. It might be due to two or three sensations if they were stimulated in the same proportions along that region, and if the identical colour could be produced by the combination of other colours. Experiment shows that a combination of two colours will make violet under certain conditions, and that instead of a simple sensation of violet we have in this region a blue sensation combined with a large proportion of red sensation. The proof of this and the estimation of the percentage composition of the violet will be given subsequently. It may, however, be here stated, that if we know the percentage composition we may provisionally use this part of the spectrum as if it excited but one sensation, and subsequently convert the results obtained with it into the true sensations. Thus in calculating the percentage of red in any colour, that existing in the provisional violet sensation would have to be added to it, and the same amount be abstracted from the violet to arrive at the true blue sensation. The green sensation would remain unaltered. In the first part of this paper the provisional violet sensation will be employed and the necessary corrections subsequently made.

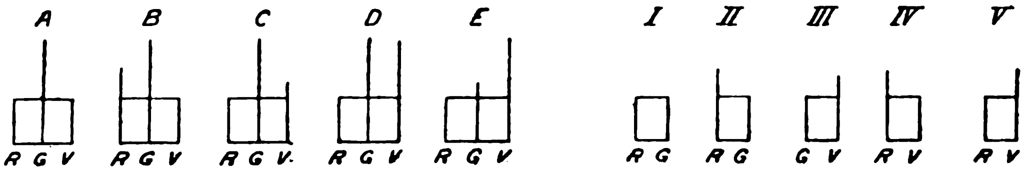
In using the violet it must be recollected that the colour is usually contaminated with the white light which illuminates the prism or grating, and that such illumination may be very appreciable at a part of the spectrum where the luminosity is very small. White light must therefore be cut off as far as practicable, and by use of an absorbing medium such as blue glass coated with a gelatine film dyed with a blue dye this is attained. The use of a second prism in front of the spectrum is inconvenient though effectual.

(III.) *Possible Mixtures of Sensations.*

Having at one end of the spectrum a pure red sensation, and at the other mixed sensations, due to the stimulation of the red and a blue sensation, it remains to isolate the green sensation. Owing to the overlapping of the curves in the green of the spectrum, due to the fact that this region stimulates all three of the sensations, the effect of the pure green sensation is never experienced by a normal eye, though presumably it is by what are termed the red-blind of the Young Theory. In any colour where the stimulation of all three sensations occurs, there must be always an admixture of white light, and we have to search for that point in the spectrum where white alone is added to the green sensation.

The following diagram, fig. 1, will show the variations in composition of a colour that may be met with. The provisional use of a violet sensation will not alter the argument, since, as before said, we may replace it by blue and red sensations. The different figures are purely diagrammatic. They are constructed on the supposition that equal heights of line above the base show the stimulation necessary to give the effect of white light. The scale applicable to each of the three lines is necessarily quite different to the scale of luminosity, that of the violet in particular is very greatly exaggerated. A, B, &c., mean that colours may exist each containing a sensation of white, the amount being shown by the portions between the horizontal parallel lines. I, II, and III, &c., show colours which are due to the mixtures of two sensations. A and D are the two most interesting colours. If we take away the green sensation from A we have the mixture of red and violet which a green-blind person would match with white. Similarly, if we take away the red sensation from D we have a mixture of green and violet which the red-blind person would match with white.

Fig. 1.



The position which D occupies in the spectrum can readily be found by the normal eye, by finding that colour which, with red alone added, matches the white employed, in other words by finding the complementary colour to the red. The position of A in the spectrum is much less readily determined by the normal eye, since it requires the addition of both red and violet to make the white, a condition which is also necessary with B and C. The position can of course be determined by the aid of the green-blind eye, but a preliminary measurement of colour sensations involving the assumption of its position enables it to be fixed with the required accuracy. Before the measurements herein recorded were made such a preliminary set of observations was gone through, and the position found which was subsequently confirmed by a green-blind person. (See XXII.)

(IV.) *Precautions to be taken.*

There were several considerations that had to be taken into account in making these measures. In the first place, the white light used in the observations had to be of the same quality, that is, the relative luminosities of the different rays of the spectrum had to be constant, for it must not be supposed that the positions in the spectrum of A and D are fixed points except for the same quality of white light. They may be separated from one another by nearly the same interval in the spectrum when different qualities of white light are measured, but the larger the proportion of blue contained in the white the more they will approach the more refrangible end of the spectrum. For instance, the positions will be nearer the red with the white light emitted by the crater of the positive pole of the electric light than they would be with light of the sun on a June day near noon. Again, the final equations, for white light, given in terms of the three sensations will vary according to the white light employed, and they will also vary according to the extent of the area of the retina on which the colour images fall. This last variation is caused by the absorption by the macula lutea, and may differ in different eyes. If, however, we can express the colours in all parts of the spectrum in *percentages* of luminosity of the three sensations, we can readily convert the equation derived for one quality into that for any other.

Bearing in mind the effect of the retinal area, it will be seen that it is necessary always to compare the patches of mixed colours when of the same size and viewed from the same distance, and with the centre of the retina. The light from the crater of the positive pole of the arc light, being always of the same quality and of great "whiteness," and giving a spectrum which is rich in blue rays, may be conveniently adopted as a standard. Moreover, this white seems to be of the same quality as the white light seen outside the colour fields, thus approaching to the fundamental white sensation.

(V.) *Method of Finding the Value of the Red and Green Sensations.*

It need only be stated that no blue sensation was found from D (sodium line) to the extreme limit of the spectrum in the red, and that in the yellow the amount found is very small compared with that of the red and green sensations. So below D to the red lithium line we have a mixture, in varying proportions, of pure red and green sensations, and from D to the yellow-green the same two sensations, but not absolutely free from the third sensation.

A colour in the spectrum which matches a solution of bichromate of potash will thus only excite the red and green sensations, and if we can find out in what proportions the two exist as luminosities in a given luminosity of the colour, we can readily determine the sensation luminosity composition of any other colours by means of ordinary colour equations expressed as comparative luminosities. To ascertain the composition of such an orange colour was the object of the first part of the investigation.

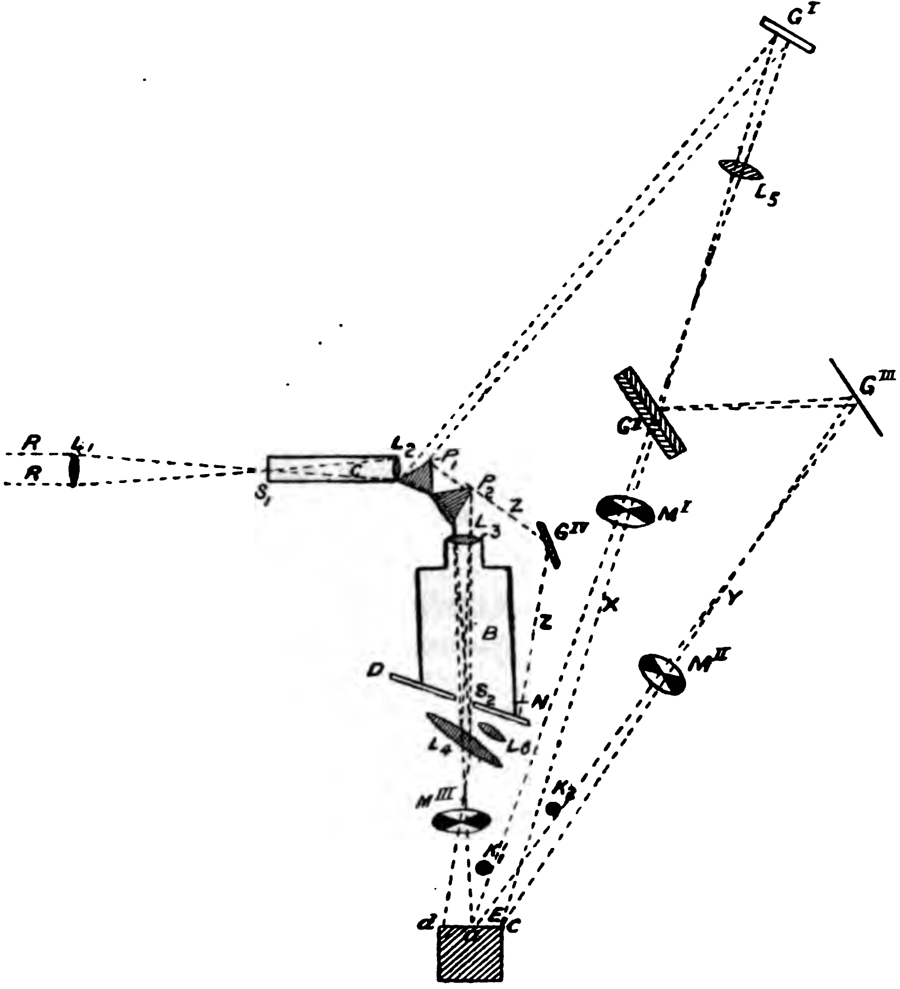
Turning to II, in fig. 1, we see that we have only to add to the colour it represents such a quantity of properly chosen violet to form white, the red and green sensations being present in the proper proportions. If, therefore, we ascertain the complementary colour to the violet, we shall find the colour which is equivalent to II, and this we find to be in the yellow. Having ascertained the luminosity of the red and green sensations in the orange, and from them the relative luminosities of the same two sensations in the complementary colour to the violet, we at once get in terms of the red, the green, and the (provisional) violet sensations the equation to the white light of the quality we may be using.

(VI.) *Apparatus Employed.*

The colour patch apparatus employed in 'Colour Photometry' was again used (fig. 2). The rays R, R, coming from the crater of the positive pole of the electric light, were collected by a lens, L_1, and an image of the crater thrown on the slit S_1.

After passing through the collimator C, the rays emerged as parallel rays; part passed through the prisms P_1 and P_2, were collected by a lens, L_3, and a spectrum was formed on a slide, D (which will be more fully described), in which slits could be placed, and an image of the surface of the first prism was formed on the white-red surface of a cube, E, by means of the lens L_4, so arranged that the image of one edge of the prism fell at *a*, the other edge falling outside *d*. The other beam which passed through the collimator was reflected from the surface of the first prism to a mirror, G^I, and passed through a lens, L_5, then through a bundle of glass, G^{II}, placed at an angle to the beam, and on to the surface *dc* of the cube, a rod, K_1, being placed in its path, to secure that this white beam did not fall on *ad*, on which the colour mixture fell. The portion of the beam which was reflected from G^{II} was again reflected by G^{III}, a silvered mirror, on to *ac*, a rod, K_2, being placed in its path to prevent it falling on *ad*. In all three beams, sectors, M^I, M^{II}, and M^{III}, were placed, to allow any or all to be reduced in intensity at pleasure. In the beams X and Y any absorbing medium desired could

Fig. 2.



be placed. A small ray of light, Z, was allowed to pass beyond P_2, and fell on a small mirror, G^{IV}, which reflected it on to the back of D, casting a shadow of a needle, N, fixed to B, the camera, on S, a scale at the back of D.

L_6 is a lens of short focus which could be moved into a fixed position behind L_4 to throw an enlarged image of the slit on a scale placed below *dc*.

In order to form colour mixtures on *ad* three slits had to be placed along the surface D. The slits were arranged in an open brass frame which slid along the plane D in grooves cut in B. At the bottom and top of the frame or slit holder two pairs of grooves were cut. In the front pair the slits could slide and be clamped in any desired position as determined by a scale engraved along the lower groove, whilst the back pair of grooves was used to hold blackened cards which filled up the intervals between the slits. The position of the slit holder was determined by the shadow cast by the needle N on the scale engraved on its back.

(VII.) *Ascertaining the Position of the Slits in the Spectrum.*

By placing one slit at some fixed number on the front scale and then causing the slit holder to move along the spectrum till known lines (due to metals vaporised in the arc) passed through the centre of the aperture, and there noting the scale number at the back, the position of the slits however placed was known. The position of the principal Fraunhofer lines in regard to the front scale was thus determined when the slit holder was placed in a fixed position as indicated by the needle shadow.

(VIII.) *Method of Determining the Colour Sensations in an Orange Ray of the Spectrum.*

It has already been stated that from a preliminary survey it was found that in the orange no blue sensation was excited, and that only red and green had to be determined. Further, it was stated that A, fig. 1, was a colour where the green sensation existed mixed only with white. If these two slits were placed in the spectrum, one in the pure red which only excited the red sensation, and the other at A, it should be possible to make such a mixture of the two colours that they should match the orange colour to which white in known quantity was added. A cell containing bichromate of potash in solution was placed in the beam X, and one slit in D (fig. 2) was caused to traverse the spectrum till the colours appeared to match. It was found, however, that the bichromate was a little paler than the orange of the spectrum, and the beam Y was diverted so that it fell only on *ad*. The white was diminished till a match was secured and the luminosities of the two were measured, when it was found that the bichromate colour contained 4·995 per cent. of white as compared with the orange that matched it.

The bichromate solution could now be used to give a colour to be matched. The slits were placed at Slit Scale Nos. (afterwards designated as SSN) 205 and 288·5 (the latter number having been found by preliminary trials to be A), and a mixture of the two beams fell on *ad*, making as near a match as possible with the colour of the bichromate solution placed in the beam X. To the latter was added white from the beam Z, and by altering the sectors and slits a perfect match could be obtained. This being effected the width of the slits was measured by the method which has been indicated in describing the apparatus. The small lens, L_6, was pushed in position to the centre of the lens, L_4, and the slits successively brought into the colour which passed through the centres of these two lenses by sliding the slit holder along the spectrum. A magnified and sharp image in monochromatic light was then thrown on the scale below, *dc*,. and the relative width of the apertures noted. This plan obviates recourse to wedges for measuring, and is very convenient. It has been employed by myself for measuring pin holes and other small apertures. The slits having been measured they were replaced in position, and the luminosities measured as has been described in 'Colour Photometry,' Part I.* Other matches were made and the aperture of the slit again measured, but the luminosity not necessarily, as the relation between width of slit and luminosity was determined from the first observations. Had the area of the retina on which the image fell been the same as that employed in 'Colour Photometry,' Part III.,† and if the quality of the white had not been slightly changed by the interposition of the bundle of glass, G^{III}, the luminosities might have been taken from the tables in that paper.

The form of the equation then becomes of this kind—

$$m(\mathrm{R}) + n(\mathrm{G}) = a[(\mathrm{Or}) + (w)] + b(\mathrm{W}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{i.}),$$

where R, G, Or, W, and w stand for the red, the green, the orange, the added white, and the white in the bichromate solution colour respectively, and m, n, a, b constants.

Now as the orange can contain but two sensations, and as the red is a pure sensation and consequently not contaminated with white, it follows that $a(w)$ and $b(\mathrm{W})$ must be in $n\mathrm{G}$, and we get

$$m(\mathrm{R}) + [n(\mathrm{G}) - a(w) - b(\mathrm{W})] = a(\mathrm{Or}) \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{ii.}).$$

That is, when the luminosity of the white is deducted from the luminosity of the green we get the green sensation left, and, finally, we get

$$m(\mathrm{RS}) + n'(\mathrm{GS}) = a(\mathrm{Or}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{iii.}).$$

where RS and GS are the red and green sensations respectively.

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* 'Phil. Trans.,' A, 1889.

† *Ibid.*, 1896.

(IX.) *Method of Determining the Red and Green Sensations in the Yellow which is complementary to the Violet.*

We are now in a position to determine the RS and GS, in that yellow is complementary to the violet, which, in this as in the previous case, we are assuming to be a simple sensation. From the preliminary observations we know that the amount of violet in the yellow is very minute, and is for our purpose here negligible. If one slit be placed in this particular yellow and another in the red near the lithium line, and the two colours be mixed to match the spectrum orange, we get

$$p\,(\mathrm{Y}) + q\,(\mathrm{R}) = c\,(\mathrm{Or}) \quad . \; . \; . \; . \; . \quad . \; . \quad \text{(iv.)},$$

where Y signifies the yellow. But

$$m\,(\mathrm{RS}) + n'\,(\mathrm{GS}) = a\,(\mathrm{Or}),$$

therefore

$$\frac{p}{c}(\mathrm{Y}) = \left(\frac{m}{a} - \frac{q}{c}\right)(\mathrm{RS}) + \frac{n'}{a}(\mathrm{GS}) \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(v.)},$$

and we have this particular yellow expressed in terms of the luminosity of the two sensations. Measurements made with a mixture of yellow and violet give the equation

$$r\,(\mathrm{Y}) + t\,(\mathrm{V}) = \mathrm{W} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(vi.)}.$$

Substituting from v. we get an equation of the form

$$\alpha\,(\mathrm{RS}) + \beta\,(\mathrm{GS}) + \gamma\,(\mathrm{V}) = 100\,(\mathrm{W}) \quad . \; . \; . \; . \; . \; . \quad \text{(vii.)},$$

and this becomes the standard equation for the particular white employed.

If we take another green which does not answer to A, fig. 1, we get

$$f\,(\mathrm{RS}) + g\,(\mathrm{G}) + k\,(\mathrm{V}) = 100\,\mathrm{W}. \quad . \; . \; . \; . \; . \; . \; . \quad \text{(viii.)}.$$

From vii. and viii. we get

$$100\,(\mathrm{G}) = \alpha'\,(\mathrm{RS}) + \beta'\,(\mathrm{GS}) + \gamma'\,(\mathrm{V}),$$

and this is the percentage composition in luminosities of this particular green.

This method applies for every spectrum colour up to that which answers to D, fig. 1, but it is not quite so well adapted for colours which lie on the blue side of this point.

(X.) *Method of Determining the Composition of the Rays in the Blue End of the Spectrum.*

The most ready method of determining the percentage composition of the colours which lie between *b* and the violet, is as follows: a slit is placed in position to allow a blue of a natural wave-length to pass, and a second slit is placed at the SSN which corresponds to the yellow, whose composition is already determined. No mixture of blue and this yellow will make a white corresponding to that we have to compare with it, but the slit-holder can be moved towards the red till a match is made, *i.e.*, where the slightly redder yellow is complementary to the blue which passes through the first slit. The shift of the slit-holder from the fixed point is noted, and from it are calculated the new positions of the two slits. From the previous measures made in the red orange and green portions of the spectrum, the percentage composition in red and green sensation-luminosity is known, and the luminosity of the light coming through the slit in the redder yellow is divided proportionally between these two sensations, and we have an equation of the form

$$\alpha''(\mathrm{RS}) + \beta''(\mathrm{GS}) + \gamma''(\mathrm{B}) = 100\,(\mathrm{W}).$$

The standard equation (vii.) is used as before, and we get the blue (B) in terms of the two sensations and the violet.

(XI.) *Method of Determining the Composition of the Violet.*

The last determination that has to be carried out is the composition of the violet. We already know that up to a point near G it is uniform in colour. But if we place a slit in the blue near the blue lithium line, and another in the red near the red lithium line, and endeavour to match the violet, we shall find that although we get a purple, yet it is too pale; a third slit is placed in the violet, and by a right-angled prism the beam is diverted and again deflected to fall on *da*, fig. 2, and the white beam Z is also reflected to fall on the same part of the white surface. The mixture of red and blue falls on *ac*. White is added to the violet till a match is made. The luminosities of all the colours and the white are compared together, and an equation is formed of this form

$$a\,(\mathrm{V}) + b\,(\mathrm{W}) = m\,(\mathrm{B}) + n\,(\mathrm{R}).$$

Now, as the red contains no white sensation, that shown on the left-hand side of the equation must be found in the blue, a proportion of blue, green and red going to form it. All the green sensation must be "used up" in forming the white, and only the blue sensation and the red sensation can remain beyond the white. We thus get

$$\begin{aligned} a\,(\mathrm{V}) &= n\,(\mathrm{RS}) + [m\,(\mathrm{B}) - b\,(\mathrm{W})] \\ &= n\,(\mathrm{RS}) + m'\,(\mathrm{BS}). \end{aligned}$$

By taking colours on each side of the blue lithium line we find that the proportions of blue and red sensations are unchanged, and always fulfil the above equation.

Having found the percentage composition of all the spectrum in terms of the red and green sensations and of violet, the last is converted into blue sensation and red sensation. The red sensation existing in the violet is then added to that already found.

(XII.) *Difficulties in making the Observations.*

The description of the nature of the observations may make it appear that they are simple, but the reverse is the case. The labour involved is very great, and the difficulty soon becomes apparent when the work has fairly started. The sensitiveness of the eye to colour varies considerably, and this in itself makes observations hard. On some mornings, when coming fresh to the laboratory, the comparisons are readily made, but those made in the evenings after a day's work are often wild at first, and much more time has to be spent in perfecting them than may be supposed. Before any match can be considered worthy of recording, the eye has to be withdrawn from the light and to look into darkness for a minute at least, when a rapid glance will show if it needs alteration. If not correct, the slits have to be opened or closed, as may be required, and again a rest given to the eye. This procedure may be repeated several times before the match is considered satisfactory. The fatigue of the retina has a good deal to say to the difficulties encountered.

(XIII.) *Order of the Observations.*

It may be as well to record the order in which the observations were made. The first are preliminary, and are as follows :—

(1.) The position of the spectrum in regard to the slit-holder is determined.
(2.) The scales at the back and front of the slit-holder are compared.
(3.) The lens with which the apertures of the slits are measured is adjusted.

The second are those taken for recorded observations :—

(1.) The slits are placed in position.
(2.) The matches are made.
(3.) The luminosities of the light coming through the slits are measured and the apertures of the sectors noted.
(4.) The widths of the slits are measured.

2 and 4 had to be repeated several times in each series of observations. I have already shown, in "Colour Measurement and Mixture," that a certain percentage of coloured light can be hidden in white without being perceived. In making a match with the white, each slit had to be opened in turn till it was evident that an excess

of coloured rays was issuing from it, when it was closed till the match was made. Again it was closed till it was evident that the colour was in defect, when it was opened till the match appeared correct. A mean of six observations was taken as being the most probable value of the mixture.

(XIV.) *The Red and Green Sensations Unmixed with Violet.*

To find the luminosities of the red and green sensations in the orange which matches a solution of bichromate of potash.

$$(\text{Bich}) + \overset{\text{W}}{16\cdot5} = \overset{\text{R}}{45\cdot5} + \overset{288\cdot5}{32}.$$

In the bichromate there is $\overset{\text{W}}{4\cdot5}$. Therefore

$$(\text{Or}) + \overset{\text{W}}{21} = \overset{\text{R}}{45\cdot5} + \overset{288\cdot5}{32}.$$

$$(\text{Or}) = \overset{\text{RS}}{45\cdot5} + \overset{\text{G}}{(32 - 21)}$$

$$= \overset{\text{RS}}{45\cdot5} + \overset{\text{GS}}{11}.$$

Therefore

$$(1)\qquad 100\ \text{Or} = \overset{\text{RS}}{80\cdot53} + \overset{\text{GS}}{19\cdot47}.$$

Taking another example in detail, the slits being at 205 (R) and 270 (G),

$$(\text{Bi}) + \overset{\text{W}}{8\cdot5} = \overset{\text{R}}{39\cdot95} + \overset{270}{24\cdot5}.$$

$$(\text{Or}) + \overset{\text{W}}{4\cdot5} + \overset{\text{W}}{8\cdot5} = \overset{\text{R}}{39\cdot95} + \overset{270}{24\cdot5}.$$

$$(\text{Or}) = \overset{\text{R}}{39\cdot95} + \overset{270}{24\cdot5} - \overset{\text{W.}}{13}$$

$$= \overset{\text{R}}{39\cdot95} + \overset{270'*}{11\cdot5}.$$

$$(2)\qquad 100\ (\text{Or}) = \overset{\text{R}}{77\cdot6} + \overset{270'}{22\cdot4}.$$

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* 270′ means 270 deprived of all violet.

It was found in forming the equations to match white that

$$\begin{array}{ccccccc} \text{W} & & \text{R} & & 288\cdot5 & & \text{V} \\ 100 & = & 47\cdot06 & + & 51\cdot79 & + & 1\cdot15. \end{array}$$

Also

$$\begin{array}{ccccccc} \text{W} & & \text{R} & & 270 & & \text{V} \\ 100 & = & 39\cdot30 & + & 59\cdot15 & + & 1\cdot55. \end{array}$$

From which equation we find that

$$\begin{array}{ccccc} 270 & & \text{R} & & 288\cdot5-\text{V} \\ 22\cdot4 & = & 2\cdot94 & + & 19\cdot46. \end{array}$$

$$\begin{array}{ccccccccc} \text{R} & & 270' & & \text{R} & & \text{R} & & 288\cdot5\ \text{V} \\ 77\cdot6 & + & 22\cdot4 & = & 77\cdot6 & + & 2\cdot94 & + & 19\cdot46. \end{array}$$

that is in (2).

Or

$$(3)\qquad \begin{array}{ccccc} & & \text{RS} & & \text{GS} \\ 100\ (\text{Or}) & = & 80\cdot54 & + & 19\cdot46. \end{array}$$

Similarly it was found from slits placed at 205 and 283 that

$$(4)\qquad \begin{array}{ccccc} & & \text{RS} & & \text{GS} \\ 100\ (\text{Or}) & = & 80\cdot50 & + & 19\cdot50. \end{array}$$

Other determinations gave a mean of

$$(5)\qquad \begin{array}{ccccc} & & \text{RS} & & \text{GS} \\ 100\ (\text{Or}) & = & 80\cdot50 & + & 19\cdot50. \end{array}$$

The (Or) has SSN (·236), therefore

$$\begin{array}{ccccc} (236) & & \text{RS} & & \text{GS} \\ 100 & = & 80\cdot50 & + & 19\cdot50. \end{array}$$

(XV.) *Determination of the General Provisional Equation for White.*

Now the complementary colour to the violet at SSN 390 is SSN 245, and it was found that

$$\begin{array}{ccccc} 236 & & \text{RS} & & 245 \\ 100 & = & 35\cdot86 & + & 64\cdot14. \end{array}$$

But

$$\begin{array}{ccccc} 236 & & \text{RS} & & \text{GS} \\ 100 & = & 80\cdot50 & + & 19\cdot50. \end{array}$$

Therefore

$$(6)\qquad \begin{array}{ccccc} 245 & & \text{RS} & & \text{GS} \\ 100 & = & 69\cdot60 & + & 30\cdot40. \end{array}$$

Also it was found that

$$\overset{\text{W}}{100} = \overset{245}{98{\cdot}32} + \overset{390}{1{\cdot}68}.$$

Therefore

$$(7)\qquad \overset{\text{W}}{100} = \overset{\text{RS}}{68{\cdot}42} + \overset{\text{GS}}{29{\cdot}90} + \overset{\text{VS}}{1{\cdot}68},$$

which is the equation to white in terms of RS, GS, and the provisional violet sensation.

(XVI.) *Determination of the Sensation Values in the Orange, Yellow, and Green of the Spectrum.*

The slits placed at SSNs 220, 285, and 390.

$$\overset{220}{48{\cdot}27} + \overset{285}{50{\cdot}52} + \overset{390}{1{\cdot}21} = \overset{\text{W}}{100}.$$

But as will be seen subsequently

$$\overset{285}{100} = \overset{\text{RS}}{42{\cdot}71} + \overset{\text{GS}}{56{\cdot}37} + \overset{390}{{\cdot}92}.$$

Therefore

$$\overset{220}{48{\cdot}27} + \overset{\text{RS}}{21{\cdot}577} + \overset{\text{GS}}{28{\cdot}472} + \overset{390}{{\cdot}465} + \overset{390}{1{\cdot}21} = \overset{\text{W}}{100}.$$

Or

$$(8)\qquad \overset{220}{100} = \overset{\text{RS}}{97{\cdot}04} + \overset{\text{GS}}{2{\cdot}96}.$$

There is a small residuum of V equal to ·01 left, but as it is non-existent at this part of the spectrum it is added to the GS.

Slits at SSNs 228, 294, and 390.

$$\overset{228}{59{\cdot}80} + \overset{294}{39{\cdot}86} + \overset{390}{{\cdot}34} = 100.$$

$$\overset{294}{100} = \overset{\text{RS}}{37{\cdot}57} + \overset{\text{GS}}{59{\cdot}08} + \overset{390}{3{\cdot}35}.$$

From which we get

$$(9)\qquad \overset{228}{100} = \overset{\text{RS}}{89{\cdot}37} + \overset{\text{GS}}{10{\cdot}63}.$$

The percentage composition of SSN 236 we have already found as

$$\overset{236}{100} = \overset{\text{RS}}{80{\cdot}50} + \overset{\text{GS}}{19{\cdot}50}.$$

Slits at SSNs 240 and 205 to match 236.

$$\overset{236}{100} = \overset{205}{22{\cdot}24} + \overset{240}{77{\cdot}76}.$$

But

$$\overset{230}{100} = 80{\cdot}50 + 19{\cdot}50.$$

Therefore

$$\overset{240}{100} = \overset{\text{RS}}{74{\cdot}92} + \overset{\text{GS}}{25{\cdot}08}. \tag{10}$$

The percentage composition of 245 has already been found.

$$\overset{245}{100} = \overset{\text{RS}}{69{\cdot}60} + \overset{\text{GS}}{30{\cdot}40}.$$

Slits placed at SSNs 205, 250 and 390.

$$\overset{\text{RS}}{13{\cdot}78} + \overset{250}{84{\cdot}56} + \overset{390}{1{\cdot}66} = \overset{\text{W}}{100}.$$

But

$$\overset{\text{RS}}{68{\cdot}42} + \overset{\text{GS}}{29{\cdot}90} + \overset{390}{1{\cdot}68} = 100.$$

Therefore

$$\overset{250}{100} = \overset{\text{RS}}{64{\cdot}62} + \overset{\text{GS}}{35{\cdot}36} + \overset{\text{V}}{{\cdot}02}. \tag{11}$$

In a similar way the following percentages were found.

$$\overset{260}{100} = \overset{\text{RS}}{55{\cdot}68} + \overset{\text{GS}}{44{\cdot}17} + \overset{\text{V}}{{\cdot}15}. \tag{12}$$

$$\overset{270}{100} = \overset{\text{RS}}{49{\cdot}23} + \overset{\text{GS}}{50{\cdot}55} + \overset{\text{V}}{{\cdot}22}. \tag{13}$$

$$\overset{275}{100} = \overset{\text{RS}}{46{\cdot}75} + \overset{\text{GS}}{52{\cdot}89} + \overset{\text{V}}{{\cdot}36}. \tag{14}$$

$$\overset{280}{100} = \overset{\text{RS}}{44{\cdot}77} + \overset{\text{GS}}{54{\cdot}94} + \overset{\text{V}}{{\cdot}54}. \tag{15}$$

$$\overset{283}{100} = \overset{\text{RS}}{43{\cdot}63} + \overset{\text{GS}}{55{\cdot}60} + \overset{\text{V}}{{\cdot}77}. \tag{16}$$

$$\overset{285}{100} = \overset{\text{RS}}{42{\cdot}71} + \overset{\text{GS}}{56{\cdot}37} + \overset{\text{V}}{{\cdot}92}. \tag{17}$$

$$\overset{288{\cdot}5}{100} = \overset{\text{RS}}{41{\cdot}24} + \overset{\text{GS}}{57{\cdot}74} + \overset{\text{V}}{1{\cdot}02}. \tag{18}$$

$$(19)\qquad \overset{290}{100} = \overset{\text{RS}}{40{\cdot}53} + \overset{\text{GS}}{58{\cdot}28} + \overset{\text{V}}{1{\cdot}19}.$$

$$(20)\qquad \overset{294}{100} = \overset{\text{RS}}{37{\cdot}57} + \overset{\text{GS}}{59{\cdot}08} + \overset{\text{V}}{3{\cdot}35}.$$

(XVII.) *Determination of the Sensation Values in the Blue-Green.*

The next determinations were made with two slits, as before described; one being at SSN 245 and the other in the blue of different hues, with the following results. The first one is shown in detail. Slits at SSNs 312 and 245.

The frame was moved ·9 division of the scale towards the red, which was equivalent to placing the slits at SSNs 302·4 and 241·4 respectively, the back scale being ·4 of the front scale.

Now the increase in red and consequent diminution in the green sensation at 241·4 is 4·54.

The proportion of RS to GS is therefore (see Equation (6) for $\overset{245}{100}$)

$$(\overset{\text{RS}}{69{\cdot}60} + \overset{\text{RS}}{4{\cdot}54}) \text{ to } (\overset{\text{GS}}{30{\cdot}40} - \overset{\text{GS}}{4{\cdot}54}) \text{ or } 74{\cdot}14 \text{ to } 25{\cdot}86.$$

The luminosity equation is

$$\overset{241\cdot4}{87} + \overset{208\cdot4}{13} = \overset{\text{W.}}{100}.$$

From this we derive that

$$(21)\qquad \overset{308\cdot4}{100} = \overset{\text{RS}}{30{\cdot}15} + \overset{\text{GS}}{56{\cdot}85} + \overset{\text{V}}{13}.$$

Slits at 245 and 320.

The slit-holder was moved 5 of the back scale, which was equivalent to moving each slit 2 units towards the red. The slits were, therefore, actually at 243 and 318.

The increase in red being 1·25 per cent. per unit of scale, at this point it became 72·1 per cent. and the green 27·9 per cent.

The equation is

$$\overset{243}{92} + \overset{318}{8} = \overset{\text{W}}{100}.$$

Hence

$$(22)\qquad \overset{318}{100} = \overset{\text{RS}}{26} + \overset{\text{GS}}{53} + \overset{\text{V}}{21}.$$

Similarly the following equations were found :—

$$\overset{\text{329}}{100} = \overset{\text{RS}}{18{\cdot}8} + \overset{\text{GS}}{45{\cdot}3} + \overset{\text{V}}{35{\cdot}9}. \tag{23}$$

$$\overset{\text{339·6}}{100} = \overset{\text{RS}}{15} + \overset{\text{GS}}{30} + \overset{\text{V}}{55}. \tag{24}$$

$$\overset{\text{349·75}}{100} = \overset{\text{RS}}{9{\cdot}7} + \overset{\text{GS}}{19{\cdot}4} + \overset{\text{V}}{70{\cdot}9}. \tag{25}$$

$$\overset{\text{359·94}}{100} = \overset{\text{RS}}{4{\cdot}6} + \overset{\text{GS}}{9{\cdot}4} + \overset{\text{V}}{86}. \tag{26}$$

$$\overset{\text{370}}{100} = \overset{\text{RS}}{0} + \overset{\text{G}}{3} + \overset{\text{V}}{97}. \tag{27}$$

At 380 the violet was of the same hue as further in the spectrum.

This completes the observations made in the spectrum with the provisional violet sensation. It now remains to show how the composition of the violet was determined.

(XVIII.) *Determination of the Composition of the Violet.*

A slit was placed at SSN 205 and another at SSN 345, and mixture made to match a violet at SSN 400, to which white was added.

The equation was

$$\overset{\text{B}}{33} + \overset{\text{R}}{15{\cdot}2} = \overset{\text{V}}{21{\cdot}2} + \overset{\text{W}}{27},$$

$$\overset{\text{V}}{21{\cdot}2} = \overset{\text{R}}{15{\cdot}2} + \overset{\text{BS}}{6},$$

or

$$\overset{\text{V}}{100} = \overset{\text{RS}}{71{\cdot}7} + \overset{\text{BS}}{28{\cdot}3}. \tag{28}$$

Other equations made at 230, 235, 240, 250, gave the following :—

$$\overset{\text{V}}{100} = \overset{\text{RS}}{70} + \overset{\text{BS}}{30}. \tag{29}$$

$$\overset{\text{V}}{100} = \overset{\text{RS}}{72} + \overset{\text{BS}}{28}. \tag{30}$$

$$\overset{\text{V}}{100} = \overset{\text{RS}}{74{\cdot}5} + \overset{\text{BS}}{25{\cdot}5}. \tag{31}$$

$$\overset{\text{V}}{100} = \overset{\text{RS}}{73{\cdot}5} + \overset{\text{BS}}{26{\cdot}5}. \tag{32}$$

The adopted reading was

$$\begin{array}{lccc} & \text{V} & \text{RS} & \text{BS} \\ (33) & 100 = & 72{\cdot}5 + & 27{\cdot}5. \end{array}$$

(XIX.) *Corrected Colour Sensations.*

TABLE I.

This equation was applied to the foregoing percentages of violet in the different colours. The next table shows the provisional and the correct percentages.

In this and the following tables the former are shown in italics.

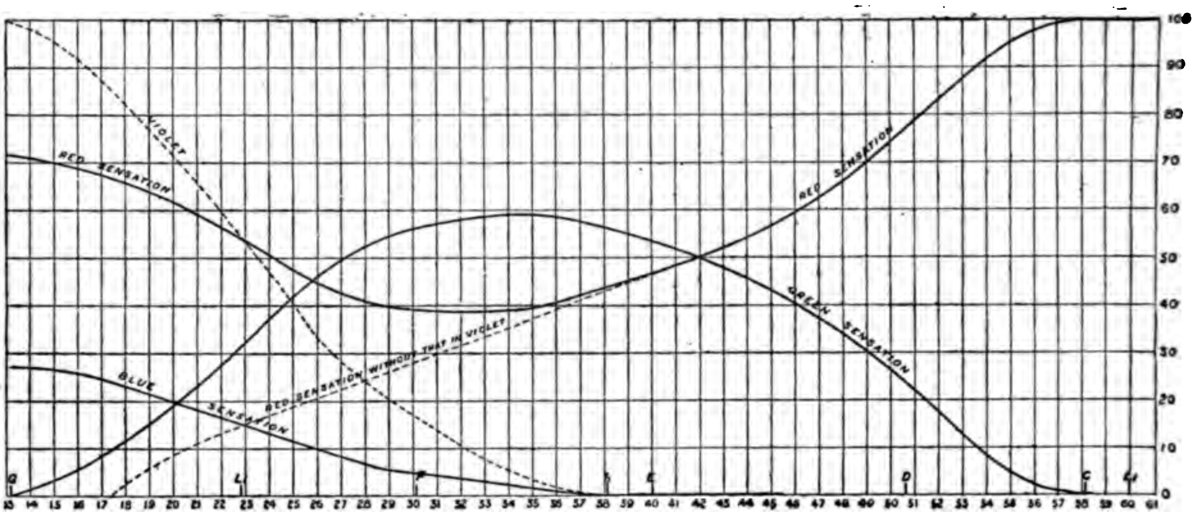
SSN.	Standard scale number.	Provisional sensations.			Corrected sensations.		
		Red.	Green.	Violet.	Red.	Green.	Blue.
205	59·83	*100*	*0*	*0*	100	0	0
215	57·07	*100*	*0*	*0*	100	0	0
220	55·70	*97·04*	*2·96*	*0*	97·04	2·96	0
228	53·48	*89·37*	*10·63*	*0*	89·37	10·63	0
236	51·27	*80·50*	*19·50*	*0*	80·50	19·50	0
240	50·17	*74·92*	*25·08*	..	74·92	25·08	..
245	48·80	*69·60*	*30·40*	..	69·60	30·40	..
250	47·42	*64·62*	*35·36*	*0·02*	64·63	35·36	0·01
260	44·67	*55·68*	*44·17*	*0·15*	55·79	44·17	0·04
270	41·92	*49·23*	*50·55*	*0·22*	49·38	50·55	0·07
275	40·55	*46·75*	*52·89*	*0·36*	47·01	52·89	0·10
280	39·17	*44·77*	*54·69*	*0·54*	45·16	54·69	0·15
283	38·35	*43·63*	*55·60*	*0·77*	44·18	55·60	0·22
285	37·80	*42·71*	*56·37*	*0·92*	43·37	56·37	0·26
288·5	36·83	*41·24*	*57·74*	*1·02*	41·97	57·74	0·29
290	36·42	*40·53*	*58·28*	*1·19*	41·39	58·28	0·33
294	35·32	*37·57*	*59·08*	*3·35*	40·00	59·08	1·92
308·4	31·37	*30·15*	*56·85*	*13·00*	39·57	56·85	3·58
318	28·73	*26·00*	*53·00*	*21·00*	41·22	53·00	5·78
329	25·71	*18·20*	*45·30*	*35·90*	44·83	45·30	9·87
339·6	22·78	*15·00*	*30·00*	*55·00*	54·87	30·00	15·13
349·75	20·00	*9·70*	*19·40*	*70·90*	61·10	19·40	19·50
359·9	17·17	*4·60*	*9·40*	*86·00*	66·95	9·4	23·65
370	14·42	*0*	*3*	*97·00*	70·32	3·0	26·68

Both sets of ordinates were drawn from the preceding table, and freehand curves drawn through them. (The standard scale is the same as in 'Colour Photometry,' Part III.) The results are shown in fig. 3. It will be noticed that, had the provisionally-employed violet sensation been correct, the percentage curves are very fairly regular, and altogether what would have been expected, but that in the true sensation curves the red sensation takes a form which is curious, and one which, from any theoretical considerations, would not be prophesied as probable. In every investigation, whether of colour fields or otherwise, the red sensation seems to be but little connected with the other two.

(XX.) *Sub-division of Luminosity into Sensations.*

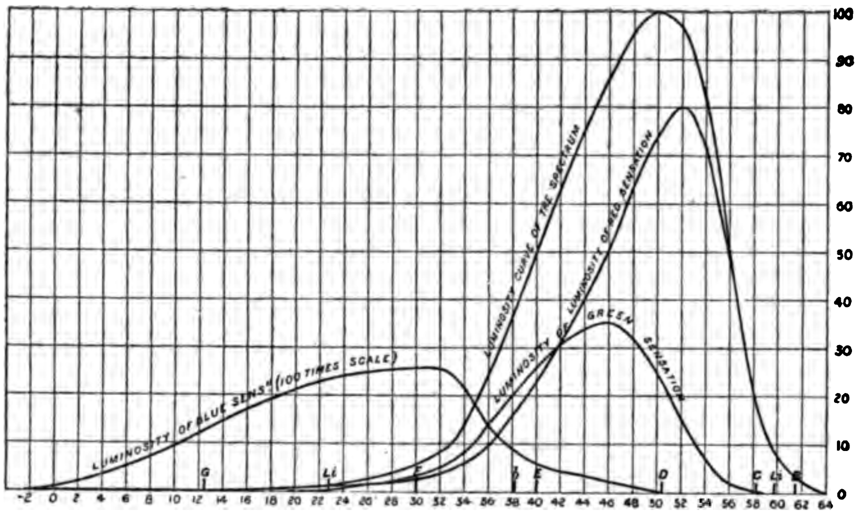
Having ascertained the percentage composition in sensation luminosity of all the spectrum, the luminosity curve of any known spectrum can be sub-divided into sensation-luminosities. This is done in the next table. Columns X., XI., XII., XIII., XIV., XV. and XVI. give the sensation-luminosity curves derived from the total luminosity curve of the spectrum of the crater of the positive pole of the arc

Fig. 3.



Percentage of colour sensations as luminosity in the prismatic spectrum colours.

Fig. 4.



Sensation-luminosities in the spectrum of the light of the crater of the positive pole of the arc light as seen with the centre of the retina.

electric light, the ordinates for which are given in 'Colour Photometry,' Part III. The graphic results are shown in fig. 4. In this figure the blue sensation curve ordinates are on a scale 100 times larger than those of the red and green sensations. The luminosity of this blue sensation is really very small, and except for the hue would be negligible.

The areas of the sensation curves in this figure, or of one constructed with the RS, GS, and VS, should, supposing the white light to be the same as that used in the observations, be proportional to the constants in the colour equations for white light. The areas of the RS, GS, and VS curves are closely 1102, 529, and 247 on an empiric scale, and these numbers, converted into percentages, give

$$\overset{\text{RS}}{66{\cdot}55} + \overset{\text{GS}}{31{\cdot}96} + \overset{\text{V}}{1{\cdot}49} = \overset{\text{W}}{100}.$$

The equation employed is

$$\overset{\text{RS}}{68{\cdot}42} + \overset{\text{GS}}{29{\cdot}90} + \overset{\text{V}}{1{\cdot}68} = \overset{\text{W}}{100}.$$

Owing to the colour of the glass interposed in the beam, and to the slightly different angular dimension on the retina of the images in the measurements on the two occasions, this small discrepancy is fully accounted for. To see whether experiments bear out this deduction, a measure was made under the conditions which were present when the curves in 'Colour Photometry,' Part III., were made, and it came out as follows—

$$\overset{\text{RS}}{66{\cdot}20} + \overset{\text{GS}}{32{\cdot}28} + \overset{\text{V}}{1{\cdot}52} = \overset{\text{W}}{100}.$$

This is sufficiently close to indicate that the measurements made are fairly exact.

TABLE II.—Table showing Results of Applying Table I. to the Luminosity Curve of the Spectrum as Measured on the Yellow Spot of the Retina.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.	XIII.	XIV.	XV.	XVI.	XVII.	XVIII.	XIX.
Scale number of spectrum.	SSN.	λ.	Percentages of luminosity of			Final percentages of			Luminosity of spectrum colours.	Luminosity of prismatic spectrum colours sub-divided into luminosities of			Luminosity of prismatic spectrum colours sub-divided into luminosities of			Equal ordinates to make white.		
			RS.	GS.	VS.	RS.	GS.	BS.		RS.	GS.	VS.	RS.	GS.	BS.	RS.	GS.	BS.
64	...	7217	*100*	*0*	*0*	100	0	0	0·5	*0·5*	*0*		0·5	...	...	0·5	...	...
63	...	7082	*100*	*0*	*0*	100	0	0	1	*1*	*0*	*0*	1	...	...	1	...	...
62	...	6957	*100*	*0*	*0*	100	0	0	2	*2*	*0*	*0*	2	...	...	2	...	...
61	198·73	6839	*100*	*0*	*0*	100	0	0	4	*4*	*0*	*0*	4	...	...	4	...	...
60	204·36	6728	*100*	*0*	*0*	100	0	0	7	*7*	*0*	*0*	7	...	...	7	...	...
59	208·00	6621	*100*	*0*	*0*	100	0	0	12·5	*12·5*	*0*	*0*	12·5	...	...	12·5	...	...
58	211·64	6520	*99·5*	*0·5*	*0*	99·5	0·5	0	21	*20·89*	*0·105*	*0*	20·89	1·05	...	20·89	0·22	...
57	215·28	6423	*99*	*1*	*0*	99	1	0	33	*32·67*	*0·330*	*0*	32·67	0·33	...	32·67	0·67	...
56	218·92	6330	*98*	*2*	*0*	98	2	0	50	*49·00*	*1·008*	*0*	49·00	1·00	...	49·00	2·13	...
55	222·55	6242	*95·5*	*4·5*	*0*	95·5	4·5	0	65	*62·05*	*2·95*	*0*	62·05	2·95	...	62·05	6·25	...
54	226·19	6152	*92*	*8·0*	*0*	92	8·0	0	80	*73·6*	*6·40*	*0*	73·6	6·40	...	73·6	13·56	...
53	229·73	6074	*88*	*12·0*	*0*	88	12·0	0	90	*79·2*	*10·80*	*0*	79·2	10·80	...	79·2	22·90	...
52	233·36	5996	*83*	*17*	*0*	83	17·0	0	96	*79·58*	*16·42*	*0*	79·58	16·42	...	79·58	34·80	...
51	237·00	5919	*79*	*21*	*0*	79	21	0	99	*78·22*	*20·79*	*0*	78·22	20·79	...	78·22	44·08	...
50	240·64	5850	*74*	*26*	*0·005*	74	26	0·001	100	*74·00*	*26·00*	*0·005*	74·00	26·00	·001	74·00	55·12	0·16
49	244·27	5783	*70*	*30*	*0·015*	70	30	0·004	99	*69·30*	*29·69*	*0·015*	69·31	29·69	·004	69·31	62·96	0·66
48	247·91	5720	*66·5*	*33·5*	*0·03*	66·51	33·5	0·008	97	*64·50*	*32·47*	*0·030*	64·52	32·47	·008	64·52	68·73	1·32
47	251·55	5658	*62·7*	*37·2*	*0·05*	62·73	37·2	0·013	92·5	*58·00*	*34·45*	*0·050*	58·03	34·45	·015	58·03	73·02	2·47
46	255·18	5596	*59·2*	*40·7*	*0·08*	59·26	40·7	0·022	87	*51·50*	*35·43*	*0·071*	51·55	35·43	·021	51·55	75·11	3·54
45	258·83	5538	*56·2*	*43*	*0·11*	56·28	43·0	0·03	81	*45·52*	*35·39*	*0·089*	45·58	35·39	·024	45·58	75·02	3·96
44	262·46	5481	*53·7*	*46·2*	*0·15*	53·81	46·2	0·04	75	*40·27*	*34·61*	*0·112*	40·35	34·61	·031	40·35	73·37	5·11
43	266·10	5427	*51·2*	*48·6*	*0·20*	51·54	48·6	0·06	69	*35·33*	*33·67*	*0·138*	35·43	33·67	·038	35·43	71·38	6·27
42	269·73	5373	*49·5*	*50·3*	*0·26*	49·69	50·3	0·07	62·5	*30·94*	*31·40*	*0·162*	31·06	31·40	·046	31·06	66·56	7·59
41	273·37	5321	*47·7*	*52·0*	*0·34*	48·50	52·0	0·095	57	*27·19*	*29·62*	*0·194*	27·33	29·62	·053	27·33	62·75	8·74
40	277·01	5270	*46·0*	*53·5*	*0·46*	46·34	53·5	0·12	50	*23·00*	*26·75*	*0·230*	23·16	26·75	·063	23·16	56·71	10·39
39	280·64	5221	*44·5*	*54·9*	*0·60*	45·44	54·9	0·165	42·5	*18·91*	*23·33*	*0·255*	19·09	23·33	·070	19·09	49·46	11·55
38	284·28	5172	*43·38*	*56·4*	*0·80*	43·38	56·4	0·22	36·0	*15·41*	*20·30*	*0·288*	15·62	20·30	·080	15·62	43·03	13·20
37	287·92	5128	*41·2*	*57·7*	*1·1*	42·00	57·7	0·30	29·5	*12·15*	*17·05*	*0·324*	12·38	17·05	·090	12·38	36·14	14·85
36	291·55	5085	*39·3*	*58·9*	*1·8*	40·6	58·9	0·50	24	*9·43*	*14·09*	*0·432*	9·74	14·09	·119	9·74	29·90	19·8

TABLE II. (continued).—Table showing Results of Applying Table I. to the Luminosity Curve of the Spectrum as measured on the Yellow Spot of the Retina.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.	XIII.	XIV.	XV.	XVI.	XVII.	XVIII.	XIX.
Scale number of spectrum.	SSN.	λ.	Percentages of luminosity of			Final percentages of			Luminosity of spectrum colours.	Luminosity of prismatic spectrum colours sub-divided into luminosities of			Luminosity of prismatic spectrum colours sub-divided into luminosities of			Equal ordinates to make white.		
			RS.	GS.	VS.	RS.	GS.	BS.		RS.	GS.	VS.	RS.	GS.	BS.	RS.	GS.	BS.
35	295·19	5043	*37·0*	*59·0*	*3·6*	39·6	59·0	1·00	18·4	*6·81*	*11·06*	*0·662*	7·28	11·04	·183	7·28	23·44	30·2
34	298·83	5002	*35·0*	*59*	*5·5*	39	59·0	1·5	14·2	*4·97*	*8·38*	*0·781*	5·53	8·38	·215	5·53	17·54	35·47
33	302·47	4963	*33·0*	*58·5*	*7·5*	39·4	52·5	2·1	10·5	*3·46*	*6·15*	*0·892*	4·10	6·15	·245	4·10	13·04	40·40
32	306·10	4924	*31·0*	*58*	*11*	39	58·0	3·0	8·5	*2·63*	*4·93*	*0·935*	3·30	4·93	·257	3·30	10·45	42·40
30	313·38	4848	*27·0*	*56·0*	*17*	39·3	56·0	4·7	5·7	*1·48*	*3·08*	*0·969*	2·18	3·08	·268	2·18	6·53	44·00
28	320·65	4776	*23·5*	*52·5*	*24*	40·9	53·5	6·6	4	*·940*	*2·10*	*0·960*	1·63	2·10	·265	1·63	4·45	43·23
26	327·92	4707	*19·7*	*46·8*	*33·5*	44·0	46·8	9·2	2·8	*·55*	*1·31*	*0·938*	1·23	1·31	·257	1·23	2·78	42·40
24	335·19	4639	*16·0*	*37*	*48*	50·3	37	12·9	1·9	*·35*	*·60*	*0·893*	·99	·60	·245	·99	1·23	40·43
22	342·47	4578	*12·5*	*27·5*	*60*	56·0	27·5	16·5	1·4	*·17*	*·29*	*0·840*	·88	·29	·233	·88	·63	38·44
20	349·73	4517	*9·5*	*19·0*	*71·5*	61·5	19·0	19·5	1·1	*·10*	*·22*	*0·786*	·67	·22	·213	·67	·47	35·14
18	357·00	4459	*6*	*11·5*	*82·5*	66	11·5	22·5	·86	*·05*	*·11*	*0·761*	·56	·11	·193	·56	·23	32·21
16	364·25	4404	*2·5*	*5·0*	*92·5*	69·5	5·0	25·5	·70	*·01*	*·05*	*0·647*	·47	·05	·178	·47	·10	29·37
14	371·50	4349	*0*	*2·0*	*98*	71	2·0	27·0	·56	*0*	*·01*	*0·548*	·39	·01	·155	·39	·02	25·57
12	378·75	4296	*0*	*0·5*	*99·5*	72·1	0·5	27·4	·45	*0*	*0*	*0·445*	·32	0	·127	·32	0	20·95
10	386·00	4245	*0*	*0*	*100*	72·5	0	27·5	·34	*0*	*0*	*0·34*	·25	0	·092	·25	0	15·1
8	393·27	4197	*0*	*0*	*100*	72·5	0	27·5	·26	*0*	*0*	*0·26*	·19	0	·075	·19	0	12·1
6	400·52	4151	*0*	*0*	*100*	72·5	0	27·5	·18	*0*	*0*	*0·18*	·13	0	·050	·13	0	8·5
4	407·25	4106	*0*	*0*	*100*	72·5	0	27·5	·14	*0*	*0*	*0·14*	·10	0	·039	·10	0	6·6
2	...	4060	*0*	*0*	*100*	72·5	0	27·5	·10	*0*	*0*	*0·10*	·07	0	·023	·07	0	4·4

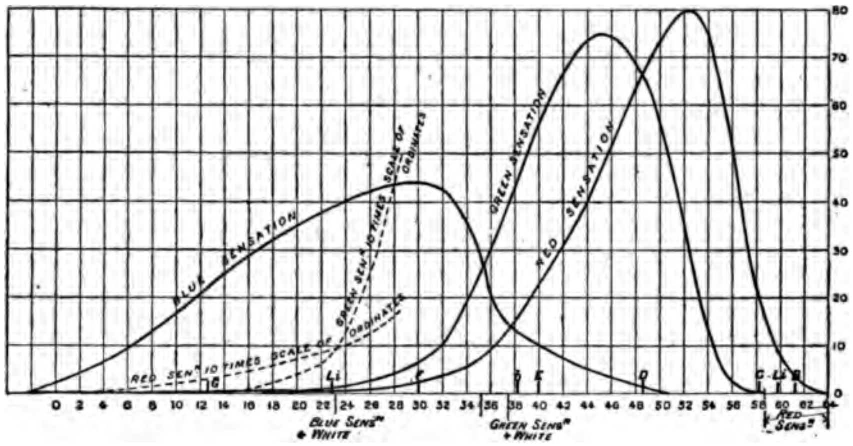
(XXI.) *Curves showing Equal Ordinates of Red, Green, and Blue Sensations to give White Light.*

The equation derived from the foregoing areas being taken as correct, when the blue sensation replaces the violet, the final equation becomes

$$\begin{array}{ccccccc} \text{RS} & & \text{GS} & & \text{BS} & & \text{W} \\ 67{\cdot}63 & + & 31{\cdot}96 & + & {\cdot}41 & = & 100, \end{array}$$

and we can find the curves which will have ordinates on such a scale that, when equal, they give white. Taking the red curve as the standard, we must multiply the GS ordinates by $\frac{67{\cdot}63}{31{\cdot}96}$, or 2·12, to make the areas of the two equal. When equal, the desired result is obtained as far as the green sensation is concerned. The BS ordinates must be multiplied by $\frac{67{\cdot}63}{{\cdot}41}$, or 165, to obtain the result for the third curve. These new ordinates are shown in Columns XVIII. and XIX., Table II., and these, with Column XVII., are shown graphically in fig. 5.

Fig. 5.



Sensation curves (prismatic spectrum) in which equal heights of ordinates form white.

From these curves can be seen at a glance the positions that A and D, fig. 1, occupy, and they show also those parts of the spectrum where the green and blue sensations are seen, unmixed with any other sensation except white. These positions of A and D we find to be about Scale Nos. 37·3 and 35·2, which are λ 5120 and 5060 respectively. The purest green sensation is felt at the former Scale Number, and the purest blue sensation is at Scale No. 23·2, or close to the blue lithium line, which is at Scale No. 22·8. As before said, the light of the crater is of that "whiteness" which closely matches the white outside the colour fields; hence it may be surmised that at these two points we have the nearest approach to the true green and blue sensations.

(XXII.) *Confirmation of the Observations by Colour-blind Persons.*

In confirmation of the positions of A and D, fig. 1, as stated before, complete red- and green-blind eyes were called in to make observations. A totally green-blind person gave the following readings for that position in the spectrum where white was matched. The Scale Numbers used are those of the diagram.

The white was placed alongside the spectrum colours, and the slit through which the colour issued was gradually moved from the red towards the neutral point. The same procedure was adopted in moving from the blue towards the same point.

The readings were

From the red.	From the blue.
38·5, 37·5, 38, 37·1	36·8, 37·1, 37·5.

The mean of these readings is 37·7 as the neutral point of the green-blind. The difference is only ·2 of the scale. A look at the curve will show that there is likely to be greater variation when moving the slit from the red, as the curve is there less steep. A mean of the readings "from the blue" gives 37·1, the position fixed by the preliminary trials, and which answers to SSN 288·5.

Two red-blind persons marked the point in the spectrum where the colour matched white. One read 34·2, 35, 35·4, or a mean of 35·1; the other read 35·2, 35, 35·6, 35·8, or a mean of 35·4. The mean of the two means is 35·2.

From the coincidence of the areas with the colour equations, and from the position of neutral points of the colour-blind with the points where the curves met, we may conclude that the observations are correct within the limits of the errors of observation.

It may be stated that the nearest approaches to the colour sensations in pigments are: Vermilion, to which a little blue has been added; emerald green and true ultramarine, to which a slight trace of red has been added. GREVILLE's cyanine blue is not far from the colour. All are slightly paler, however, and in using them as colour discs this paleness must be allowed for.

(XXIII.) *The Observations applied to the Normal Spectrum of the Electric Arc Light.*

Before proceeding further, I have thought that it would be of interest to show the colour sensations of a normal spectrum. The compression of the red in the prismatic spectrum and the extension of the blue does not enable a comparison to be easily made between the sensation curves of this spectrum and the results obtained by KŒNIG, which are based on the normal spectrum.

The following table is calculated from observations made with a grating spectrum in 1891. The grating was ruled on speculum metal, and had about 14,000 lines to the inch.

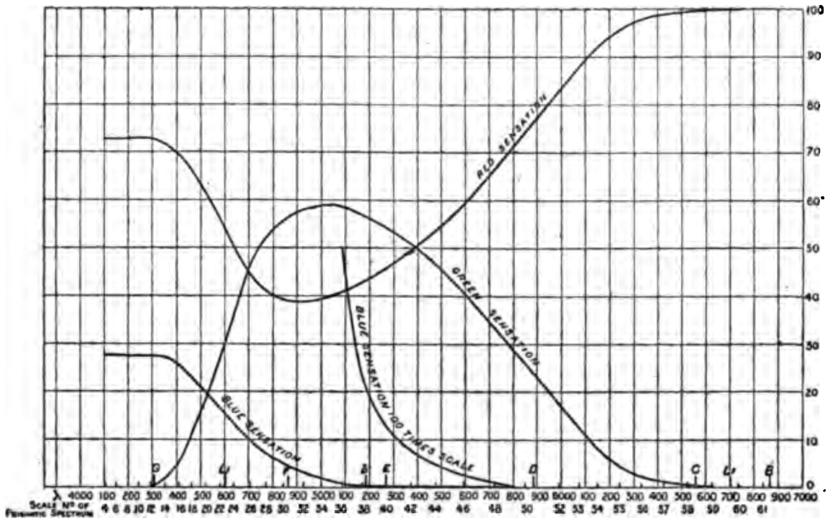
I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.
λ.	Percentage luminosity.			Luminosity of spectrum.	Sub-division of spectrum luminosity into luminosities of			Equal ordinates to give white.		
	RS.	GS.	BS.		RS.	GS.	BS.	RS.	GS.	BS.
7100	100	0	0	0·5	0·5	...	...	0·5	...	...
7000	100	0	0	0·9	0·9	...	...	0·9	...	...
6900	100	0	0	1·6	1·6	...	...	1·6	...	...
6800	100	0	0	3·2	3·2	...	...	3·2	...	...
6700	100	0	0	6·0	6·0	...	...	6·0	...	...
6600	99·8	0·2	0	10·0	9·9	0·1	...	9·9	0·22	...
6500	99	1·0	0	17·0	16·8	0·2	...	16·8	0·44	...
6400	98	2·0	0	26·0	25·48	0·52	...	25·48	1·13	...
6300	97	3·0	0	41	39·77	1·23	...	39·77	2·67	...
6200	94	6·0	0	59	55·46	3·54	...	55·46	7·68	...
6100	89·5	10·5	0	75	67·13	7·87	...	67·13	17·08	...
6000	83·3	16·7	0	85	70·81	14·19	...	70·81	30·79	...
5900	77·5	22·5	0	93	72·08	20·92	...	72·08	45·39	...
5800	71·3	28·7	0·005	99	70·38	28·42	·005	70·38	61·67	0·7
5700	65·3	35·0	0·01	100	65·00	35·00	·01	65·00	75·95	1·4
5600	59·3	40·5	0·023	95	56·52	38·46	·02	56·52	83·45	2·9
5500	54·3	45·7	0·040	89	48·33	40·64	·035	48·33	90·19	4·9
5400	49·7	50·2	0·065	80	39·76	40·16	·052	39·76	87·15	7·38
5300	47·2	52·7	0·11	70	33·04	36·58	·077	33·04	79·37	10·93
5200	53·6	56·2	0·18	56	24·00	29·90	·099	24·00	64·88	14·10
5100	41·2	58·4	0·42	35	14·42	20·43	·147	14·42	44·33	20·87
5000	39·3	59·0	1·6	18	7·09	10·61	·300	7·09	23·02	40·89
4900	39·0	57·5	3·5	11	4·48	6·50	·385	4·48	14·10	54·60
4800	40·0	54·0	6·0	7·5	3·00	4·05	·456	3·00	8·79	64·75
4700	45·0	44·8	10·2	5·0	2·25	2·24	·510	2·25	4·88	71·00
4600	54·5	30·0	15·5	3·5	1·90	1·06	·542	1·90	2·30	76·9
4500	63·5	16·0	20·5	2·7	1·71	0·44	·553	1·71	0·95	78·5
4400	69·8	4·5	25·7	2·1	1·47	0·09	·540	1·47	0·20	76·68
4300	72·5	0	27·5	1·7	1·23	0	·470	1·23	0	66·74
4200	72·5	0	27·5	1·3	·94	0	·357	·94	0	50·69
4100	72·5	0	27·5	1·0	·72	0	·275	·72	0	39·05
4000	72·5	0	27·5	·75	·54	0	·206	·54	0	29·25
3900	72·5	0	27·5	·50	·36	0	·137	·36	0	19·45
3800	72·5,	0	27·5	·25	·18	0	·068	·18	0	9·22

Column I. shows the wave-lengths; Columns II., III., and IV. the percentage luminosity of the sensations (see fig. 6); Column V. the luminosity; Columns VI., VII., and VIII. the luminosity of the sensations in the normal spectrum. The approximate areas of the three curves shown in Columns VI., VII., and VIII., and graphically in fig. 7, are 747·6, 344·3, and 5·3 for the RS, GS, and BS respectively, and the equation for white is

$$\overset{\text{RS}}{68\cdot14} + \overset{\text{GS}}{31\cdot38} + \overset{\text{BS}}{\cdot48} = \overset{\text{W}}{100}.$$

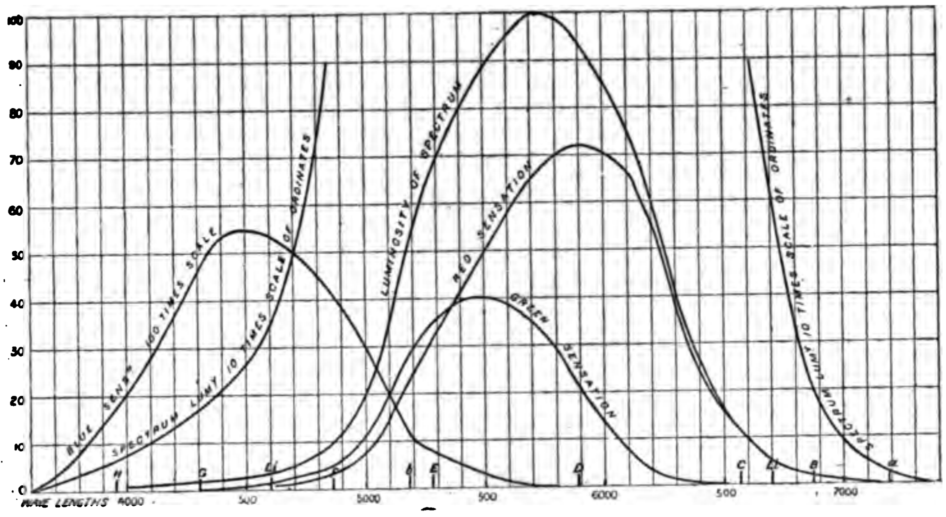
Columns VII. and VIII. have to be multiplied by $\frac{68\cdot14}{31\cdot32}$ and $\frac{68\cdot14}{\cdot48}$, or 2·17 and 142 respectively, to give equal areas with the RS area in Column VI. Column IX. is Column VI. repeated, and Columns X. and XI. show the curves for the GS and BS when of equal areas, hence the ordinates, when equal, give white. These last curves are shown in fig. 8. The above equation is not very different from that obtained from the prismatic spectrum of the same light. Any difference is caused in all probability by the colour of the light reflected by the grating, together with that caused by the necessary error of observation.

Fig. 6.



Percentage composition in sensations of the spectrum colours, normal spectrum.

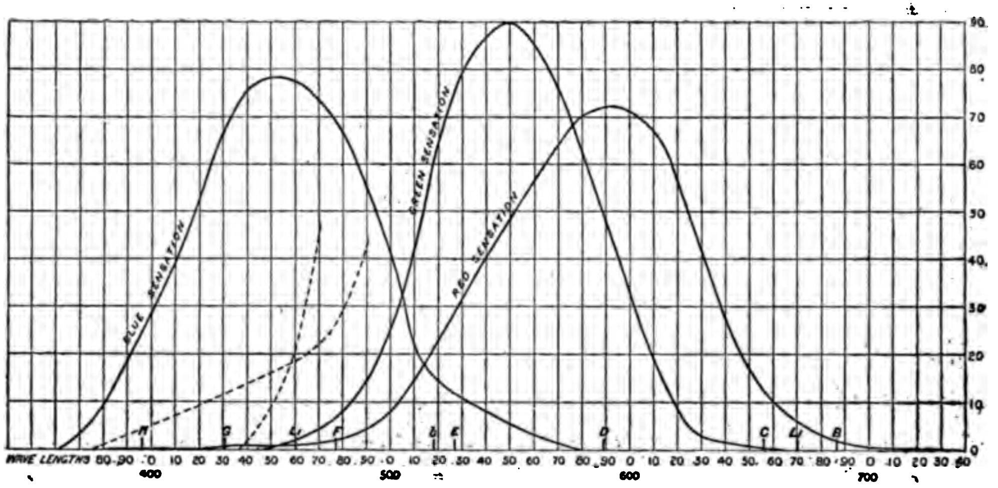
Fig. 7.



Normal luminosity curve in wave-lengths (electric light). Divided into sensation luminosities.

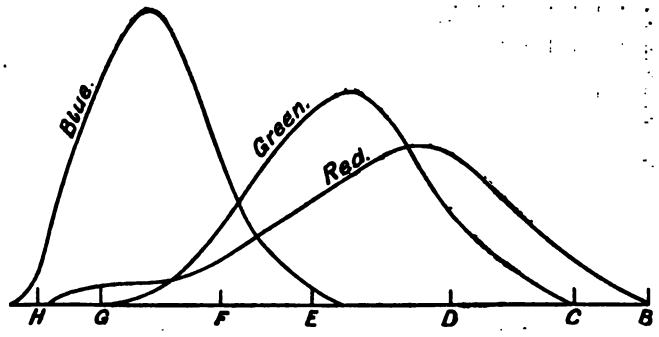
A comparison of the curves in figs. 5 and 8 shows what an effect the compression and extension of the red and blue of the spectrum has on the shape and appearance of the curves, and how they approach in some respects those of KŒNIG'S, of which fig. 9 is a rough representation taken from one of his diagrams. There is a marked difference, however, in the amount of red shown as existing in the violet. Whatever source of light is taken as the standard the same proportion will exist.

Fig. 8



Sensation curves (normal spectrum) in which equal heights of ordinates form white (electric light crater).

Fig. 9.



(XXIV.) *Observation applied to the Solar Spectrum.*

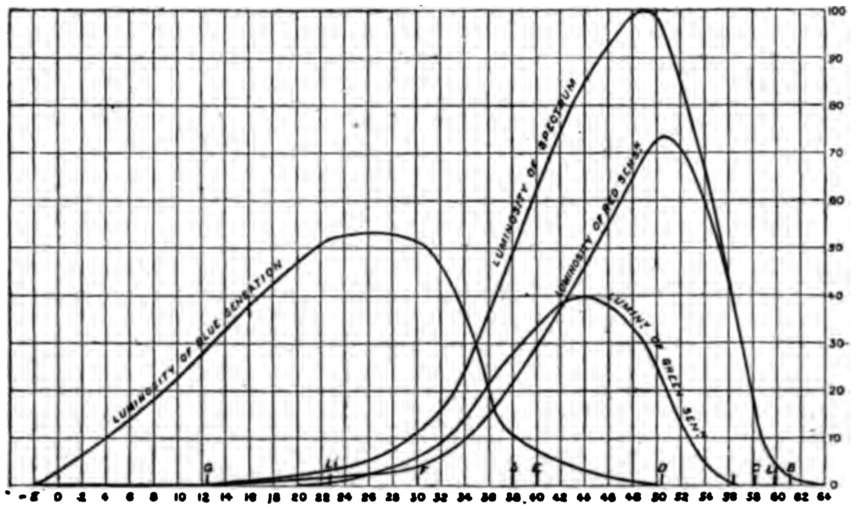
One more example may be given. The sun's spectrum in the month of September, at noon, was measured several years ago, and this has been divided up into luminosities. It is very similar in characteristics to the electric light spectrum. The equation to the white light derived from the areas is

$$\overset{\text{RS}}{65{\cdot}73} + \overset{\text{GS}}{33{\cdot}83} + \overset{\text{BS}}{{\cdot}44} = \overset{\text{W}}{100.}$$

The green sensation is more stimulated by this light than it is in the electric light, whilst curiously enough the blue sensation appears to be very much the same (fig. 10).

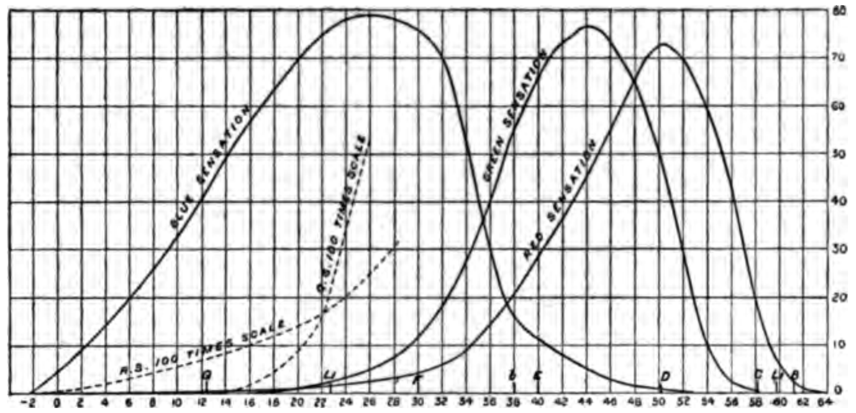
To reduce the curves to equal areas the green sensation has to be multiplied by 1·94 and the blue by 149. These are given in Columns VI., VII., and VIII. of the table and are graphically shown in fig. 11.

Fig. 10.



Sensation luminosities in sunlight of noon in September.

Fig. 11.



Sensation curves (prismatic spectrum) in which equal heights of ordinates form white (sun light).

SENSATION Curves in Sun's Spectrum.

I. Scale number.	II. Luminosity.	III.	IV.	V.	VI.	VII.	VIII.
		Percentage composition.			Curves of equal areas.		
		RS.	GS.	BS.	RS.	GS.	BS.
64	0	0	...	...	0	...	...
63	0·5	0·5	...	...	0·5	...	...
62	0·74	0·74	...	...	0·74	...	...
61	2·5	2·5	...	...	2·5	...	...
60	4·5	4·5	...	...	4·5	...	...
59	9·0	9·0	...	...	9·0	...	...
58	19·0	18·9	0·1	...	18·9	0·19	...
57	32·0	31·68	0·32	...	31·68	0·62	...
56	43·0	42·14	0·86	...	42·14	1·67	...
55	55	52·52	2·48	...	52·52	4·81	...
54	64	58·88	5·12	...	58·88	9·93	...
53	75	66·00	9·00	...	66·00	17·46	...
52	85	70·55	14·45	...	70·55	28·03	...
51	93	73·47	19·53	...	73·47	37·89	...
50	98	72·62	25·38	·001	72·62	48·24	0·148
49	100	70·00	30·00	·004	70·00	58·20	0·592
48	98·5	65·42	33·00	·008	65·42	64·02	1·184
47	95·5	59·94	35·55	·012	59·94	68·97	1·776
46	93·5	55·40	38·08	·021	55·40	73·87	3·108
45	90	50·34	39·63	·027	50·34	76·88	3·996
44	85	45·74	39·23	·034	45·74	76·10	5·032
43	80·2	41·17	38·98	·048	41·17	75·62	7·104
42	75·5	37·31	38·14	·053	37·31	73·99	7·844
41	69·5	33·32	36·11	·066	33·32	70·05	9·768
40	63·0	28·52	33·73	·075	28·52	65·44	11·100
39	56·5	25·69	31·02	·093	25·69	60·18	13·764
38	50	21·69	28·20	·110	21·69	54·71	16·280
37	43·5	18·27	25·10	·130	18·27	48·69	19·244
36	36·5	14·82	21·03	·200	14·82	40·80	29·600
35	30·0	11·88	17·82	·300	11·88	34·57	44·40
34	24	9·36	14·28	·360	9·36	27·70	53·28
33	19	7·48	11·12	·399	7·48	21·57	59·05
32	16	6·24	9·28	·480	6·24	18·00	71·04
30	11	4·32	6·16	·517	4·32	11·95	76·52
28	7·8	3·19	4·08	·529	3·19	7·91	78·29
26	5·85	2·57	2·75	·533	2·57	5·33	78·88
24	4·1	2·06	1·51	·529	2·06	2·93	78·29
22	3·1	1·73	0·85	·511	1·73	1·64	75·63
20	2·3	1·41	0·42	·470	1·41	0·81	69·56
18	1·9	1·25	0·22	·427	1·25	0·43	63·20
16	1·5	1·04	0·08	·382	1·04	0·15	56·54
14	1·24	0·88	0·03	·335	0·88	0·08	49·58
12	1·03	0·73	0·02	·282	0·73	0·04	41·74
10	0·80	0·58	0	·220	0·58	0	32·56
8	0·66	0·48	0	·180	0·48	0	26·60
6	0·51	0·37	0	·140	0·37	0	20·70
4	0·36	0·26	0	·100	0·26	0	14·80
2	0·23	0·16	0	·070	0·16	0	10·40

In fig. 10 it will be noticed that A and D of fig. 1 are somewhat closer together, and that the place where the blue sensation exists mixed with white alone, is again close to the blue lithium line. The blue sensation curve is also somewhat higher.

Confirmatory of these curves are the luminosity curves given for the red blind in 'Colour Photometry,' Part III. If the curves there given be reduced to a little more than one-half the scale of ordinates, they will be found to closely correspond to those of fig. 4. It must be remembered that to them the red luminosity is non-existent throughout the spectrum, hence the luminosity of the violet end is much diminished. For this reason the luminosity curves were made on too high a scale when drawn for that paper, and they should be corrected accordingly. It need not follow from this investigation that the colour-blind see less light than those having the three full sensations, though the "extinction" readings given in that same paper seem to indicate that such is the case. A further investigation into this is in hand, and for the present the question must be left an open one.

I have to thank my assistant, Mr. WALTER BRADFIELD, for the aid he has given me in this investigation. The actual observations recorded were made by myself, but all preparatory work was done by him.

tively and Negatively

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IX. *On the Comparative Efficiency as Condensation Nuclei of Positively and Negatively Charged Ions.*

By C. T. R. Wilson, *M.A.*

Communicated by the Meteorological Council.

Received May 11,—Read June 15, 1899.

The work, of which the results are given in this communication, forms part of an investigation on atmospheric electricity, on which I am engaged on behalf of the Meteorological Council.

The relation between rain and atmospheric electricity was one of the problems it was suggested I should study experimentally. The importance in connection with that question of the subject dealt with in this paper has already been noticed by Professor J. J. Thomson, who points out* that "if the negative ions, say, were to differ in their power of condensing water around them from the positive, then we might get a cloud formed round one set of ions and not round the other. The ions in the cloud would fall under gravity, and thus we might have separation of positive and negative ions and the production of an electric field, the work required for the production of the field being done by gravity."

To make this process worthy of consideration as a source of atmospheric electricity, it would be necessary to show reason for believing (1) that atmospheric air in the regions in which rain is formed is likely to contain free ions, (2) that the positively and negatively charged ions differ in their efficiency as condensation nuclei.

With respect to the first of these questions, former experiments furnish considerable evidence in favour of an affirmative answer. When moist dust-free air is allowed to expand suddenly, a slight rain-like condensation always takes place if the maximum supersaturation attained exceeds a certain limit. This limit is identical with that which is necessary for the formation of fogs in air, in which a supply of ions has been produced by the action of Röntgen rays or other ionising agent. The nuclei, on which the drops are formed in air exposed to the rays, were shown experimentally to be identical with the ions to which the conductivity of the gas when exposed to the rays is due. The equality of the expansion required to give the comparatively few drops in the absence of the rays, with that required to cause water to condense on the ions, is so exact as to furnish what is at first sight almost convincing evidence that ordinary moist air is always to a very slight extent ionised. The number of these nuclei is so small that the absence of any sensible conductivity in air under ordinary conditions is in no way inconsistent with the view that they are ions. In the latter part of this paper are described some attempts which I have made to decide experimentally whether these nuclei are charged or not.

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* 'Phil. Mag.,' December, 1898, p. 533

It is mainly, however, with the second of the above questions that this paper deals. The experiments to be described prove that there is a great difference between positively and negatively charged ions, with respect to their power of serving as nuclei for the condensation of water vapour; a much smaller degree of supersaturation sufficing to cause water to condense on the negative ions than is required in the case of positively charged ions. They therefore furnish a possible explanation of the preponderance of negatively electrified rain,* which is required by theories which attribute the normal positive potential of the air to the action of precipitation.

I have shown, in a previous paper,† that the ions produced by various agents (X-rays, uranium-rays, negatively charged zinc exposed to ultra-violet light) are identical with respect to the minimum supersaturation required to make water condense on them. They are also identical in this respect, with the few nuclei apparently always present in moist air. In the present investigation I have therefore felt justified in using exclusively the Röntgen rays as being the most convenient ionising agent, and in assuming that the same results would be obtained with ions from other sources.

To compare the efficiency as condensation nuclei of the positive and negative ions respectively, expansion experiments were made with moist air containing ions all, or nearly all, charged with electricity of one sign, alternately positive and negative in successive experiments.

To enable a supply of ions nearly all positive or nearly all negative to be produced at will in the air under observation, this was enclosed between two parallel metal plates, and a narrow beam of Röntgen rays was made to pass between the plates parallel to and almost in contact with the surface of one of them. Under these conditions a supply of positive and negative ions is produced in the thin lamina of air exposed to the rays, and when a difference of potential is maintained between the plates, the two sets of ions move in opposite directions, the positive towards the negative plate and *vice versâ.* If we neglect the slight difference in the velocity of positive and negative ions, shown to exist by the experiments of ZELENY,‡ the number of ions in unit volume of the positive and negative streams will be the same, assuming (an assumption which later experiments justify) that equal numbers of positive and negative ions are produced, and that the ionisation does not, for example, consist in the breaking up of the neutral molecules into a certain number of positive ions and half as many negative ions, each carrying twice as large a charge as the positive. It is plain, therefore, that there must at any moment be a great excess of the ions which have the greater distance to travel; in other words, of the ions charged with electricity of the same sign as that on the plate nearest the layer of air exposed to the rays. The expansion may either be made while this layer is exposed to the rays, or the rays may be cut off before the expansion. If the interval, between cutting off the rays and making the expansion, lies within certain limits, it is plain that all the ions travelling to the plate next the ionised layer may have been removed, while only a small proportion of those travelling towards the more distant plate have reached it before the expansion is made. In this way we would therefore expect to get positive or negative ions with almost complete absence of ions of the other kind.

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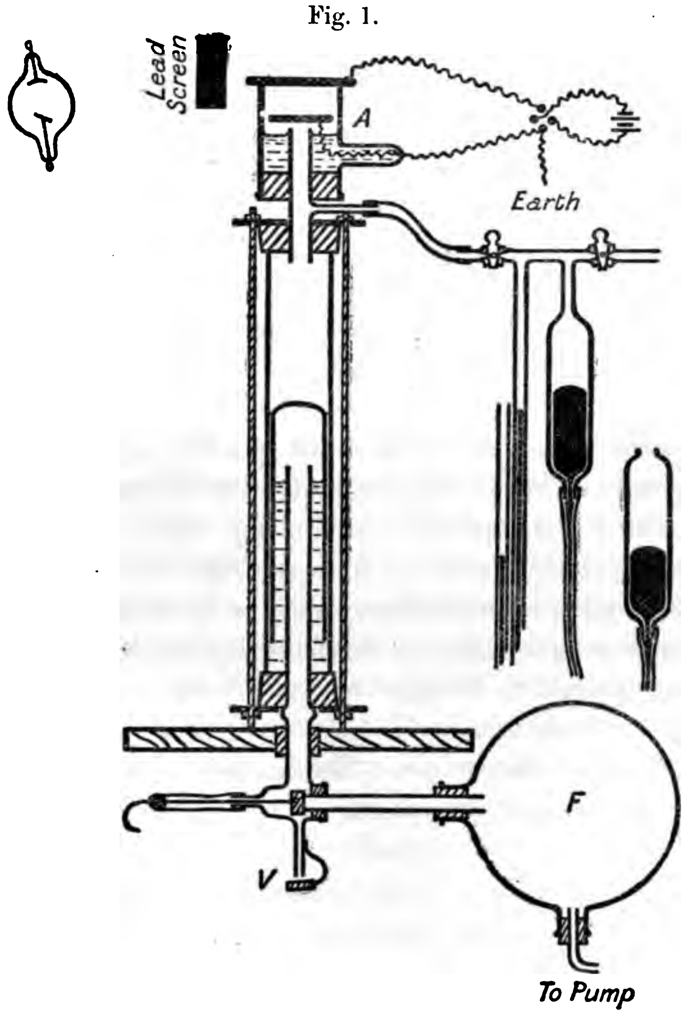
* The earlier observations of ELSTER and GEITEL appeared to show a preponderance of negatively electrified rain ('Sitzungsber. d. k. Akad. in Wien.,' 99, IIA, p. 421), but this is not shown in their later observations ('Terrestrial Magnetism,' vol. 4, p. 15).

† 'Phil. Trans.,' A, vol. 192, pp. 403–453, 1899.

‡ 'Phil. Mag.,' vol. 46, p. 120, 1898.

Fig. 1.



The method of producing sudden expansion of any desired amount was that which I have described in a previous paper.* Such differences as there were in the details of the apparatus are sufficiently indicated in fig. 1. The glass cylinder and piston of the expansion apparatus were larger than before, the former being 3·7 centims. in internal diameter. The arrangements for connecting the air-space below the piston with the vacuum in F have been improved in form, and a self-acting indiarubber valve, V, has been substituted for the stop-cock, formerly used to cut off communication with the atmosphere immediately before making this connection. The addition of this valve was found to be a great convenience.

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* 'Camb. Phil. Soc. Proc.,' vol. 9, p. 333, 1897.

The method of arranging for an expansion of any desired amount is described in the paper to which reference has just been made. In the new apparatus, however, an error is introduced into the measurements of expansion by the yielding of the indiarubber stoppers closing the top of the expansion cylinder and the bottom of the vessel, A, in which the clouds are observed, as well as to some extent probably by the momentum acquired by the air in the connecting tube. Both sources of error tend to make the actual maximum expansion of the air in A greater than that obtained by calculation from its pressure before and after expansion.

Now the older experiments,* made with apparatus suitable for absolute measurements, showed that there are two well-defined critical values of the expansion, which we may use as fixed points to standardise an expansion apparatus unsuited for absolute measurements. At the first point the expansion (measured by v_2/v_1, the ratio of the final to the initial volume) is equal to 1·25; it is the minimum expansion required to make condensation in the form of drops begin in dust-free air initially saturated, whether the air be exposed to an ionising agent, such as Röntgen rays, or not. At the second point the transition from rain-like to cloud-like condensation (in the absence of ionising agents) takes place; v_2/v_1, is equal here to 1·38.

On testing the apparatus in this way the following results were obtained:—

(1.) The air being exposed to Röntgen rays, condensation was first observed when the apparent value of $v_2/v_1 = 1·22$ instead of 1·25. Error = − ·03.

(2.) The change from rain-like to cloud-like condensation (in the absence of ionising agents) took place when the apparent value of $v_2/v_1 = 1·35$ instead of 1·38. Error = − ·03.

Thus the correction to be added to the apparent values of v_2/v_1 is the same at both points and equal to + ·03. I have, therefore, assumed that the same correction holds for intermediate values of v_2/v_1. Throughout the paper the corrected values of v_2/v_1 are given.

The vessel, A, in which the ions were produced and the clouds formed upon them observed, consisted of a wide glass tube, 4·2 centims. in diameter, closed above by a brass plate, cemented to the ground top of the tube with sealing wax; 2 centims. below this plate was a smaller circular brass plate, 3·7 centims. in diameter, fixed horizontally within the tube by means of three projecting tags cemented to the sides of the tube. In the side of the tube a horizontal slot was cut, 3 centims. long and 3·5 millims. in diameter, the lower edge of the slot being on a level with the upper surface of the lower plate. The slot was closed externally by a strip of thin aluminium, cemented to the glass with shellac. A focus bulb, giving out Röntgen rays, was fixed with the anticathode on a level with the upper surface of the lower plate, the anticathode being placed with its plane almost horizontal, so that the effective source of the rays which entered the slot was almost linear. A thick lead screen was fixed, as shown in the figure, to prevent any rays reaching the interior of the vessel above the level of the slot. There is thus only a thin layer of air, close to the surface of the lower plate, exposed to the direct action of the rays from the bulb. The lower plate was kept at zero potential, while the upper plate could be connected either to the positive or negative terminal of a battery, of which the other terminal was earthed. The vessel could thus be charged at will with an excess of either positive or negative ions.

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* 'Phil. Trans.,' A, vol. 189, p. 265, 1897.

All the metal surfaces within the tube were covered with wet filter paper, to keep the air saturated with water vapour, and to prevent nuclei being produced by the action of the metal itself.

A considerable advantage is gained by having the plates horizontal, and the ionised layer in contact with the lower plate, for any drops formed on the ions which are in a minority (these being confined to the lower part of the tube), have thus only a short distance to fall, and if condensation takes place on these ions only, the drops will be confined to the lower part of the vessel.

In the experiments first performed with this apparatus the expansions were made without previously cutting off the rays.

The apparatus being adjusted to give expansions somewhat exceeding the limit $v_2/v_1 = 1{\cdot}25$, comparatively dense fogs were obtained when the upper plate was maintained at a potential a few volts higher than the lower, so that negative ions were present in excess; whereas, when the field was reversed (the positive ions being now in excess) only a slight condensation could be observed, and this was mainly confined to the region immediately over the lower plate, where a considerable number of negative ions must have been present. With expansions as great as $v_2/v_1 = 1{\cdot}35$ the appearance of the fogs obtained was independent of the direction of the field, and this continued to be the case up to the limit 1·38, at which dense fogs appear even in the absence of ions. With the field in the direction which gives an excess of negative ions, the density of the fogs which result from expansion is practically the same for all values of v_2/v_1 between 1·28 and the above-mentioned limit 1·38. When, on the other hand, the upper plate is connected to the negative pole of the battery, so that the positive ions are in excess, the drops remain few till v_2/v_1 amounts to about 1·31, when the number of the drops begins to increase as the expansion is increased. With $v_2/v_1 = 1{\cdot}33$, we obtain, with the positive ions, comparatively dense fogs, still, however, considerably less dense than those obtained with negative ions. Finally, above 1·35 the positive and negative fogs are indistinguishable.

These results admit of only one interpretation; condensation takes place on some of the negative ions when v_2/v_1 amounts to 1·25, practically all the negative ions are caught when v_2/v_1 exceeds 1·28 ; while to make water condense on any of the positive ions v_2/v_1 must exceed about 1·31 ; all the ions positive and negative being caught when v_2/v_1 exceeds 1·35.

Experiments were now made in which the expansion did not take place till after the rays had been cut off.

The ions which are being attracted from the ionised layer to the lower plate, have at the most about 3½ millims. to travel, while the upward moving ones have from 17 to 20 millims. to travel. One would expect, therefore, about six times as long a time to be taken for the removal of the latter as is required for the removal of the former. There is thus a considerable range of time available for making the expansion, so that the majority of the ions of one kind shall still be present, while all the ions of the other kind have reached the lower plate. Using one LECLANCHÉ cell to maintain the electric field between the plates, an interval of about 1 second between cutting off the rays and making the expansion was found to be suitable.

The results were in agreement with those previously obtained ; the drops, when the field was such as to give positive ions and v_2/v_1 was less than 1·31, were now no more numerous than if the expansion were made without exposure to the rays at all. The method, therefore, enables us to obtain ions entirely of one kind.

To test to what degree accidental variations in the time allowed to elapse between cutting off the rays and making the expansion could affect the result, some experiments were made in which this interval was varied. A metronome, giving 90 ticks per minute, was used ; the interval before the expansion was varied by switching off the current from the coil, as the metronome made a tick, and pulling the trigger of the expansion apparatus at the moment of the first, second, or any subsequent tick thereafter. The following results were obtained, one LECLANCHÉ cell being used to maintain the field. The expansion was such as to catch negative ions only.

(1) NEGATIVE Ions moving upward.

v_2/v_1.	Interval in seconds.	Result.
1·27	⅔	Dense fog
1·27	1⅓	Dense fog
1·27	2	Fog
1·27	2⅔	Slight fog
1·27	3⅓	Dense shower
1·27	4	Very few drops (no more than without rays)

The negative ions thus take between 3 and 4 seconds to travel from the ionised layer to the upper plate, and a large proportion of them are still present more than 2 seconds after turning off the rays. The field was now reversed.

(2) NEGATIVE Ions moving downwards.

v_2/v_1.	Interval in seconds.	Result.
1·27	$\frac{2}{3}$	Very few drops
1·30	$1\frac{1}{3}$	Very few drops (no more than without rays)

All the negative ions have now reached the lower plate in two-thirds of a second after turning off the rays. That the positive ions are still present was proved by increasing the expansion to 1·34 (which is sufficient to cause water to condense on them also); a fog was now obtained when the interval was $1\frac{1}{3}$ seconds. The actual time taken for the positive ions to disappear was not measured, but as ZELENY's experiments show that they move more slowly than the negative, they would doubtless have been found to take at least as long as the negative to travel to the upper plate.

The above results show that any accidental differences in the length of the time between cutting off the rays and making the expansion are of little importance when the interval amounts to about 1 second.

The time taken for the ions to be removed, when they have to travel to the upper plate, is somewhat longer than the time we obtain by calculation, using RUTHERFORD's* value for the velocity of the ions. The time calculated in this way amounts to about 2 seconds. The want of uniformity in the field is sufficient to account for the actual time being longer than the calculated.

The following tables contain the results of observations in which the expansion was effected about 1 second after cutting off the supply of Röntgen rays.

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* 'Phil. Mag.,' vol. 44, p. 422, 1897

Rays turned off before expansion. Difference of potential between plates, 1 Leclanché cell.

v_2/v_1.	Result.	
	Upper plate negative (positive ions in excess).	Upper plate positive (negative ions in excess).
1·28	1 or 2 drops	Fog
1·29	1 or 2 drops	Fog
1·31	Very few drops	Fog
1·32	Drops few	Fog
1·33	Shower	Fog
1·34	Slight fog	Fog
1·36	Fog as dense as negative fogs	Fog

The effect of the positive ions begins to be detected when v_2/v_1 amounts to about 1·32 or 1·33; below that point no more drops are seen than in the absence of the rays.

The next series shows the point at which the positive ions first begin to be detected more sharply defined.

Rays turned off before expansion. Difference of potential between the plates, 1 Leclanché cell.

v_2/v_1	Result.	
	Upper plate negative (positive ions in excess).	Upper plate positive (negative ions in excess).
1·265	1 or 2 drops	Fog
1·31	1 or 2 drops	Fog
1·32	Slight fog	Fog
1·35	Fog as dense as negative	Fog

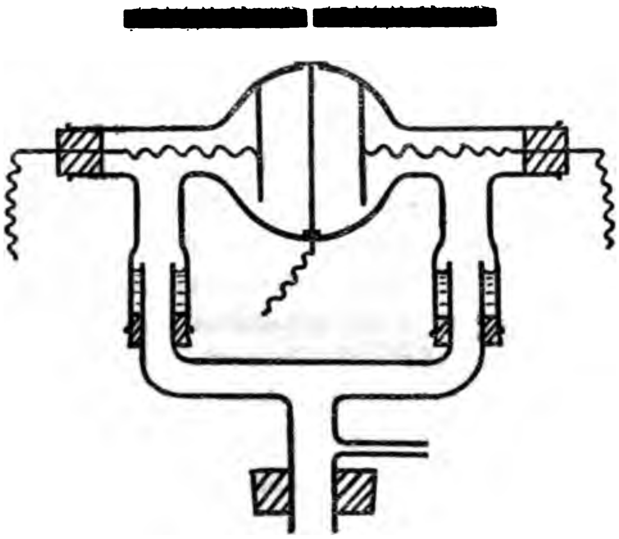
Positive ions begin to be caught when v_2/v_1 is between 1·31 and 1·32.

A double apparatus was now constructed, to allow of the comparison of fogs produced simultaneously in the presence of positive and negative ions respectively.

The arrangements for causing the sudden expansion were the same as before. The general form of the rest of the apparatus is indicated in the diagram (fig. 2) which is drawn approximately to scale.

The vessel in which the clouds were observed was nearly spherical, and about 5·8 centims. in diameter. It was divided into two equal chambers by a partition of brass (about 1 millim. thick) in the equatorial plane; the vessel was cut in two to allow of its insertion, the edges of the two halves being ground smooth, to allow them to be easily cemented against the faces of the partition. The latter was circular, its diameter being equal to the outside diameter of the vessel. A narrow strip of brass, 2½ millims. in thickness, was soldered to each face, extending all round the circumference, except for a gap of 4·5 centims. at the top. When the ground edges of the two halves of the vessel were cemented against these strips there was left at the top a slit on each side of the brass plate, 4·5 centims. long and 2·5 millims. wide. The double slit was closed by covering it with a strip of thin aluminium cemented to the outer surface of the glass and to the edge of the brass partition. A thin layer of air in contact with each surface could thus be exposed to the Röntgen rays from a source placed vertically above the partition.

Fig 2.



Each half of the apparatus contained a second brass plate fixed parallel to the central plate at a distance of 1·8 centim. from it. These had the form of equilateral triangles, and were fixed to the glass with sealing wax at the corners. There was room between the sides of the triangle and the walls of the vessel for the air to escape from between the plates at the moment of expansion.

The metal plates were, as in the former apparatus, covered with wet filter paper.

The anticathode of the focus tube, which generated the rays, was fixed by eye vertically above the central plate of the apparatus. A lead screen connected to earth and provided with a slit, 4 millims. in width, was placed about 2 centims. above the aluminium window of the cloud vessel. The final adjustments were made by moving the screen until, when both side plates were kept at the same potential (higher than that of the central plate, which was always earthed), exactly equal fogs were obtained on the two sides, with expansions sufficient to catch the negative ions.

To make the fogs on the two sides readily visible simultaneously, a horizontal stratum of the air in both halves was illuminated by the light from a horizontally placed luminous gas flame brought to a focus within the apparatus, the source being behind the apparatus with its centre in the plane of the central plate. The fogs were best seen when the eye was placed sufficiently below the level of the illuminated stratum to receive none of the direct light, but only that scattered by the drops.

To compare the effects of positive and negative ions, expansions were made with the left-hand plate at a certain positive potential (generally about 1 volt), the right-hand plate being at an equal negative potential and the central plate earthed. The appearance of the fogs on the two sides was noted, the direction of the fields reversed, and the effect of an expansion of the same amount as before again noted. Any effect due to want of symmetry in the apparatus was thus eliminated.

To secure equality in the electric fields in the two halves of the apparatus, the following arrangement was used. The source of potential (generally two Leclanché cells in series) had its terminals connected by a resistance of 200 ohms. The middle point of this resistance was earthed, and the two extremities were connected through a commutator to the outer plates of the cloud apparatus, the central plate being earthed.

The correction to be applied to obtain the true values of v_2/v_1 was found to be the same as before; to the apparent values of v_2/v_1 ·03 must be added. The error was, as before, found to be the same at both the points, $v_2/v_1 = 1{\cdot}25$ and $v_2/v_1 = 1{\cdot}38$.

To obtain an approximation to quantitative comparison between the number of drops produced in the two halves of the apparatus, measurements were made on the time taken by the upper surface of the clouds on the two sides to sink below the level of the beam of light, which was used to make them visible. The expansion and consequent cooling on both sides being the same, the same quantity of water is condensed in each half; the quantity available for each drop is thus inversely proportional to the number of drops which are formed. Now the radii of the drops on the two sides can be compared by measuring the velocity with which they fall; a comparison of the rates of fall on the two sides will thus enable us to determine the ratio between the numbers of the drops produced in the two halves of the apparatus, or will at least serve as a test for equality between the numbers. Professor J. J. THOMSON* has in fact used the rate of fall of the drops, formed on the ions as the result of expansion, to determine the number of the ions present, from which he obtains an estimate of the charge carried by each.

The results of measurements with this double apparatus are given in the tables which follow. After what has already been said in connection with the other apparatus, it is hardly necessary to point out that, when the right-hand plate is connected to the positive terminal of the battery, and the left-hand plate to the negative terminal (the central plate being at zero potential), there will be a great excess of negative ions on the right, and of positive ions on the left. The terms "positive" and "negative" in the tables refer to the sign of the charge carried by the majority of the ions, not to the potentials of the plates; the corrected values of v_2/v_1 are given.

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* 'Phil. Mag.,' *loc. cit.*

FIELDS produced by 4 Leclanché cells, terminals joined by resistance of 200 ohms, the middle point of which was earthed.

v_2/v_1.	Time taken by fogs to fall in seconds.		Ratio of times, negative / positive.
	Left side.	Right side.	
1·28	positive 5 negative 15	negative 16 positive 3	$\left.\begin{matrix}3{\cdot}2\\5{\cdot}0\end{matrix}\right\}4{\cdot}1$
1·30	negative 15 positive 5 negative 10 positive 2	positive 2 negative 15 positive 2 negative 10	$\left.\begin{matrix}7{\cdot}5\\3{\cdot}0\\5{\cdot}0\\5{\cdot}0\end{matrix}\right\}5{\cdot}1$
1·31	positive 7 negative 14	negative 12 positive 7	$\left.\begin{matrix}1{\cdot}7\\2{\cdot}0\end{matrix}\right\}1{\cdot}8$
1·32	negative 8 positive 8 negative 14 positive 12	positive 5 negative 10 positive 8 negative 17	$\left.\begin{matrix}1{\cdot}6\\1{\cdot}2\\1{\cdot}7\\1{\cdot}4\end{matrix}\right\}1{\cdot}5$
1·33	negative 12 positive 12	positive 10 negative 13	$\left.\begin{matrix}1{\cdot}2\\1{\cdot}1\end{matrix}\right\}1{\cdot}15$
1·35	negative 10 positive 10	positive 10 negative 10	$\left.\begin{matrix}1{\cdot}0\\1{\cdot}0\end{matrix}\right\}1{\cdot}0$

The results of another series, in which only two cells were used to produce the fields, are given in the next table.

v_2/v_1.	Time taken by fogs to fall in seconds.		Ratio of times, negative / positive.
	Left side.	Right side.	
1·28	positive 4 negative 14	negative 15 positive 4	$\left.\begin{matrix}3{\cdot}7\\3{\cdot}5\end{matrix}\right\}3{\cdot}6$
1·30	negative 14 positive 3	positive 4 negative 17	$\left.\begin{matrix}3{\cdot}5\\5{\cdot}7\end{matrix}\right\}4{\cdot}6$
1·32	positive 10 negative 20	negative 17 positive 12	$\left.\begin{matrix}1{\cdot}7\\1{\cdot}7\end{matrix}\right\}1{\cdot}7$
1·34	negative 20 positive 12	positive 17 negative 17	$\left.\begin{matrix}1{\cdot}2\\1{\cdot}4\end{matrix}\right\}1{\cdot}3$
1·36	negative 20 positive 20	positive 20 negative 20	$\left.\begin{matrix}1{\cdot}0\\1{\cdot}0\end{matrix}\right\}1{\cdot}0$
1·37	positive 18 negative 20	negative 18 positive 20	$\left.\begin{matrix}1{\cdot}0\\1{\cdot}0\end{matrix}\right\}1{\cdot}0$

The tables show plainly that with the smaller expansions the number of drops on the side which contains chiefly positive ions is very small compared with the number of drops on the side containing mainly negative ions. No difference can be detected between the fogs on the two sides when v_2/v_1 exceeds 1·35; the transition from the one kind of result to the other begins when v_2/v_1 is about 1·31. This is well shown by the ratio of the times taken by the fogs on the negative and positive sides to fall the same distance; increasing v_2/v_1 from 1·30 to 1·31 brings down the ratio from 5·1 to 1·8 in the first series, and in the second series the ratio diminishes from 4·6 to 1·7 as v_2/v_1 is increased from 1·30 to 1·32. The results are therefore in complete agreement with those previously obtained.

There can be no doubt that the drops which are formed when v_2/v_1 is less than 1·31 on the side containing mainly positive ions are deposited on negative ions, of which a considerable number must unavoidably be present, when the expansion is made while the rays are acting. The number of these was exceedingly small when the rays were turned off before the expansion was made.

There is no evidence of any increase in the number of the negative ions caught as v_2/v_1 passes through the region in which the positive ions begin to be caught. The following is an example of observations with expansions of different amount made in rapid succession (so that the radiation might not have time to change in intensity):—

v_2/v_1	Time taken by fogs to fall.	
	Left side.	Right side.
1·30	positive 2 seconds	negative 12 seconds
1·30	negative 12 ,,	positive 3 ,,
1·37	negative 12 ,,	positive 12 ,,
1·37	positive 12 ,,	negative 12 ,,

The rate of fall on the negative side is the same when $v_2/v_1 = 1·37$ as when it $= 1·30$. It is true that we would expect the drops if equally numerous to fall slightly quicker with the greater expansion, since a rather larger quantity of water would be condensed; against this is to be set the fact that the positive minority on the negative side are caught by means of the greater expansion, and not by the less.

It will be convenient to consider here what light the new knowledge now acquired throws upon phenomena previously observed in the course of experiments with expansion apparatus.

All the former measurements which I have published upon the least expansion required to make condensation take place in presence of ions, are now seen to have been concerned with negative ions only. One example of interest is the case of nuclei produced by a zinc plate exposed to ultra-violet light. When the plate was negatively charged, nuclei were found to be present, requiring an expansion, $v_2/v_1 = 1·25$, to make water condense on them; a fact in agreement with the results of the present investigation, since negative ions were, in this case, plainly in question. The absence of fog when the expansion was made with the field reversed was, however, as we now see, no proof that no ions escape from the positively charged plate, for the expansions used were insufficient to cause water to condense on positive ions had they been present. Another case of interest is that of the nuclei produced by the discharge from a point. Even when positive electricity was escaping from the point, fogs were obtained with expansions which are now seen to have been insufficient to catch positive ions. We are therefore driven to the conclusion that the positive discharge does not consist simply in the escape of positive ions from the point of the wire, but that negative ions (or nuclei of some other kind than ions) are present as well, possibly produced by the action upon the moist air of the radiation from the glowing point of the wire.

Indications had already been noticed of an increase in the number of drops, produced in ionised air as v_2/v_1 was increased beyond the point now shown to be that at which the positive ions first begin to act as condensation nuclei. Professor THOMSON was indeed led to make the suggestion contained in the words quoted at the beginning of this paper by noticing indications of such an increase in the neighbourhood of the point $v_2/v_1 = 1{\cdot}3$. My own observations had also previously led me to believe that there was an increase at the point $v_2/v_1 = 1{\cdot}31$; for example, my notes for March 4, 1898, contain the remark, "Many experiments with air, as well as H, seem to show that there are nuclei requiring expansion $= 1{\cdot}31$ to catch them in addition to those appearing at 1·25."

Positive and negative ions (at least those produced in air by Röntgen rays) have now been proved to differ in their efficiency as condensation nuclei; they also differ, as ZELENY has shown, in the velocity with which they move in an electric field of given strength. The negative ions move the faster and are the more efficient as nuclei for the condensation of water vapour.

A possible way of accounting for both differences is to suppose that the charge carried by the negative ions is greater than that carried by the positive, the number of the latter being, of course, correspondingly greater. We might, for example, take the view that the ionisation consists in the breaking-up of the neutral molecules into a certain number of positive ions and half as many negative ions, each carrying twice as large a charge as the positive, a process which we can readily imagine to take place in the case of water molecules.

The experiments already described make this view hardly tenable, for we have seen that when the expansion is sufficient to make water condense on all the ions ($v_2/v_1 = 1{\cdot}35$ or more) the fogs in the two halves of the apparatus are indistinguishable in appearance and in the rate of fall of the drops. We have still to consider to what extent this proves equality in the number of positive and negative ions produced.

The velocity with which the drops fall is proportional to the square of the radius, that is, to the two-thirds power of the volume of each drop, or for a given expansion to $1/n^{2/3}$ where n is the number per cubic centimetre. The time, t, taken to fall a given distance is thus proportional to $n^{2/3}$, or the number of drops is proportional to $t^{3/2}$. Now, when v_2/v_1 exceeds 1·35 the times taken by the drops on the two sides to fall a given distance certainly do not differ by as much as 1 part in 10. If $t_1/t_2 = 1{\cdot}1$ then $n_1/n_2 = 1{\cdot}1^{3/2} = 1{\cdot}15$. Thus the number of ions is the same on the two sides to within 15 per cent.

The equality of the fogs on the two sides is, in fact, rather more exact than we would expect; for the positive ions, according to ZELENY, take 1·25 times as long as the negative to travel a given distance. We would expect then (the strength of the field on each side being the same) that, if equal numbers of positive and negative ions were produced in a given time, the negative ions would be more quickly removed, and a somewhat larger number of drops should have been produced in the half containing mainly positive ions.

The absence of any indication of this slight excess of positive ions is of the less consequence for the present purpose, since it strengthens rather than weakens the evidence against the view that a larger number of positive than of negative ions is produced. As a test of the trustworthiness of the method for detecting a difference in the number of drops produced on the two sides, experiments were made in which the direction of the field on both sides of the central plate was such as to drive negative ions outwards towards the side plates; the strength of the field being, however, different on the two sides. The ratio of the fields was as 3 to 2, the stronger field being produced by two Leclanché cells. The following results were obtained:—

$v_2/v_1 = 1{\cdot}30$.

	Left side.	Right side.
Relative strength of fields	3	2
Time taken to fall	10 seconds	12 seconds
Relative strength of fields	2	3
Time taken to fall	12 seconds	10 seconds

The excess in the number of drops on the side of the weaker field is plainly shown.

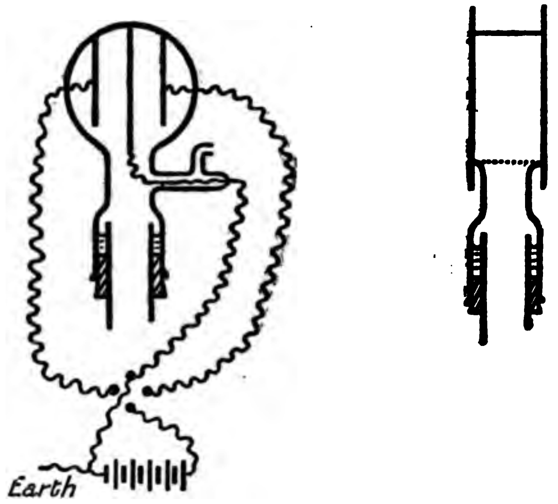
It appears, therefore, that the difference in the behaviour of the positive and negative ions is not to be explained by a difference in the quantity of electricity carried by positively and negatively charged ions respectively.

Some experiments were now tried with the object of determining whether the rain-like condensation, which takes place in dust-free air, even when not exposed to any ionising agents, is due to ions.

As already pointed out, the least expansion necessary to produce such showers in air initially saturated is identical with that required to cause condensation on ions —really on negative ions, as the experiments described in this paper show. This is a remarkable coincidence if we are really concerned with nuclei of entirely different kinds. Further, it was during experiments made in the absence of ionising agents that I first noticed indications of an increase in the number of the drops about the point $v_2/v_1 = 1{\cdot}31$, the expansion now proved to be that required to cause water to condense on positive ions.

These considerations seemed to furnish strong ground for believing that the very few nuclei always present actually are ions. In a recent paper,* it is true, I described some unsuccessful attempts which I had made to remove the nuclei by applying a strong electric field; I did not, however, consider these experiments to be conclusive evidence against the ionic nature of the nuclei. I have, therefore, recently subjected them to a much more severe test by means of a differential apparatus (fig. 3).

Fig. 3.



The mechanism for causing the sudden expansion was the same as in the other experiments, and is not shown in the figure. A front view of the apparatus is given on the left, a side view on the right. It consists of a short glass cylinder, 4 centims. in diameter and 2 centims. long, the ends being ground smooth and closed by plates, that forming the front face being of glass, the other of quartz. (The quartz was for experiments with ultra-violet light described below.) A thin brass plate, 2 centims. wide (reaching, therefore, from back to front of the apparatus), divided the vessel into two equal chambers. On each side of this, at a distance of 8 millims., was a parallel brass plate of the same width, 2 centims., but not reaching to the lower wall of the cylinder. The brass plates were covered with wet filter paper.

A difference of potential of 320 volts could be maintained, by means of a series of secondary cells, between the central plate and one of the side plates, the other being connected to the central plate, which was earthed. In one half, therefore, there was a strong electric field, in the other half, none. A lens was used to examine simultaneously the drops formed on the two sides of the central plate. Any difference in the number of the drops could thus readily be detected.

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* 'Phil. Trans.,' A, vol. 192, pp. 403–453, 1899.

In the strong field the maximum length of life of an ion will be the time taken to travel from one plate to the other under the action of the field; in the present case, a distance of 8 millims., with a potential gradient of 320 volts in 8 millims., or 400 volts per centimetre. The velocity of the ions produced by X rays would in such a field be about 400 × 1·6 centim. per second, if we take RUTHERFORD's value of the velocity for a potential gradient of 1 volt per centimetre. The 8 millims. would therefore be traversed in $\cdot 8/(400 \times 1 \cdot 6) = 1/800$ of a second. (The time would really be somewhat longer on account of the plates being too small to give a sufficiently uniform field.) Now the average length of life of the ions due to Röntgen rays when they are destroyed by recombination alone is of the order of 1 second.[*] The fewer the ions also the less rapid is the rate at which they recombine; we would, therefore, expect the average life of the very few nuclei with which we are now concerned, if they really are ions, to be at least as long as 1 second in the absence of any electric field. If then we have here simply a case of spontaneous ionisation due to molecular encounters, we ought to obtain something like 800 times as many drops without the field as with it. In similar experiments[†] made with the ions produced by Röntgen or uranium rays, much weaker fields were in fact found sufficient to prevent almost completely the production of fogs by expansion.

In the experiments now made without external ionising agents, not the slightest difference could be detected between the appearance of the showers on the two sides of the apparatus. All degrees of expansion from $v_2/v_1 = 1 \cdot 25$ to $v_2/v_1 = 1 \cdot 38$ were tried.

If then we have here to do with a case of ionisation, it differs completely from the ionisation produced by Röntgen rays.

Very similar nuclei, requiring practically the same supersaturation to make water condense on them as the ions, are produced by the action on moist air of sunlight and of weak ultra-violet light. Former experiments[‡] showed that the nuclei produced by this volume effect of ultra-violet light (unlike those produced by its action on a negatively charged zinc plate) are unaffected by electric fields strong enough to remove the ions produced by Röntgen rays as fast as they are produced. More severe tests were now made with the double apparatus, to see whether they are altogether uninfluenced by the electric field.

The ultra-violet light was produced by the spark discharge between aluminium terminals, at a distance of 55 centims. from the quartz plate in one series of experiments, at 180 centims. in another. In neither case was the radiation strong enough to cause drops to be produced with expansions appreciably below $v_2/v_1 = 1{\cdot}25$. The spark-terminals were placed in the plane of the central plate, so that the air in both halves was equally exposed to the rays. The wet filter paper which covered the brass plates prevented any surface effect from the light which might reach the plates. As before, the air on one side was between plates at the same potential, that on the other side in a strong field. The central plate was earthed and one of the side plates kept at a positive potential of 320 volts, the other being earthed; the connections were interchanged after each observation, so that the air in each half alternately was subjected to the action of the field.

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[*] RUTHERFORD, *loc. cit.*

[†] 'Phil. Trans.,' A, vol. 192, pp. 403–453, 1899. Also J. J. THOMSON, 'Phil. Mag.,' *loc. cit.*

[‡] 'Phil. Trans.,' *loc. cit.*

When the expansions were made while the air was exposed to the rays, no difference could be detected between the fogs in the two halves of the apparatus. Moreover, even when the rays were turned off 10 seconds before the expansion was made, a slight fog was obtained, equally dense on both sides of the central plate. Thus some of the nuclei appear to persist for 10 seconds, and even in that time the field has had no sensible effect in reducing the number of the nuclei.

A field, therefore, of 400 volts per centimetre causes the nuclei to move in 10 seconds, a distance small compared with 8 millims., the distance between the plates. This gives, for the velocity under a potential gradient of 1 volt per centimetre, less than 1/4000 centim. per second, whereas the ions produced by Röntgen rays travel under these conditions between 1 and 2 centims. per second.

The slight rain-like condensation which takes place, when v_2/v_1 lies between 1·25 and 1·38, in the absence of all radiation, as well as the much denser condensation produced by the same expansions when the air is exposed to weak ultra-violet light, are thus essentially different phenomena from the apparently similar condensation produced in air ionised by Röntgen rays.

We might perhaps most naturally conclude that we are in these cases not concerned with ions at all. There is, however, the difficulty of the unlikelihood of two entirely different classes of nuclei being so exactly identical in the degree of supersaturation necessary to cause water to condense on them. The apparent existence of a second coincidence (an increase of the number of drops when v_2/v_1 exceeds 1·31) is still harder to explain on this view.

It is possible that condensation in these cases really does take place on ions carrying the same charge as those produced by Röntgen rays. There are, in fact, several ways in which we may account for the fact that an electric field does not remove them.

We might suppose that the nuclei differ from those produced by Röntgen rays merely in being so much larger, that their velocity in a given field is diminished enormously, the charge in each nucleus remaining the same. It is difficult, however, to believe that the efficiency of the nucleus in helping condensation would remain unaltered by such an increase in size as would reduce the velocity to anything like such a small value as 1/4000 centim. per second for a potential gradient of 1 volt per centimetre. In fact, if we can trust to obtaining approximately correct results by applying to drops as small as 2×10^{-6} centim., the formulæ which hold for larger drops, we can see* that these nuclei must amount to at least 2×10^{-6} centim. in radius; the supersaturation required to cause condensation to take place on drops of this radius being that produced by an expansion, $v_2/v_1 =$ about 1·01, which is very far removed from that actually required, $v_2/v_1 = 1{\cdot}25$.

The ionisation may be a result of the expansion. This is the view I am inclined to take.

It is easy to understand, according to this view, how the number of drops produced is entirely uninfluenced by even a strong electric field, for the whole time for which the ions would be free to move under the action of the field, before the formation of drops upon them, would be exceedingly short.

Uncharged nuclei are probably present before the expansion. Some change is produced in the air by weak ultra-violet light independently of the expansion, for, as we have seen, the fogs may be obtained even when the expansion is made some seconds after the light has been cut off. The behaviour of moist air exposed to stronger ultra-violet light, and especially the extreme case where we get visible particles produced without expansion, almost compel us to conclude that even weak ultra-violet light produces nuclei before the expansion is made. I have already suggested† that these nuclei consist of what we may regard as minute water drops containing hydrogen peroxide in solution. The difference between the effects of the strongest and weakest ultra-violet light would consist in a difference in the size of these drops due to the larger quantity of hydrogen peroxide produced in each by the stronger radiation. We may suppose the very minute molecular aggregates due to weak ultra-violet light to be of themselves too small to act directly as condensation nuclei with the expansion $v_2/v_1 = 1{\cdot}25$; in other words, the growth which results from the supersaturation corresponding to this expansion may be insufficient to bring them up to the critical size beyond which the unstable condition is reached, where increase in size is accompanied by a diminution of the equilibrium vapour pressure. But if, as a consequence of the increase in size which results from the supersaturation, the nucleus becomes charged, an ion carrying electricity of opposite sign to that left on the original nucleus being thrown off, the result actually met with would be explained. Judging from the behaviour of hydrogen peroxide solutions, with respect to the electricity developed by splashing, we would expect the original nucleus to become negatively charged; for the splashing results in a negative electrification of the drops‡ indicating that the inner coat of the double layer originally covering the drops is negative; in other words, that solutions of hydrogen peroxide in water, surrounded by air, attract negative electricity more than positive.

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* Using the value found by Professor THOMSON (*loc. cit.*) for the charge on one ion, and applying the formula for the steady motion of a sphere through a viscous fluid (LAMB, 'Hydrodynamics,' p. 532).

† 'Phil. Trans.,' *loc. cit.*

‡ J. J. THOMSON, 'Phil. Mag.,' vol. 37, p. 341, 1894.

The view here suggested is, therefore, that the nuclei requiring the definite expansions 1·25 or 1·31 to make water condense on them are always really ions, the cases in which an electric field is without influence upon the result of expansion being explained by supposing that the ionisation does not in such cases take place until supersaturation is produced.

The separation of the positive and negative electricity by the formation of drops on the negative ions only, as soon as the supersaturated state reaches the necessary limit, will take place equally well whether the ions exist before the supersaturation is produced or are the result of the supersaturation. Moreover, if the initial growth of a drop, as above suggested, is able to cause it to acquire a charge equal to that of one ion, the further growth of the drop may result in an increase of the charge. The drops may thus acquire a charge considerably exceeding that of one ion, even if there be no coalescence of small drops to form large ones.

Further experiments on this point are evidently required.

I do not propose to discuss here the meteorological bearings of the results obtained. The questions with which the experiments are concerned are, I think, fundamental ones in connection with the electrical effects of precipitation. From this point of view the principal results of this investigation are:—

(1.) To cause water to condense on negatively charged ions, the supersaturation must reach the limit corresponding to the expansion $v_2/v_1 = 1{\cdot}25$ (approximately a fourfold supersaturation). To make water condense on positively charged ions, the supersaturation must reach the much higher limit corresponding to the expansion $v_2/v_1 = 1{\cdot}31$ (the supersaturation being then nearly sixfold).

(2.) The nuclei, of which a very small number can always be detected by expansion experiments with air in the absence of external ionising agents, and which require exactly the same supersaturation as ions to make water condense on them (as well as the similar nuclei produced in much greater numbers by the action of weak ultra-violet light on moist air) cannot be regarded as free ions, unless we suppose the ionisation to be developed by the process of producing the supersaturation.

We see, then, that if ions ever act as condensation nuclei in the atmosphere, it must be mainly or solely the negative ones which do so, and thus a preponderance of negative electricity will be carried down by precipitation to the earth's surface.

The experiments described in this paper were carried out at the Cavendish Laboratory.

NOTE ADDED SEPTEMBER 25, 1899.

In order that the results of these investigations should have a direct bearing on the subject of atmospheric electricity, it is necessary to assume that condensation in the atmosphere frequently takes place from the supersaturated condition. There is very little direct evidence of the existence of supersaturation in the atmosphere, but there is at least an equal lack of evidence against its existence above the lower cloud layers even as a normal accompaniment of precipitation. That supersaturation occurs in connection with thunderstorms is held by v. BEZOLD and others ('Sitzungsb. Akad. d. Wissenschaft. zu Berlin,' 1892).

In the lower dust-charged layers of the atmosphere supersaturation is not to be expected. When there is an ascending air current, however, the dust particles may be retained in the lower cloud layers, through each becoming loaded with water, and ceasing to rise as soon as a certain critical size, depending on the upward velocity of the air, is attained. Supersaturation will exist under these conditions in the air which has left its dust particles behind, and if the ascending current reaches a sufficient elevation a second condensation will take place at a higher level, as was pointed out in a former paper ('Phil. Trans.,' A, vol. 189, p. 286, 1897); the conditions will then be such that the experimental results obtained in the present investigation may be applied.

Whether the drops will from the first be too large to be supported by the upward current and therefore at once begin to fall as raindrops, growing rapidly as they fall through the supersaturated layers, or will still continue to be supported by the ascending current and form an upper cloud layer, depends on the upward velocity of the air, the number of nuclei (negative ions ?) and other conditions. In either case we should expect the drops to be negatively charged, the air rising above them carrying a corresponding excess of positive ions.

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throughout the section; and

$$dw/dn + (mx - ly)\tau = 0$$

along the boundary, where dn = an element of the outwards normal to the boundary, τ is the angle of torsion per unit length, and l, m are the direction-cosines of dn.

Now in the present case

$$dn = \pm d\xi \times (c\sqrt{\mathrm{J}})$$

where

$$\mathrm{J} = \partial\left(\frac{x}{c}, \frac{y}{c}\right) \Big/ \partial(\xi, \eta)$$

at the boundary where ξ = const., and

$$dn = \pm d\eta \times (c\sqrt{\mathrm{J}})$$

at the boundary where η = const., the sign being so determined that dn is positive.

By adding suitable terms to w, we can reduce one or other of the boundary conditions to the form

$$dw_1/dn = 0,$$

where

$$w = w_1 + \text{suitable terms.}$$

Suppose we make

$$\left(\frac{dw_1}{d\xi}\right)_{\xi=\pm\alpha} = 0.$$

Expanding now w_1 in the form of a series,

$$w_1 = \sum_{n=0}^{n=\infty} \mathrm{A}_n \sinh\left\{\frac{2n+1}{2\alpha}\pi(\eta+\kappa)\right\} \sin\frac{\overline{2n+1}\pi\xi}{2\alpha},$$

the differential equation and the first boundary condition are identically satisfied.

When this value is substituted in the second boundary condition, we get an equation expressing a given function of ξ in a series of sines of odd multiples of $\frac{\pi\xi}{2\alpha}$, between the limits $+\alpha$ and $-\alpha$.

But such an expression can be definitely obtained by a method analogous to that for FOURIER'S series. Comparing coefficients, we obtain relations which determine completely all the constants in the expression of w_1.

w is then known. The shears and torsion moment are then deduced by differentiation and a double integration.

§ 2. *Summary of the Results.*

The cross-sections which are dealt with in the present paper are of very great generality, and they include as special cases many of the cross-sections which SAINT-VENANT has worked out, for instance the rectangle and the sector of a circle.

The first section of which I treat is that bounded by an ellipse and two confocal hyperbolas. Although the analysis is worked out for the case where the two hyperbolic segments are not symmetrical, I have not given any numerical examples of this case, as the sections obtained by taking two hyperbolas curved the same way, as in fig. 1, do not correspond to any intersecting practical case: the section is too broad at the ends and too narrow at the bend to be any fair representation of the angle iron.

The section (fig. 2) bounded by an ellipse and the two branches of a confocal hyperbola is, on the other hand, an approximate representation of a well-known section, much used in engineering practice, the rail section.

This section I have worked out for various values of the eccentricity of the ellipse and of the angle between the asymptotes of the hyperbola.

The four sections in fig. 2, where this angle is 120°, give the best representation of the rail section.

The numerical results are tabulated so as to show the ratio of the torsional rigidity of this section to that of the circular section of the same area, and also the same ratio for the maximum stress.

The ratio of these two ratios gives us a kind of measure of the usefulness or "efficiency" of the section.

In the case of the sections of fig. 2 I have investigated at length the position of the *fail-points*, or points of maximum strain and stress, the maximum strain, in the case of torsion, being coincident with the maximum stress. It is found that for the two smaller ellipses the maximum stress occurs at the point B where the section is thinnest. For the two larger ellipses the maximum stress occurs at points F, F, F, F, symmetrically distributed round the contour, and lying on the broad sides of the section. The critical section, when these two cases pass into one another, can be calculated and is shown as qq, qq in fig. 2. In figs. 2–5 the corresponding points belonging to the different sections are distinguished by suffixes.

The changes in the stresses are shown by the curves in fig. 9, (p. 340) in which the abscissa represents the quantity α whose hyperbolic cosine and sine are proportional to the major and minor axes of the ellipse respectively, and in which the ordinates represent the stresses at A, B, F, divided by the maximum stress of the circular section of equal area. The curves are in certain parts only roughly drawn, but they suffice to show the manner in which the stresses vary. It is seen that the stress at B separates from the maximum stress after the critical value $\alpha = 1\cdot225$, and gradually diminishes, compared with the stresses at A and F.

This result might have been expected from the investigations of DE SAINT-VENANT upon certain sections bounded by curves of the fourth degree. These investigations appear, however, not to have been sufficiently noticed. THOMSON and TAIT, in their 'Natural Philosophy,' and BOUSSINESQ, in his researches on torsion ('Journal de Mathématiques,' Série II., vol. 16, p. 200), both conclude that the fail-points are at the points of the cross-section nearest to the centre, and BOUSSINESQ even gives an apparently general proof of this proposition. His proof, however, is subject to certain restrictions which I point out, and which prevent it from being applied to the sections I am dealing with.

The sections are sensibly less useful than the circular section, their torsional rigidity being always diminished and the maximum stress very often increased. This remark, I may add, applies to all the sections dealt with in this paper.

Fig. 1.

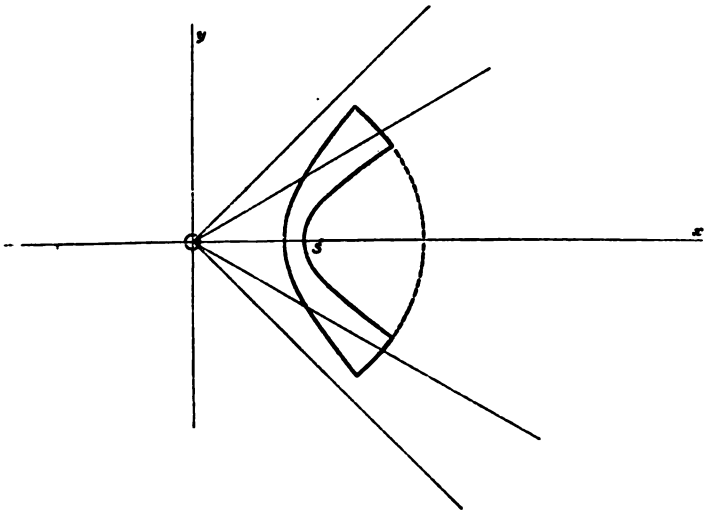
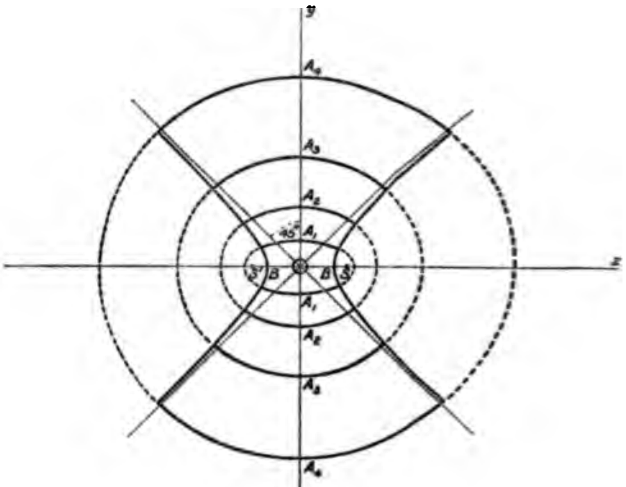


Fig. 3.



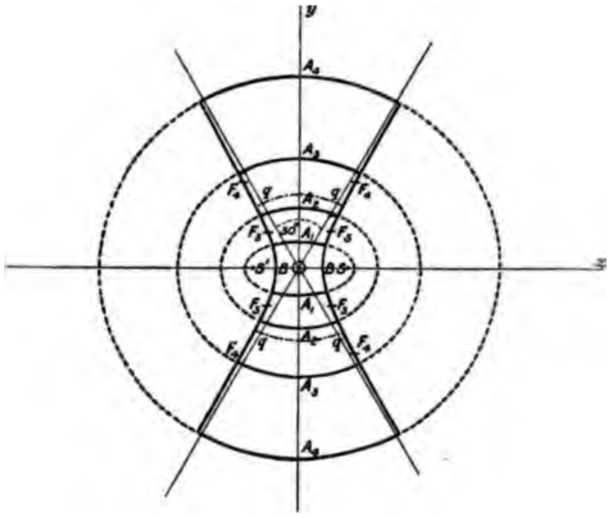


Fig. 2.

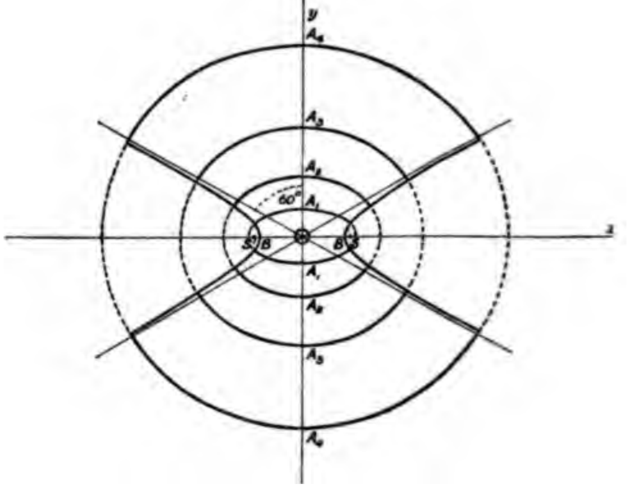


Fig. 4.

This usefulness or efficiency decreases as the neck of the section becomes more narrow, as, indeed, might have been anticipated.

Other sections worked out are those corresponding to angles between the asymptotes of 90° (fig. 3), 60° (fig. 4), and 0° (fig. 5); in the latter case the sections degenerate into ordinary elliptic sections with two straight slits, or indefinitely thin keyways, cut into them along the major axis, as far as the foci. The stress at the foci, however, is then theoretically infinite.

It is interesting to see how, as we make the bend round the foci sharper, the values of α, for which the two fail-points break up into four, become larger and larger, until, when the angle between the asymptotes of the hyperbolas is less than 73°, the greatest stress always occurs at the neck of the section.

Fig. 5.

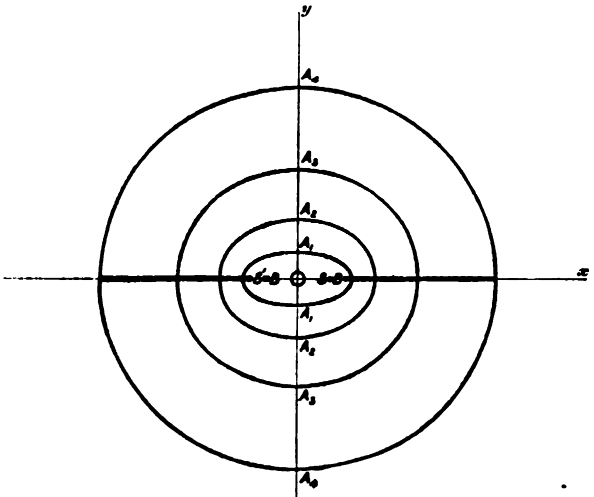
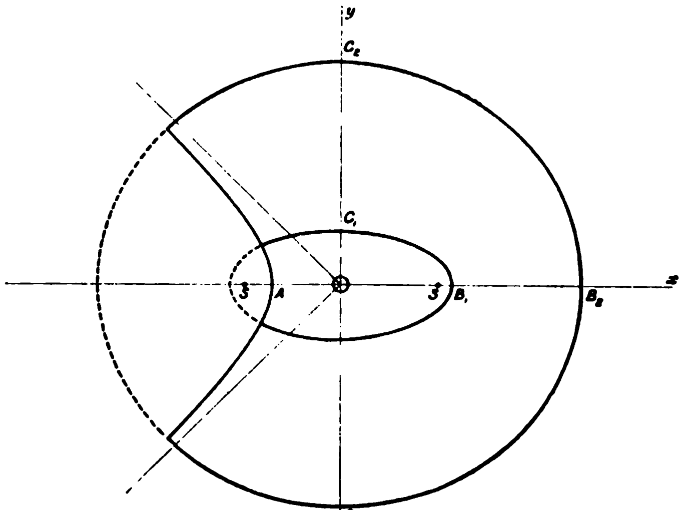


Fig. 7.



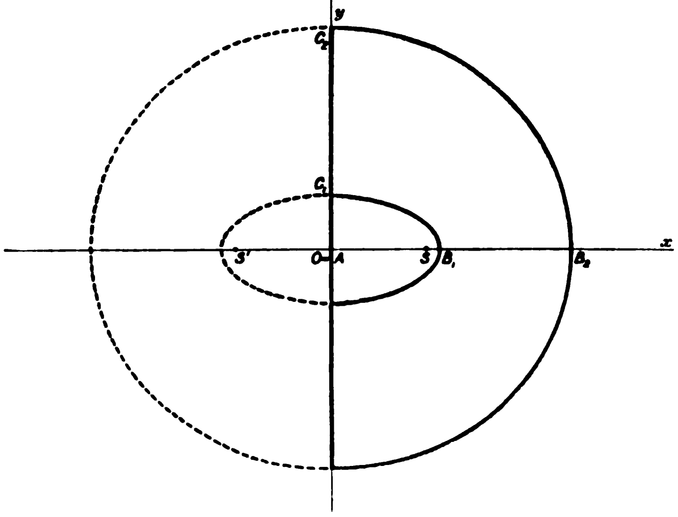


Fig. 6.

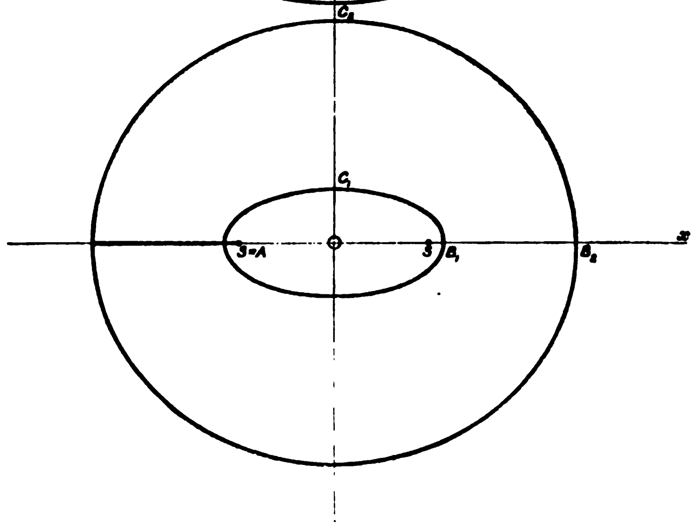


Fig. 8.

The limiting case of such sections, when the angle between the asymptotes is very small and the eccentricity of the ellipse nearly unity, the distance between the foci being very great, gives us the rectangle.

I then pass on to the section bounded by one ellipse and one confocal hyperbola. In the limiting case when the foci coincide, we obtain the sector of a circle.

Of this I have worked out numerically three cases, in each case taking two ellipses

(1.) The semi-ellipse (fig. 6).

(2.) The ellipse with a keyway cut into it of the shape of a rectangular confocal hyperbola (fig. 7).

(3.) The ellipse with a single slit cut into it (fig. 8).

The most striking of the results is in reference to the reduction of the torsional rigidity of the ellipse in case (3). This reduction of rigidity decreases rapidly as the depth of the notch decreases.

The rigidity, which is reduced by as much as 23 per cent. when the depth of the keyway is as great as ·6 (semi-major axis), falls only about 1 per cent. when this depth is ·12 (semi-major axis).

Possibly this may throw some light on the fact that the effect of cutting such slits into the material does not always give in practice the reduction in the torsional rigidity which should have been expected from SAINT-VENANT'S results for the circle. Clearly the depth of the keyway is a factor of the very first importance, and keyways of moderate depth will produce a comparatively small effect on the torsional rigidity.

It is also shown that the effect of cutting two equal and opposite slits is practically equal, in the two cases which I have calculated (namely, $\alpha = \pi/6$ and $\alpha = \pi/2$), to twice the effect of a single slit.

It seems, therefore, that the study of these sections brings to light several interesting facts in the theory of elasticity, and will well repay the trouble involved in dealing with the long and somewhat tedious algebra and arithmetic which lead to these results.

§ 3. *Statement of Notation, &c.*

In what follows the axis of z will be taken parallel to the generators, the axes of x and y in one of the terminal cross-sections.

The origin, however, will not necessarily be at the centroid of the cross-section.

The shifts of any point of the material parallel to the axis will be denoted as usual by u, v, w, and for the stresses I shall use the notation of TODHUNTER and PEARSON'S 'History of Elasticity,' $\widehat{rs}$ denoting the stress, parallel to r, across a plane element perpendicular to s.

Then if, following SAINT-VENANT, we suppose the terminal cross-section $z = 0$ to be fixed (that is to say, $u = 0$, $v = 0$, but $w \neq 0$), then if τ be the angle of torsion per unit length,

$$v = \tau xz, \qquad u = -\tau yz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

If μ be the modulus of rigidity,

$$\frac{\widehat{yz}}{\mu} = \frac{dw}{dy} + \tau x, \qquad \frac{\widehat{xz}}{\mu} = \frac{dw}{dx} - \tau y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

and all the other stresses are zero.

w is then determined from the body stress equation

$$\frac{d\widehat{xz}}{dx} + \frac{d\widehat{yz}}{dy} = 0,$$

or

$$\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3),$$

which holds at all points of the cross-section, and from the surface-stress equation

$$l\,(\widehat{xz}) + m\,(\widehat{yz}) = 0,$$

that is,

$$\frac{dw}{dn} + (mx - ly)\,\tau = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

at the boundary, where dn is an element of the outwards normal, and (l, m) are its direction-cosines.

The above conditions allow us to determine w uniquely. They are associated with the condition that the parallel to the generators through the origin remains fixed. If, however, we take any other parallel to the generators through the point (a, b) to remain fixed, then

$$u = -\tau\,(y - b)\,z, \qquad v = \tau\,(x - a)\,z,$$

$$\frac{\widehat{yz}}{\mu} = \frac{dw}{dy} + \tau\,(x - a), \qquad \frac{\widehat{xz}}{\mu} = \frac{dw}{dx} - \tau\,(y - b),$$

and the equation at the boundary becomes

$$\frac{dw}{dn} + (mx - ly)\,\tau - \tau\,(am - lb) = 0.$$

Now, if instead of w we write

$$w = w' + \tau\,(ay - bx),$$

then

$$\frac{\widehat{yz}}{\mu} = \frac{dw'}{dy} + \tau x, \qquad \frac{\widehat{xz}}{\mu} = \frac{dw'}{dx} - \tau y,$$

and the equations to determine w' are the same which we had before for w. It follows that the *stresses* in the cylinder are unaltered, whatever be the parallel to the generators about which it is twisted, the effect of the change being merely to introduce a term $\tau\,(ay - bx)$ into w, which corresponds to a rigid rotation about an axis joining (a, b) to the origin.

It follows from this that in dealing with stresses due to torsion we may take our origin wherever it is most convenient.

§ 4. *Analytical Work for Sections Bounded by One Elliptic and Two Hyperbolic Arcs.*

Consider now the transformation

$$x = c \cosh \xi \sin \eta,$$

$$y = c \sinh \xi \cos \eta.$$

If we allow η to vary between β and β', and ξ between $+\alpha$ and $-\alpha$, the point (x, y) will move within the space contained between the ellipse

$$\frac{x^2}{c^2 \cosh^2 \alpha} + \frac{y^2}{c^2 \sinh^2 \alpha} = 1,$$

and the two hyperbolas

$$\frac{x^2}{c^2 \sin^2 \beta} - \frac{y^2}{c^2 \cos^2 \beta} = 1, \qquad \frac{x^2}{c^2 \sin^2 \beta'} - \frac{y^2}{c^2 \cos^2 \beta'} = 1.$$

Using then the coordinates (ξ, η) instead of (x, y) we find that our equations for w become

$$\frac{d^2 w}{d\xi^2} + \frac{d^2 w}{d\eta^2} = 0 \quad \text{for} \quad \left\{ \begin{matrix} \beta' < \eta < \beta \\ -\alpha < \xi < \alpha \end{matrix} \right\} \quad . \quad . \quad . \quad . \quad . \quad (3').$$

Also

$$\left. \begin{aligned} &\frac{dw}{d\xi} + \tfrac{1}{2}\tau c^2 \sin 2\eta = 0, \quad \text{when} \quad \xi = \pm \alpha, \qquad \beta' < \eta < \beta \\ &\frac{dw}{d\eta} - \tfrac{1}{2}\tau c^2 \sinh 2\xi = 0, \quad \text{when} \quad \eta = \beta \text{ or } \beta', \quad -\alpha < \xi < \alpha \end{aligned} \right\} \quad . \quad . \quad (4').$$

Write now

$$w = w_1 - \tfrac{1}{4}\tau c^2 \frac{\sinh 2\xi \sin 2\eta}{\cosh 2\alpha}.$$

Then

$$dw_1/d\xi = 0, \qquad \xi = \pm \alpha, \qquad \beta' < \eta < \beta \quad . \quad . \quad . \quad . \quad . \quad (5).$$

$$\left. \begin{aligned} &\frac{dw_1}{d\eta} - \tfrac{1}{2}\tau c^2 \sinh 2\xi \left(1 + \frac{\cos 2\beta}{\cosh 2\alpha}\right) = 0, \quad \eta = \beta, \quad -\alpha < \xi < \alpha \\ &\frac{dw_1}{d\eta} - \tfrac{1}{2}\tau c^2 \sinh 2\xi \left(1 + \frac{\cos 2\beta'}{\cosh 2\alpha}\right) = 0, \quad \eta = \beta', \quad -\alpha < \xi < \alpha \end{aligned} \right\} \quad . \quad . \quad (6).$$

Let us assume

$$w_1 = \sum_{n=0}^{n=\infty} \left(A_n \sinh \frac{\overparen{2n+1}\pi}{2\alpha} (\eta - \epsilon) + B_n \cosh \frac{\overparen{2n+1}\pi(\eta - \epsilon)}{2\alpha} \right) \sin \frac{\overparen{2n+1}\pi\xi}{2\alpha},$$

where $\epsilon = \frac{1}{2}(\beta + \beta')$.

Then conditions (5) and (3′) are identically satisfied. Let us now determine the coefficients A and B so that (6) shall also be satisfied.

We have to expand $\sinh 2\xi$ between the limits $\pm\alpha$ in a series of sines as follows :

$$\sinh 2\xi = a_0 \sin\frac{\pi\xi}{2\alpha} + a_1 \sin\frac{3\pi\xi}{2\alpha} + \ldots + a_n \sin\frac{\overparen{2n+1}\pi\xi}{2\alpha} + \ldots$$

The coefficients $a_0, a_1, \ldots a_n \ldots$ are found in the usual way

$$a_n = \frac{1}{\alpha}\int_{-\alpha}^{\alpha} \sinh 2\xi \sin\frac{\overparen{2n+1}\pi\xi}{2\alpha}\, d\xi = (-1)^n \frac{16\alpha\cosh 2\alpha}{\pi^2(2n+1)^2+16\alpha^2},$$

also we have, substituting in (6), equating coefficients of $\sin\frac{\overparen{2n+1}\pi\xi}{2\alpha}$, and writing $\beta-\beta'=2\gamma$,

$$\mathrm{A}_n \cosh\frac{\overparen{2n+1}\pi\gamma}{2\alpha} + \mathrm{B}_n \sinh\frac{\overparen{2n+1}\pi\gamma}{2\alpha} = \frac{\tau c^2\alpha}{(2n+1)\pi} a_n \left(1+\frac{\cos 2\beta}{\cosh 2\alpha}\right),$$

$$\mathrm{A}_n \cosh\frac{\overparen{2n+1}\pi\gamma}{2\alpha} - \mathrm{B}_n \sinh\frac{\overparen{2n+1}\pi\gamma}{2\alpha} = \frac{\tau c^2\alpha}{(2n+1)\pi} a_n \left(1+\frac{\cos 2\beta'}{\cosh 2\alpha}\right),$$

whence, solving for A_n and B_n, we find

$$\mathrm{A}_n = \frac{16\tau c^2\alpha^2}{(2n+1)\pi}(-1)^n \frac{\operatorname{sech}\frac{\overparen{2n+1}\pi\gamma}{2\alpha}}{\pi^2(2n+1)^2+16\alpha^2}(\cosh 2\alpha + \cos 2\epsilon\cos 2\gamma) \quad . \quad . \quad (7),$$

$$\mathrm{B}_n = -\frac{16\tau c^2\alpha^2}{(2n+1)\pi}(-1)^n \frac{\operatorname{cosech}\frac{\overparen{2n+1}\pi\gamma}{2\alpha}}{\pi^2(2n+1)^2+16\alpha^2}\sin 2\epsilon\sin 2\gamma \quad . \quad . \quad . \quad . \quad . \quad (8),$$

whence

$$w = -\tfrac{1}{4}\tau c^2 \frac{\sinh 2\xi \sin 2\eta}{\cosh 2\alpha}$$

$$+16\tau c^2\alpha^2\{\cosh 2\alpha + \cos 2\epsilon\cos 2\gamma\}\sum_{n=0}^{\infty}(-1)^n \frac{\sinh\frac{\overparen{2n+1}\pi}{2\alpha}(\eta-\epsilon)\sin\frac{\overparen{2n+1}\pi\xi}{2\alpha}}{\pi(2n+1)[\pi^2(2n+1)^2+16\alpha^2]\cosh\frac{\overparen{2n+1}\pi\gamma}{2\alpha}}$$

$$-16\tau c^2\alpha^2\sin 2\gamma\sin 2\epsilon\sum_{n=0}^{\infty}(-1)^n \frac{\cosh\frac{\overparen{2n+1}\pi}{2\alpha}(\eta-\epsilon)\sin\frac{\overparen{2n+1}\pi\xi}{2\alpha}}{\pi(2n+1)[\pi^2(2n+1)^2+16\alpha^2]\sinh\frac{\overparen{2n+1}\pi\gamma}{2\alpha}} \quad . \quad . \quad (9).$$

Having obtained w, the shears are easily deduced by simple differentiation

$$\left.\begin{aligned}\frac{\widehat{xz}}{\mu} &= \frac{1}{cJ}\left(\sinh\xi\sin\eta\frac{dw}{d\xi} + \cosh\xi\cos\eta\frac{dw}{d\eta}\right) - \tau c\sinh\xi\cos\eta \\ \frac{\widehat{yz}}{\mu} &= \frac{1}{cJ}\left(\cosh\xi\cos\eta\frac{dw}{d\xi} - \sinh\xi\sin\eta\frac{dw}{d\eta}\right) + \tau c\cosh\xi\sin\eta\end{aligned}\right\} \quad . \quad . \quad (10),$$

where $J = \cosh^2\xi\cos^2\eta + \sinh^2\xi\sin^2\eta$.

These again may be put into the slightly different form

$$\frac{\widehat{xz}}{\mu} = \frac{1}{cJ}\left(\sinh\xi\sin\eta\frac{dw_1}{d\xi} + \cosh\xi\cos\eta\frac{dw_1}{d\eta}\right) - \tau c\sinh\xi\cos\eta\,(1 + \operatorname{sech} 2\alpha)$$

$$\frac{\widehat{yz}}{\mu} = \frac{1}{cJ}\left(\cosh\xi\cos\eta\frac{dw_1}{d\xi} - \sinh\xi\sin\eta\frac{dw_1}{d\eta}\right) + \tau c\cosh\xi\sin\eta\,(1 - \operatorname{sech} 2\alpha).$$

The next quantity which we require is the moment of the shears

$$M = \int(x\widehat{yz} - y\widehat{xz})\,dx\,dy$$

$$= \frac{\mu\tau c^4}{2}\int_{\beta'}^{\beta}d\eta\int_{-\alpha}^{\alpha}d\xi\,[(\cosh 2\xi - \cos 2\eta) - \operatorname{sech} 2\alpha\,(1 - \cosh 2\xi\cos 2\eta)] \times J$$

$$+ \frac{\mu c^2}{2}\int_{\beta'}^{\beta}d\eta\int_{-\alpha}^{\alpha}d\xi\left[\frac{dw_1}{d\xi}\sin 2\eta - \frac{dw_1}{d\eta}\sinh 2\xi\right]\cdot$$

$$= \frac{\mu\tau c^4}{8}\int_{\beta'}^{\beta}d\eta\int_{-\alpha}^{\alpha}d\xi\,(\cosh 4\xi - \cos 4\eta - \operatorname{sech} 2\alpha\,\{\cosh 2\xi + \cos 2\eta\}$$

$$+ \operatorname{sech} 2\alpha\,\{\cosh 4\xi\cos 2\eta + \cosh 2\xi\cos 4\eta\})$$

$$+ \frac{\mu c^2}{2}\int_{\beta'}^{\beta}d\eta\sin 2\eta\Big[w_1\Big]_{-\alpha}^{\alpha} - \frac{\mu c^2}{2}\int_{-\alpha}^{\alpha}d\xi\sinh 2\xi\Big[w_1\Big]_{\beta'}^{\beta}\cdot$$

$$= \frac{\mu\tau c^4}{8}\left[\begin{array}{l}\gamma\sinh 4\alpha - \alpha\sin 4\gamma\cos 4\epsilon - \operatorname{sech} 2\alpha\,(2\gamma\sinh 2\alpha + 2\alpha\sin 2\gamma\cos 2\epsilon) \\ + \dfrac{\operatorname{sech} 2\alpha}{2}(\sinh 4\alpha\cos 2\epsilon\sin 2\gamma + \sinh 2\alpha\cos 4\epsilon\sin 4\gamma)\end{array}\right]$$

$$+ \sum_{n=0}^{n=\infty}\frac{16\mu\tau c^4\alpha^2(\cosh 2\alpha + \cos 2\epsilon\cos 2\gamma)}{\pi(2n+1)[\pi^2(2n+1)^2 + 16\alpha^2]\cosh\dfrac{\overparen{2n+1}\pi\gamma}{2\alpha}}\int_{\beta'}^{\beta}\sinh\frac{\overparen{2n+1}\pi}{2\alpha}(\eta - \epsilon)\sin 2\eta\,d\eta$$

$$- \sum_{n=0}^{n=\infty}\frac{16\mu\tau c^4\alpha^2\sin 2\gamma\sin 2\epsilon}{\pi(2n+1)[\pi^2(2n+1)^2 + 16\alpha^2]\sinh\dfrac{\overparen{2n+1}\pi\gamma}{2\alpha}}\int_{\beta'}^{\beta}\cosh\frac{\overparen{2n+1}\pi}{2\alpha}(\eta - \epsilon)\sin 2\eta\,d\eta$$

$$- \sum_{n=0}^{n=\infty}\frac{16\mu\tau c^4\alpha^2(\cosh 2\alpha + \cos 2\epsilon\cos 2\gamma)(-1)^n\tanh\dfrac{\overparen{2n+1}\pi\gamma}{2\alpha}}{\pi(2n+1)[\pi^2(2n+1)^2 + 16\alpha^2]}\int_{-\alpha}^{\alpha}\sin\frac{\overparen{2n+1}\pi\xi}{2\alpha}\sinh 2\xi\,d\xi.$$

$$= \frac{\mu\tau c^4}{8}\left[\begin{array}{l} \gamma \sinh 4\alpha - \alpha \sin 4\gamma \cos 4\epsilon - \operatorname{sech} 2\alpha \, \{2\gamma \sinh 2\alpha + 2\alpha \sin 2\gamma \cos 2\epsilon\} \\ + \frac{\operatorname{sech} 2\alpha}{2} \{\sinh 4\alpha \sin 2\gamma \cos 2\epsilon + \sinh 2\alpha \sin 4\gamma \cos 4\epsilon\} \end{array}\right]$$

$$- 256\mu\tau c^4\alpha^4 (\cosh 2\alpha + \cos 2\epsilon \cos 2\gamma)^2 \sum_{n=0}^{n=\infty} \frac{\tanh \frac{\overline{2n+1}\pi\gamma}{2\alpha}}{\pi(2n+1)[\pi^2(2n+1)^2 + 16\alpha^2]^2}$$

$$- 256\mu\tau c^4\alpha^4 (\sin 2\epsilon \sin 2\gamma)^2 \sum_{n=0}^{n=\infty} \frac{\coth \left(\frac{\overline{2n+1}\pi\gamma}{2\alpha}\right)}{\pi(2n+1)[\pi^2(2n+1)^2 + 16\alpha^2]^2}$$

$$+ 64\mu\tau c^4\alpha^3 \sin 2\gamma (\cosh 2\alpha \cos 2\epsilon + \cos 2\gamma \cos 4\epsilon) \sum_{n=0}^{n=\infty} \frac{1}{[(2n+1)^2\pi^2 + 16\alpha^2]^2}.$$

But the series $\sum_{0}^{\infty} \frac{1}{[(2n+1)^2\pi^2 + 16\alpha^2]^2}$ can be summed in finite terms,* and it is found to be

$$- \frac{1}{512\alpha^3} (2\alpha \operatorname{sech}^2 2\alpha - \tanh 2\alpha) \cdot$$

Substituting this value in the expression for the torsion moment we find

$$M = \frac{\mu\tau c^4}{8}\left[\begin{array}{l} (\gamma \sinh 4\alpha - \alpha \sin 4\gamma \cos 4\epsilon)(1 - \operatorname{sech}^2 2\alpha) \\ - \alpha \sin 4\gamma \cos 4\epsilon \operatorname{sech}^2 2\alpha - 2\alpha \operatorname{sech} 2\alpha \sin 2\gamma \cos 2\epsilon \\ + \operatorname{sech} 2\alpha \, [\sinh 2\alpha \sin 2\gamma] (\cosh 2\alpha \cos 2\epsilon + \cos 2\gamma \cos 4\epsilon) \\ - \sin 2\gamma (2\alpha \operatorname{sech}^2 2\alpha - \tanh 2\alpha)(\cosh 2\alpha \cos 2\epsilon + \cos 2\gamma \cos 4\epsilon) \end{array}\right]$$

— the two series terms.

Whence, after some obvious reductions

$$\frac{M}{\mu\tau c^4} = \frac{\tanh^2 2\alpha}{8} (\gamma \sinh 4\alpha - \alpha \sin 4\gamma \cos 4\epsilon)$$

$$- \tfrac{1}{4} \sin 2\gamma (\cosh 2\alpha \cos 2\epsilon + \cos 2\gamma \cos 4\epsilon)(2\alpha \operatorname{sech}^2 2\alpha - \tanh 2\alpha)$$

$$- 256\alpha^4 (\cosh 2\alpha + \cos 2\epsilon \cos 2\gamma)^2 \sum_{n=0}^{n=\infty} \frac{\tanh \frac{\overline{2n+1}\pi\gamma}{2\alpha}}{\pi(2n+1)[16\alpha^2 + \overline{2n+1}^2\pi^2]^2}$$

$$- 256\alpha^4 (\sin 2\gamma \sin 2\epsilon)^2 \sum_{n=0}^{n=\infty} \frac{\coth \frac{\overline{2n+1}\pi\gamma}{2\alpha}}{\pi(2n+1)[16\alpha^2 + \overline{2n+1}^2\pi^2]^2} \quad . \quad . \quad . \quad . \quad . \quad (11).$$

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* See CHRYSTAL'S 'Algebra,' vol. 2 (Differentiating the result marked (8) on p. 338).

§ 5. *Alternative Solution for the same Sections.*

There exists also an alternative solution ; it is generally of a less convenient form than the one last given. It may, however, be useful in certain cases.

Return now to the boundary conditions (4′) and write

$$w = w_1 + \tfrac{1}{4}\tau c^2 \frac{\sinh 2\xi \sin 2(\eta - \epsilon)}{\cos 2\gamma}$$

we then obtain

$$\frac{dw_1}{d\eta} = 0 \quad \text{when} \quad \eta = \beta \quad \text{or} \quad \eta = \beta', \quad -\alpha < \xi < +\alpha$$

and

$$\frac{dw_1}{d\xi} + \tfrac{1}{2}\tau c^2 \left\{\sin 2\eta + \frac{\cosh 2\alpha}{\cos 2\gamma} \sin 2(\eta - \epsilon)\right\} = 0,$$

when

$$\xi = \pm \alpha, \beta' < \eta < \beta.$$

The latter condition may be written

$$\frac{dw_1}{d\xi} + \tfrac{1}{2}\tau c^2 \left[\sin 2(\eta - \epsilon) \frac{(\cosh 2\alpha + \cos 2\gamma \cos 2\epsilon)}{\cos 2\gamma} + \cos 2(\eta - \epsilon) . \sin 2\epsilon\right] = 0.$$

Now let us write

$$w_1 = \varpi_1 + \varpi_2$$

where

$$\frac{d^2\varpi_1}{d\xi^2} + \frac{d^2\varpi_1}{d\eta^2} = 0, \quad \frac{d^2\varpi_2}{d\xi^2} + \frac{d^2\varpi_2}{d\eta^2} = 0,$$

and

$$d\varpi_1/d\eta = 0, \quad \eta - \epsilon = \pm\gamma \qquad d\varpi_2/d\eta = 0, \quad \eta - \epsilon = \pm\gamma.$$

But

$$\frac{d\varpi_1}{d\xi} + \tfrac{1}{2}\tau c^2 \sin 2(\eta - \epsilon)\left(\frac{\cosh 2\alpha + \cos 2\gamma \cos 2\epsilon}{\cos 2\gamma}\right) = 0 \quad . \quad . \quad . \quad . \quad (12)$$

$$\frac{d\varpi_2}{d\xi} + \tfrac{1}{2}\tau c^2 \cos 2(\eta - \epsilon) \sin 2\epsilon = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

when

$$\xi = \pm\alpha \quad \text{and} \quad -\gamma < \eta - \epsilon < +\gamma.$$

Assume

$$\varpi_1 = \sum_{n=0}^{\infty} A_n \sinh \frac{\overparen{2n+1}\pi\xi}{2\gamma} \sin \frac{\overparen{2n+1}\pi(\eta - \epsilon)}{2\gamma}$$

$$\varpi_2 = B_0\xi + \sum_{n=1}^{n=\infty} B_n \sinh \frac{n\pi\xi}{\gamma} \cos \frac{n\pi(\eta - \epsilon)}{\gamma},$$

we have now to express between limits ± γ

$$\cos 2\theta = b_0 + b_1 \cos \frac{\pi\theta}{\gamma} + b_2 \cos \frac{2\pi\theta}{\gamma} + \ldots + b_n \cos \frac{n\pi\theta}{\gamma} + \ldots$$

$$\sin 2\theta = a_1 \sin \frac{\pi\theta}{2\gamma} + a_2 \sin \frac{3\pi\theta}{2\gamma} + \ldots + a_n \sin \frac{\widehat{2n+1}\pi\theta}{2\gamma} + \ldots$$

We find

$$b_n = (-1)^{n+1} \frac{4\gamma \sin 2\gamma}{n^2\pi^2 - 4\gamma^2}, \quad a_n = (-1)^n \frac{16\gamma \cos 2\gamma}{(2n+1)^2\pi^2 - 16\gamma^2}$$

$$b_0 = \frac{1}{2\gamma} \sin 2\gamma$$

Substituting in (12) and (13) and equating coefficients we obtain

$$A_n = -\frac{16\tau c^2\gamma^2 (\cosh 2\alpha + \cos 2\gamma \cos 2\epsilon)(-1)^n}{(2n+1)\pi[(2n+1)^2\pi^2 - 16\gamma^2]} \operatorname{sech} \frac{\widehat{2n+1}\pi\alpha}{2\gamma},$$

$$B_n = \frac{2\tau c^2\gamma^2 \sin 2\epsilon \sin 2\gamma (-1)^n}{n\pi(n^2\pi^2 - 4\gamma^2)} \operatorname{sech} \frac{n\pi\alpha}{\gamma},$$

$$B_0 = -\frac{1}{4\gamma} \tau c^2 \sin 2\gamma \sin 2\epsilon,$$

whence

$$\begin{aligned} w = {} & \tfrac{1}{4}\tau c^2 \frac{\sinh 2\xi \sin 2(\eta - \epsilon)}{\cos 2\gamma} \\ & + 16\tau c^2\gamma^2 \sin 2\epsilon \sin 2\gamma \left[-\frac{\xi}{64\gamma^3} + \sum_{n=1}^{n=} (-1)^n \frac{\sinh \frac{n\pi\xi}{\gamma} \cos \frac{n\pi(\eta - \epsilon)}{\gamma}}{2n\pi(4n^2\pi^2 - 16\gamma^2) \cosh \frac{n\pi\alpha}{\gamma}} \right] \\ & - 16\tau c^2\gamma^2 (\cosh 2\alpha + \cos 2\gamma \cos 2\epsilon) \\ & \times \left[\sum_{n=0}^{n=\infty} (-1)^n \frac{\sinh \frac{\widehat{2n+1}\pi\xi}{2\gamma} \sin \frac{\widehat{2n+1}\pi(\eta - \epsilon)}{2\gamma}}{(2n+1)\pi[(2n+1)^2\pi^2 - 16\gamma^2] \cosh \frac{\widehat{2n+1}\pi\alpha}{2\gamma}} \right] \quad . \quad . \quad . \quad . \quad . \quad (14). \end{aligned}$$

It may be noted here that γ may vary between 0 and $\pi/2$. It follows that γ may have the value $\pi/2$, and in that case the denominator of the first term under the first Σ becomes zero. The same happens to the denominator of the first term of the second Σ when γ has the value $\pi/4$. Further, in this latter case the first term in w also becomes infinite, so that the expression (14) is apparently no longer applicable.

It is easy to see, however, that the terms, which are apparently infinite in (14), exactly cancel each other. If we write $\gamma = \pi/4 - \zeta$, where ζ is small, simplify and proceed to the limit, we find that when $\gamma = \pi/4$ the two infinite terms reduce to:

$$\begin{aligned} & \tfrac{1}{4}\tau c^2 \sinh 2\xi \sin 2(\eta - \epsilon) \left[\frac{\cos 2\epsilon}{\cosh 2\alpha} - \frac{4}{\pi} - \frac{4\alpha}{\pi} \tanh 2\alpha \right] \\ & + \frac{\tau c^2}{\pi} [\xi \cosh 2\xi \sin 2(\eta - \epsilon) + (\eta - \epsilon) \cos 2(\eta - \epsilon) \sinh 2\xi] \quad . \quad . \quad . \quad (15), \end{aligned}$$

which is finite.

In like manner when $\gamma = \pi/2$ the limit of w is easily evaluated.

For all *other* values of γ, ϵ and α the series in (14) are absolutely convergent as they stand, for all points (ξ, η) *within* or *on* the boundary of the section. The same holds of all the series in the last paragraph for *all* values of α, γ, ϵ.

The expression for w being given, the shears and torsion moment are obtained as before

$$\mathrm{M} = \frac{\mu\tau c^4}{8}\int_{\beta'}^{\beta} d\eta \int_{-\alpha}^{\alpha} d\xi\,(\cosh 4\xi - \cos 4\eta) + \frac{\mu c^2}{2}\int_{\beta'}^{\beta}\Big[w\Big]_{-\alpha}^{\alpha} \sin 2\eta\, d\eta - \frac{\mu c^2}{2}\int_{-\alpha}^{\alpha}\Big[w\Big]_{\beta'}^{\beta} \sinh 2\xi\, d\xi.$$

The integrations are all easily effected, and we find

$$\begin{aligned}\mathrm{M} = {} & \frac{\mu\tau c^4}{8}(\gamma \sinh 4\alpha - \alpha \sin 4\gamma \cos 4\epsilon) \\ & + \frac{\mu\tau c^4}{4}\frac{(\alpha \sin 2\gamma + \gamma \sinh 2\alpha \cos 2\epsilon)}{\cos 2\gamma} - \frac{1}{16}\frac{\mu\tau c^4(\sinh 4\alpha \sin 2\gamma + \sinh 2\alpha \sin 4\gamma \cos 2\epsilon)}{\cos 2\gamma} \\ & + 64\mu\tau c^4\gamma^3(\cosh 2\alpha + \cos 2\gamma \cos 2\epsilon)\sinh 2\alpha \sum_{n=0}^{n=\infty} \frac{1}{[(2n+1)^2\pi^2 - 16\gamma^2]^2} \\ & - 256\mu\tau c^4\gamma^4(\cosh 2\alpha + \cos 2\gamma \cos 2\epsilon)^2 \sum_{n=0}^{n=\infty} \frac{\tanh\dfrac{(2n+1)\pi\alpha}{2\gamma}}{(2n+1)\pi[\pi^2(2n+1)^2 - 16\gamma^2]^2} \\ & - 256\mu\tau c^4\gamma^4(\sin 2\epsilon \sin 2\gamma)^2 \left\{\sum_{n=1}^{n=\infty} \frac{\tanh\dfrac{n\pi\alpha}{\gamma}}{2n\pi(4n^2\pi^2 - 16\gamma^2)^2} + \frac{\alpha}{1024\gamma^5}\right\}.\end{aligned}$$

Remembering that $\sum\limits_{n=0}^{n=\infty}\dfrac{1}{\{(2n+1)^2\pi^2 - 16\gamma^2\}^2} = \dfrac{2\gamma \sec^2 2\gamma - \tan 2\gamma}{512\gamma^3}$, reducing, and regrouping the terms, we find finally

$$\begin{aligned}\frac{\mathrm{M}}{\mu\tau c^4} = {} & \tfrac{1}{8}(\alpha \sin 4\gamma - \gamma \sinh 4\alpha)\tan^2 2\gamma \\ & + \tfrac{1}{4}(\cosh 2\alpha + \cos 2\gamma \cos 2\epsilon)(2\gamma \sec^2 2\gamma - \tan 2\gamma)\sinh 2\alpha \\ & + \frac{\alpha}{8\gamma}\sin 4\gamma \sin^2 2\epsilon\,(2\gamma - \tan 2\gamma) \\ & - 256\gamma^4(\cosh 2\alpha + \cos 2\gamma \cos 2\epsilon)^2 \sum_{n=0}^{n=\infty} \frac{\tanh\dfrac{\overline{2n+1}\,\pi\alpha}{2\gamma}}{\left.2n+1\right|\pi[\pi^2(2n+1)^2 - 16\gamma^2]^2} \\ & - 256\gamma^4(\sin 2\epsilon \sin 2\gamma)^2 \sum_{n=1}^{n=\infty} \frac{\tanh\dfrac{n\pi\alpha}{\gamma}}{2n\pi(4n^2\pi^2 - 16\gamma^2)^2} \quad \cdots\cdots\cdots \quad (16).\end{aligned}$$

We may test the correctness of the expressions (11) and (16) by remembering that when we make the distance c between the foci very great, and γ and α very small, the section reduces to a rectangle, of which the half sides a and b are given by

$$a = c\alpha\cos\epsilon, \qquad b = c\gamma\cos\epsilon.$$

The first and last terms in (16) are ultimately of negligible order, when multiplied by c^4.

The second and third reduce to

$$\tfrac{8}{3}\gamma^3\alpha\,(1 + \cos 2\epsilon) - \tfrac{2}{3}\gamma^3\alpha\,(1 - \cos 4\epsilon) = \tfrac{2}{3}\gamma^3\alpha\,(3 + 4\cos 2\epsilon + \cos 4\epsilon)$$
$$= \tfrac{16}{3}\gamma^3\alpha\cos^4\epsilon.$$

The fourth term gives

$$-\,1024\,(\gamma\cos\epsilon)^4\sum_{n=0}^{n=\infty}\frac{\tanh\dfrac{\overparen{2n+1}\pi a}{2b}}{(2n+1)^5\pi^5}.$$

Hence

$$\frac{\mathrm{M}}{\mu\tau b^4} = \tfrac{16}{3}\frac{a}{b} - \left(\frac{4}{\pi}\right)^5\sum_{n=0}^{n=\infty}\frac{\tanh\dfrac{\overparen{2n+1}\pi a}{2b}}{(2n+1)^5}\quad . \quad . \quad .,$$

which is one of SAINT-VENANT's expressions for the torsion moment of a rectangle of sides $2a$, $2b$.

If we treat in a similar manner expression (11), neglecting terms of order greater than four in α, γ, we get the other expression for the torsion moment of the rectangle.

§ 6. *Recapitulation of Results for the Symmetrical Case.*

By far the most important case we have to deal with is that in which the sections are symmetrical.

We have then $\beta' = -\beta$, and therefore $\epsilon = 0$, $\gamma = \beta$.

Both solutions then simplify a good deal, and we have the equivalent expressions

$$w = -\tfrac{1}{4}\tau c^2\sinh 2\xi\sin 2\eta\,\mathrm{sech}\,2\alpha$$
$$+\,16\tau c^2\alpha^2(\cosh 2\alpha + \cos 2\beta)\sum_{n=0}^{n=\infty}\frac{(-1)^n\sinh\dfrac{\overparen{2n+1}\pi\eta}{2\alpha}\sin\dfrac{\overparen{2n+1}\pi\xi}{2\alpha}}{\pi(2n+1)\left[(2n+1)^2\pi^2 + 16\alpha^2\right]\cosh\dfrac{\overparen{2n+1}\pi\beta}{2\alpha}}$$
$$= \tfrac{1}{4}\tau c^2\sinh 2\xi\sin 2\eta\sec 2\beta$$
$$-\,16\tau c^2\beta^2(\cosh 2\alpha + \cos 2\beta)\sum_{n=0}^{n=\infty}\frac{(-1)^n\sinh\dfrac{\overparen{2n+1}\pi\xi}{2\beta}\sin\dfrac{\overparen{2n+1}\pi\eta}{2\beta}}{\pi(2n+1)\left[(2n+1)^2\pi^2 - 16\beta^2\right]\cosh\dfrac{\overparen{2n+1}\pi\alpha}{2\beta}}. \quad (17).$$

$$\frac{\widehat{xz}}{\mu} = -\tau c \sinh\xi\cos\eta\,(1 + \operatorname{sech} 2\alpha) + 32\tau c\alpha^2\,\frac{(\cosh 2\alpha + \cos 2\beta)}{(\cosh 2\xi + \cos 2\eta)}$$

$$\times\sum_{n=0}^{\infty}\frac{(-1)^n\left(\sinh\xi\sin\eta\,\dfrac{d}{d\xi} + \cosh\xi\cos\eta\,\dfrac{d}{d\eta}\right)\sinh\dfrac{\widehat{2n+1}\pi\eta}{2\alpha}\sin\dfrac{\widehat{2n+1}\pi\xi}{2\alpha}}{\pi(2n+1)[(2n+1)^2\pi^2 + 16\alpha^2]\cosh\dfrac{\widehat{2n+1}\pi\beta}{2\alpha}}$$

$$= -\tau c\sinh\xi\cos\eta\,(1 - \sec 2\beta) - 32\tau c\beta^2\,\frac{(\cosh 2\alpha + \cos 2\beta)}{(\cosh 2\xi + \cos 2\eta)}$$

$$\times\sum_{n=0}^{\infty}\frac{(-1)^n\left(\sinh\xi\sin\eta\,\dfrac{d}{d\xi} + \cosh\xi\cos\eta\,\dfrac{d}{d\eta}\right)\sinh\dfrac{\widehat{2n+1}\pi\xi}{2\beta}\sin\dfrac{\widehat{2n+1}\pi\eta}{2\beta}}{\pi(2n+1)[(2n+1)^2\pi^2 - 16\beta^2]\cosh\dfrac{\widehat{2n+1}\pi\alpha}{2\beta}}\ . \quad (18).$$

$$\frac{\widehat{yz}}{\mu} = \tau c\cosh\xi\sin\eta\,(1 - \operatorname{sech} 2\alpha) + 32\tau c\alpha^2\left(\frac{\cosh 2\alpha + \cos 2\beta}{\cosh 2\xi + \cos 2\eta}\right)$$

$$\times\sum_{n=0}^{n=}\frac{(-1)^n\left(\cosh\xi\cos\eta\,\dfrac{d}{d\xi} - \sinh\xi\sin\eta\,\dfrac{d}{d\eta}\right)\sinh\dfrac{\widehat{2n+1}\pi\eta}{2\alpha}\sin\dfrac{\widehat{2n+1}\pi\xi}{2\alpha}}{\pi(2n+1)[\pi^2(2n+1)^2 + 16\alpha^2]\cosh\dfrac{\widehat{2n+1}\pi\beta}{2\alpha}}$$

$$= \tau c\cosh\xi\sin\eta\,(1 + \sec 2\beta) - 32\tau c\beta^2\left(\frac{\cosh 2\alpha + \cos 2\beta}{\cosh 2\xi + \cos 2\eta}\right)$$

$$\times\sum_{n=0}^{n=\infty}\frac{(-1)^n\left(\cosh\xi\cos\eta\,\dfrac{d}{d\xi} - \sinh\xi\sin\eta\,\dfrac{d}{d\eta}\right)\sinh\dfrac{\widehat{2n+1}\pi\xi}{2\beta}\sin\dfrac{\widehat{2n+1}\pi\eta}{2\beta}}{\pi(2n+1)[\pi^2(2n+1)^2 - 16\beta^2]\cosh\dfrac{\widehat{2n+1}\pi\alpha}{2\beta}}\ . \quad (19).$$

$$\frac{\mathrm{M}}{\mu\tau c^4} = \frac{\tanh^2 2\alpha}{8}(\beta\sinh 4\alpha - \alpha\sin 4\beta)$$

$$- \tfrac{1}{4}\sin 2\beta\,(\cosh 2\alpha + \cos 2\beta)(2\alpha\operatorname{sech}^2 2\alpha - \tanh 2\alpha)$$

$$- 256\alpha^4(\cosh 2\alpha + \cos 2\beta)^2\sum_{n=0}^{n=\infty}\frac{\tanh\dfrac{\widehat{2n+1}\pi\beta}{2\alpha}}{\pi(2n+1)[16\alpha^2 + \overline{2n+1}|^2\pi^2]^2}$$

$$= \frac{\tan^2 2\beta}{8}(\alpha\sin 4\beta - \beta\sinh 4\alpha)$$

$$+ \tfrac{1}{4}\sinh 2\alpha\,(\cosh 2\alpha + \cos 2\beta)(2\beta\sec^2 2\beta - \tan 2\beta)$$

$$- 256\beta^4(\cosh 2\alpha + \cos 2\beta)^2\sum_{n=0}^{n=\infty}\frac{\tanh\dfrac{\widehat{2n+1}\pi\alpha}{2\beta}}{\pi(2n+1)[\overline{2n+1}|^2\pi^2 - 16\beta^2]^2}\ . \quad . \quad (20).$$

§ 7. *Importance of the Maximum Stress. Application to Rupture.*

I now pass on to the numerical determination of the torsion moment and stresses, and in particular of the maximum stress.

SAINT-VENANT has shown, in his memoir on torsion,* that if we assume an ellipsoidal distribution of *limiting* stretch (*i.e.*, stretch such that, when it is exceeded, the elasticity of the material is impaired), then, in the case of a shaft under torsion, we have at any point

$$s/\bar{s} = \sqrt{(\sigma_{xz}^2 + \sigma_{yz}^2)/\bar{\sigma}^2},$$

where σ_{yz}, σ_{xz} are the shearing strains in the planes yz, xz respectively, $\bar{\sigma}$ is the value of the *limiting shearing strain* of the material, and $s/\bar{s}$ is the maximum value of the ratio of the stretch in any direction to the *limiting* stretch in that direction.

The condition that there should be no failure of elasticity is therefore that $s/\bar{s} < 1$. Therefore

$$\sigma_{yz}^2 + \sigma_{zx}^2 < \bar{\sigma}_0^2,$$

and, since $\sigma_{yz} = \widehat{yz}/\mu$, $\sigma_{zx} = \widehat{xz}/\mu$,

$$\widehat{xz}^2 + \widehat{yz}^2 < \mu^2\bar{\sigma}_0^2.$$

The points where this condition will first be broken are called by SAINT-VENANT the *fail-points* ("*points dangereux*"). They are clearly the points where $\widehat{xz}^2 + \widehat{yz}^2$ is a maximum, *i.e.*, where the resultant stress across an element of the section is a maximum. Hence the importance of determining the points of maximum stress. Strictly speaking, the latter give us no certain information as to where, or how, rupture will actually take place: all that they tell us is where linear elasticity begins to fail. But they will, in general, give us a useful clue to the regions where breaking may be expected to occur, and, in the absence of any definite theory of plastic deformation and rupture, we must be content to be guided by the results of elastic theory.

I have worked out numerically the values of the stresses at the points $\xi = \pm\alpha$, $\eta = 0$ and $\xi = 0$, $\eta = \pm\beta$. These give the four points in which the axes meet the boundary of the cross-section. I have denoted them by A and B respectively. The boundary is convex at A and concave at B From considerations of symmetry it follows that A and B must be points of *maximum* or *minimum* stress, and the stress being zero both at the corners and at the centre, it will often happen that they are points of *maximum* stress. When this is the case those of the points A and B, where the stress is numerically greater, will give us the *fail-points*. But there is an obvious exception, when there are two points of maximum stress on either side of the mid-point, and this is a case which, we shall see, does occur in these sections.

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* 'Mémoires des Savants Étrangers,' 1855, vol. xiv., pp. 278–288. See also TODHUNTER and PEARSON, Hist. Elast.,' vol. ii., part i., pp. 7–10.

§ 8. *Methods of Calculation.*

The symmetrical sections selected for numerical treatment are those for which $\beta = \pi/6$, $\pi/4$, $\pi/3$, $\pi/2$ and $\alpha = \pi/6$, $\pi/3$, $\pi/2$, and $2\pi/3$, sixteen in all. These sections are shown in figs. 2 to 5. β of course is the complement of the half angle between the asymptotes, and all sections having the same β have been collected in one figure.

The numerical calculations were generally based upon the formulæ of § 4, as they did not require modification for the value $\pi/4$ of β. In many cases, however, the alternative series were used also, in order to test the results obtained.

The calculation of the terms of the series in the expression for the torsion moment was carried on until $\frac{\overparen{2n+1}\pi\alpha}{2\beta}$ was so great that $\tanh\frac{\overparen{2n+1}\pi\alpha}{2\beta}$ could be taken sensibly equal to unity. The remainder of the series, namely,

$$\sum_{n}^{\infty}\frac{1}{\pi(2n+1)\,[\pi^2(2n+1)^2+16\alpha^2]^2},$$

was then obtained by expanding the denominator by the binomial theorem, thus

$$\sum_{n}^{\infty}\frac{1}{\pi^5(2n+1)^5} - 2\,(16\alpha^2)\sum_{n}^{\infty}\frac{1}{\pi^7(2n+1)^7} + 3\,(16\alpha^2)^2\sum_{n}^{\infty}\frac{1}{\pi^9(2n+1)^9} - \dots$$

The successive terms were easily calculated from the values of $\sum\limits_{0}\frac{1}{n^{2s+1}}$ given in CHRYSTAL'S 'Algebra,' vol. 2, chapter XXX, § 15.

The stresses at A and B were calculated from formulæ (18) and (19). If S_A, S_B denote these stresses, we find easily

$$\frac{S_A}{\mu} = -\tau c\tanh 2\alpha\cosh\alpha + 8\tau c\alpha\left(\frac{\cosh 2\alpha+\cos 2\beta}{\cosh\alpha}\right)\sum_{n=0}^{n=\infty}\frac{\operatorname{sech}\frac{\overparen{2n+1}\pi\beta}{2\alpha}}{\pi^2(2n+1)^2+16\alpha^2}$$

$$= \tau c\sinh\alpha\,(\sec 2\beta-1) - 8\tau c\beta\left(\frac{\cosh 2\alpha+\cos 2\beta}{\cosh\alpha}\right)\sum_{n=0}^{n=\infty}\frac{(-1)^n\tanh\frac{\overparen{2n+1}\pi\alpha}{2\beta}}{\pi^2(2n+1)^2-16\beta^2}\,. \qquad (21),$$

$$\frac{S_B}{\mu} = \tau c\sin\beta\,(1-\operatorname{sech}2\alpha) + 8\tau c\alpha\left(\frac{\cosh 2\alpha+\cos 2\beta}{\cos\beta}\right)\sum_{n=0}^{n=\infty}\frac{(-1)^n\tanh\frac{\overparen{2n+1}\pi\beta}{2\alpha}}{\pi^2(2n+1)^2+16\alpha^2}$$

$$= \tau c\tan 2\beta\cos\beta - 8\tau c\beta\left(\frac{\cosh 2\alpha+\cos 2\beta}{\cos\beta}\right)\sum_{n=0}^{n=\infty}\frac{\operatorname{sech}\frac{\overparen{2n+1}\pi\alpha}{2\beta}}{(2n+1)^2\pi^2-16\beta^2}\quad\cdot\;\cdot\;\cdot \qquad (22).$$

The first expressions for S_A, S_B were used in each case as the main basis of the calculation, but the results were partly verified by means of the second expressions.

With regard to the accuracy obtained, I may say that, where I was able to verify, I found the values of the stresses correct to five, and sometimes to six, significant figures. In the values of the torsion moment the first five figures generally agreed, so that, on the whole, the results may be considered correct to the number of figures given.

The first series for S_B and the second series for S_A are very slowly convergent indeed. The method adopted in dealing with them was the following:

The series

$$\sum_{n=0}^{n=\infty} \frac{(-1)^n \tanh \frac{\widehat{2n+1}\pi\beta}{2\alpha}}{\pi^2(2n+1)^2+16\alpha^2}$$

was broken up into two series

$$\sum_{n=0}^{n=\infty} \frac{(-1)^n}{\pi^2(2n+1)^2+16\alpha^2} - \sum_{n=0}^{n=\infty} \frac{(-1)^n\left(1-\tanh \frac{\widehat{2n+1}\pi\beta}{2\alpha}\right)}{\pi^2(2n+1)^2+16\alpha^2}.$$

The latter series converges rapidly, and was easily calculated. The former was calculated to an even number of terms, and the remainder obtained by means of the Euler-Maclaurin sum-formula, thus:

$$\sum_{n=2x-1}^{n=\infty} \frac{(-1)^n}{\pi^2(2n+1)^2+16\alpha^2} = \sum_x^{\infty} \frac{1}{\pi^2(4x+1)^2+16\alpha^2} - \sum_x^{\infty} \frac{1}{\pi^2(4x-1)^2+16\alpha^2}.$$

Now

$$\Sigma u_x = \mathrm{C} + \int u_x\,dx - \tfrac{1}{2}u_x + \frac{\mathrm{B}_1}{2!}\frac{du_x}{dx} - \frac{\mathrm{B}_2}{4!}\frac{d^3u_x}{dx^3} + \ldots,$$

where $\mathrm{B}_1 = \frac{1}{6}$, $\mathrm{B}_2 = \frac{1}{30}$, &c.

Let me write

$$\rho^2 = (4x+1)^2\pi^2 + 16\alpha^2,$$

$$\theta = \tan^{-1}\frac{4\alpha}{(4x+1)\pi}, \qquad u_x = \frac{1}{\rho^2}.$$

Then

$$\int u_x\,dx = -\frac{\theta}{16\alpha\pi} \qquad \frac{d^nu_x}{dx^n} = \frac{(-1)^n \sin \widehat{n+1}\theta}{\rho^{n+1}}\frac{(4\pi)^n\,n\,!}{4\alpha},$$

whence

$$\sum^{x-1} u_x = \mathrm{C} - \frac{\theta}{16\alpha\pi} - \frac{1}{2\rho^2} - \frac{\mathrm{B}_1}{2}\frac{\sin 2\theta}{\rho^2}\cdot\frac{(4\pi)}{4\alpha} + \frac{\mathrm{B}_2}{4}\frac{\sin 4\theta}{\rho^4}\frac{(4\pi)^3}{4\alpha} - \ldots$$

Put $x=\infty$, $\theta = 0$, $\rho = \infty$,

$$\sum^{\infty} u_x = \mathrm{C},$$

therefore,

$$\sum_x^{\infty} u_x = \frac{\theta}{16\alpha\pi} + \frac{1}{2\rho^2} + \frac{\mathrm{B}_1}{2}\frac{\sin 2\theta}{\rho^2}\left(\frac{\pi}{\alpha}\right) - \frac{\mathrm{B}_2}{4}\cdot\frac{\pi}{\alpha}\cdot\frac{\sin 4\theta}{\rho^4}\,16\pi^2 + \ldots$$

In like manner if $v_r = \dfrac{1}{\rho'^2}$,

$$\rho'^2 = \pi^2(4x-1)^2 + 16\alpha^2,$$

$$\theta' = \tan^{-1}\frac{4\alpha}{(4x-1)\pi},$$

$$\sum_r^\infty v_r = \frac{\theta'}{16\alpha\pi} + \frac{1}{2\rho'^2} + \frac{B_1}{2}\frac{\sin 2\theta'}{\rho'^2}\left(\frac{\pi}{\alpha}\right) - \frac{B_2}{4}\left(\frac{\pi}{\alpha}\right)\frac{\sin 4\theta'}{\rho'^4}16\pi^2 + \ldots$$

These series converge fairly rapidly if x is at all large, and thus the remainder can be obtained.

Even with the help of all these devices the labour of calculating the moment and stress for the sixteen sections was considerable.

The values of the hyperbolic functions were taken from GUDERMANN'S Tables ('Theorie der Potenzial oder Cyklisch-hyperbolischen Functionen'), and from GLAISHER'S and NEUMAN'S Tables of the Exponential Function ('Cambridge Phil. Trans.,' vol. 13).

§ 9. *Values of the Torsional Rigidity.*

The first quantity calculated was the torsion moment. The values found are shown in the table below.

TABLE of $M/\mu\tau c^4$.

	$\beta = \pi/6$.	$\pi/4$.	$\pi/3$.	$\pi/2$.
$\alpha = \pi/6$	·1710	·3116	·4055	·4676
$\alpha = \pi/3$	·8764	2·0317	3·2205	4·8117
$\alpha = \pi/2$	3·8798	9·4161	16·442	29·912
$\alpha = 2\pi/3$	22·898	54·824	96·411	194·18

It is interesting to compare this table with the table of values of the torsion moment, as given by DE SAINT-VENANT'S empirical formula, viz.,

$$\text{torsion moment} = \frac{\mu\tau}{40}\frac{A^4}{I},$$

where A = area, I = moment of inertia of section about its centroid.

Calling M′ this value of the torsion moment, we have

$$\frac{M'}{\mu\tau c^4} = \frac{1}{5}\frac{(\beta\sinh 2\alpha + \alpha\sin 2\beta)^4}{\beta\sinh 4\alpha - \alpha\sin 4\beta},$$

whence we obtain the following set of values :—

TABLE of $M'/\mu\tau c^4$.

	$\beta = \pi/6$.	$\pi/4$.	$\pi/3$.	$\pi/2$.
$\alpha = \pi/6$	·1835	·3266	·4152	·4723
$\alpha = \pi/3$	·9914	2·3759	3·8023	6·0121
$\alpha = \pi/2$	4·3368	12·195	23·260	51·500
$\alpha = 2\pi/3$	23·289	71·826	152·96	420·95

If we compare this with the preceding table, we see at once that although the agreement is fairly good for the more compact sections, SAINT-VENANT'S empirical formula utterly breaks down for deeply indented sections. That, indeed, might have been expected, since it takes no account of slits cut into the material. One rather noticeable feature in the comparison is that SAINT-VENANT'S formula always gives too high a value for the torsional rigidity.

§ 10. *Comparison with the Circle. "Relative" Torsional Rigidity.*

In order, however, to compare properly the efficiency or usefulness of these various sections, it was found advisable to refer each of them to some kind of standard, or unit. The most obvious standard, as I thought, was the circular section, this being the one whose torsion obeys the most simple laws. I determined, therefore, to compare every section with the circular section of the same area.

Now if r be the radius of this circular section, its torsion moment $M_0 = \frac{1}{2}\pi\mu\tau r^4$, and the maximum stress $S_0 = \mu\tau r$. To find r, we have the equation :

$$\pi r^2 = \text{area of given section} = c^2(\beta \sinh 2\alpha + \alpha \sinh 2\beta),$$

whence

$$\frac{M_0}{\mu\tau c^4} = \frac{\pi}{2}\left(\frac{\beta \sinh 2\alpha + \alpha \sin 2\beta}{\pi}\right)^2,$$

$$\frac{S_0}{\mu\tau c} = \left(\frac{\beta \sinh 2\alpha + \alpha \sin 2\beta}{\pi}\right)^{\frac{1}{2}}.$$

The values of $\left(\frac{\beta \sinh 2\alpha + \alpha \sin 2\beta}{\pi}\right)$ for the various sections are easily found.

TABLE of $\left(\dfrac{\beta \sinh 2\alpha + \alpha \sin 2\beta}{\pi}\right)$.

	$\beta = \pi/6$.	$\pi/4$.	$\pi/3$.	$\pi/2$.
$\alpha = \pi/6$	·3526	·4790	·5608	·6247
$\alpha = \pi/3$	·9551	1·3330	1·6216	1·9993
$\alpha = \pi/2$	2·3578	3·3872	4·2826	5·7744
$\alpha = 2\pi/3$	6·0713	8·9076	11·565	16·482

whence we obtain the following table giving us M/M_0:—

	$\beta = \pi/6$.	$\pi/4$.	$\pi/3$.	$\pi/2$.
$\alpha = \pi/6$	·8756	·8644	·8208	·7628
$\alpha = \pi/3$	·6116	·7279	·7797	·7663
$\alpha = \pi/2$	·4443	·5225	·5707	·5711
$\alpha = 2\pi/3$	·3955	·4399	4590	·4551

When we look at this table, we observe immediately that the torsional rigidity decreases, compared with the torsional rigidity of the circular section, as we increase α, that is to say, as we decrease the thickness of the neck with regard to the other linear dimension. This indeed might have been expected, for it is clear that such a process must weaken the rigidity enormously, inasmuch as it tends to render the two halves of the section independent of each other.

When $\alpha = \pi/6$, $\beta = \pi/6$, the ratio M/M_0 is greatest. In this case the section does not deviate very much from a square. (For very small values of α and β the section is, of course, a rectangle.) This result shows us therefore that, so far as torsional rigidity is concerned, the square is a more efficient form of section than any one of those dealt with in the present paragraph.

That the circle is a more efficient type of section for rigidity is quite evident from the table, since all the values in it are less than unity.

It may be interesting to note what are the values of M/M_0 for the full ellipse. When we use the values given by SAINT-VENANT in his memoir on torsion

$$M = \frac{\mu\tau\pi a^3 b^3}{a^2 + b^2} \quad \text{and} \quad M_0 = \frac{\mu\tau\pi a^2 b^2}{2},$$

$$\frac{M}{M_0} = \frac{2ab}{a^2 + b^2} = \tanh 2\alpha, \quad \text{if} \quad b/a = \tanh \alpha.$$

Hence we have, for the full ellipse

	$\alpha = \pi/6$.	$\alpha = \pi/3$.	$\alpha = \pi/2$.	$\alpha = 2\pi/3$.
M/M_0	·7807	·9701	·9963	·9995

If we compare this with the previous table, we see that for the flattest ellipse, $\alpha = \pi/6$, the ratio of the torsional rigidity to the torsional rigidity of the circular section of equal area, which I propose to call for brevity the *relative* torsional rigidity, is greater for the truncated than for the full ellipse, except in the last case, $\beta = \pi/2$. This last must necessarily be, since the strength of the section should be reduced by cutting two slits into it along the major axis. For the higher values of α we see that the relative torsional rigidity is always greater for the full than for the truncated ellipse.

§ 11. *Values of the Stresses at the Points of Symmetry.*

Passing now to the values of the stress, the values of $S_B/\mu\tau c$ are given, for the sixteen symmetrical sections, in the table below.

TABLE of $S_B/\mu\tau c$.

	$\beta = \pi/6$.	$\beta = \pi/4$.	$\beta = \pi/3$.	$\beta = \pi/2$.
$\alpha = \pi/6$	·7594	·8421	·8949	∞
$\alpha = \pi/3$	1·1482	1·6735	2·2690	∞
$\alpha = \pi/2$	1·3084	2·2798	3·6780	∞
$\alpha = 2\pi/3$	1·3897	2·7955	5·2484	∞

The values of $S_A/\mu\tau c$ were only calculated for $\beta = \pi/6$ and $\beta = \pi/4$, it being clear that for the given values of α, S_A would be less than S_B for $\beta = \pi/3$. The values of $S_A/(\mu\tau c)$ are given in the following table:—

TABLE of $S_A/\mu\tau c$.

	$\beta = \pi/6$.	$\pi/4$.
$\alpha = \pi/6$	·6730	·8126
$\alpha = \pi/3$	·8394	1·0975
$\alpha = \pi/2$	1·1971	1·5176
$\alpha = 2\pi/3$	1·8987	2·3650

The values of $S_A/\mu\tau c$, as calculated from the formula (21) above, actually turn out to be negative. The sign is, however, of no importance.

Several important results are seen to follow at once from these tables.

In the first place, when a keyway cut into a shaft of elliptic cross-section reduces to a mere slit, the stress at the inner extremity of the keyway is seen to be infinite, although this keyway is the limit of a single continuous curve and not of two curves making a sharp angle, as is the case for a slit along a radius of a circle, obtained as the limit of a keyway of the shape of a sector of the circle.

Such slits are thus bound to produce rupture or plastic flow of the material at their deepest points, whatever be the manner in which we approximate to them in practice.

The second point of importance, which these tables bring out clearly, is that the maximum strain and stress do not always occur, as most of the results obtained by DE SAINT-VENANT would lead one to suppose, and as THOMSON and TAIT ('Natural Philosophy,' vol. 1, Part II., § 710), and BOUSSINESQ ('Journal de Mathématiques,' Série II., vol. 16, p. 200) assumed, at the point of the boundary nearest the centre.

Indeed SAINT-VENANT himself, in his edition of NAVIER'S 'Leçons de Mécanique' (§ 33, p. 313), has given an example to the contrary, and it happens that the section dealt with in this example is closely analogous to the sections of fig. 2 in this paper. The shape of the section is reproduced in fig. 11 (p. 342), from SAINT-VENANT'S 'Leçons de NAVIER.' He calls it a "*section en double spatule analogue à celle d'un rail de chemin de fer.*" He gives two numerical examples in which the ratio of breadth to length of the section is ·20 and ·14, corresponding for our sections, when $\beta = \pi/6$, to $\alpha = 1·647$ and $\alpha = 1·985$ respectively. He finds in these two cases that the fail-points are *not* at the point of symmetry on the contour which is closest to the centre, but at points on the contour at a distance from the axis of symmetry of ·46 and ·52 of the half-length respectively.

Now it is easy to see from the tables above that the result which one would expect according to the ordinary rule, namely $S_B > S_A$, does hold in fifteen out of the sixteen cases, but there is one exception in the case of the section $\alpha = 2\pi/3$, $\beta = \pi/6$, when the greater of the two stresses is found to occur at A, the point further from the centre.

I was much struck at first by this apparently solitary deviation from the rule, and was inclined to ascribe it to some error in the arithmetic.

In order to test this, I calculated the values of S_A and S_B for the neighbouring section, $\beta = \pi/6$, $\alpha = 3\pi/4$. I found

$$\frac{S_A}{\mu\tau c} = 2·4333, \qquad \frac{S_B}{\mu\tau c} = 1·4144,$$

confirming the previous exception.

I then took the expressions (21) and (22) for S_A and S_B, and tried to determine the limits to which they tended, when α was made very great.

Clearly, when we look at the second of expressions (22), we see that the second term must ultimately become vanishingly small, provided $\pi\alpha/2\beta > 2\alpha$, or $\beta < \pi/4$. For such values of β, then, S_B tends to the definite limit $\mu\tau c \tan 2\beta \cos\beta$, *i.e.*, for $\beta = \pi/6$, $S_B/\mu\tau c$ tends to the limit 1·5 for large values of α.

For values of $\beta > \pi/4$ it would seem that S_B increases numerically to an indefinite extent.

Consider now the second expression for S_A. Clearly, if α exceed a certain value, $\tanh \frac{\widehat{2n+1}\pi\alpha}{2\beta}$ approximates so closely to unity that we may replace it in the series by unity, and the error will only be a small fraction of the series itself. We then find

$$\frac{S_A}{\mu\tau c} \text{ tends to } \sinh\alpha\,(\sec 2\beta - 1) - \frac{8\beta(\cosh 2\alpha + \cos 2\beta)}{\cosh\alpha} \sum_0^\infty \frac{(-1)^n}{\pi^2(2n+1)^2 - 16\beta^2},$$

when α is large.

Substituting $\beta = \pi/6$, and using the Euler-Maclaurin sum-formula to calculate the series, I find

$$\left(\frac{S_A}{\mu\tau c}\right)_{\substack{\beta=\pi/6\\ \alpha \text{ large}}} = \sinh\alpha - \frac{12}{\pi}\frac{(\cosh 2\alpha + \frac{1}{2})}{\cosh\alpha}(\cdot 190086),$$

and remembering that

$$\frac{\cosh 2\alpha + \frac{1}{2}}{\cosh\alpha} = 2\cosh\alpha - \frac{1}{2\cosh\alpha} = 2\cosh\alpha, \text{ if } \alpha \text{ large},$$

$$\sinh\alpha = \cosh\alpha - \frac{1}{\cosh\alpha + \sinh\alpha} = \cosh\alpha, \text{ if } \alpha \text{ large},$$

we see that, when α is large,

$$\left(\frac{S_A}{\mu\tau c}\right)_{\beta=\pi/6} = -\cosh\alpha\,(\cdot 452147),$$

and, therefore, increases numerically indefinitely.

On the other hand, if α be very small, we get a flat section, A being now the point nearest to the centre. Looking at expressions (21) and (22) we see easily that

$$\left(\frac{S_A}{\mu\tau c}\right) \text{ tends to } -2\alpha$$

and

$$\left(\frac{S_B}{\mu\tau c}\right) \text{ to } (1 - \operatorname{sech} 2\alpha)\sin\beta + \frac{16}{\pi^2}\alpha\cos\beta \sum_0^\infty \frac{(-1)^n}{(2n+1)^2},$$

or, neglecting squares of α compared with the first power,

$$\left(\frac{S_B}{\mu\tau c}\right)_{\alpha \text{ small}} = \frac{16}{\pi^2}\alpha\cos\beta \sum_0^\infty \frac{(-1)^n}{(2n+1)^2}.$$

Now $\sum_0^\alpha \frac{(-1)^n}{(2n+1)^2}$ lies between 1 and 8/9, hence $\left(\frac{S_B}{\mu\tau c}\right) < \left(\frac{16}{\pi^2}\cos\beta\right)\alpha$, and $\frac{16}{\pi^2}\cos\beta$ being always < 2, S_A is always numerically greater than S_B for small values of α. This confirms the usual rule, which we should expect, since the section approximates in this case to a flat rectangular section.

§ 12. *Discussion of the Variations in these Stresses.*

The variations in the stresses S_A, S_B are shown in fig. 9 (p. 340). This figure gives, not the values of the stresses themselves, but the ratio of the stresses to the maximum stress S_0 in the circular section of equal area. This ratio is plotted as ordinate to the various values of α as abscissæ. Of course, S_A, as given by the expressions (21) being negative throughout, the curve shows the ratio $(-S_A/S_0)$ and not S_A/S_0.

The diagram is comparatively rough, especially near the origin, owing to the very limited number of points which I could calculate. The value of β selected was $\beta = \pi/6$.

When α is small, it is not difficult to show that, for $\beta = \pi/6$

$$S_B/S_0 = 1{\cdot}799 \times \sqrt{\alpha}, \qquad S_A/S =_0 - 2{\cdot}563 \times \sqrt{\alpha},$$

and when α is large

$$S_B/S_0 = 5{\cdot}196e^{-\alpha}, \qquad S_A/S_0 = -{\cdot}7831.$$

These last enable us to see the form of the curves near the origin and at a great distance from it. They are perpendicular to the axis of α at the origin, and at first the curve of S_B lies below that of S_A. At some point between $\alpha = 0$ and $\alpha = {\cdot}5$ they cross. This corresponds to the case of the square for rectangular cross-sections. S_A is now less than S_B, but instead of its remaining so, as we should have expected, the curves cross again near the value of $\alpha \doteq 1{\cdot}7$. The curve of S_A/S_0 now tends to become practically a straight line parallel to the axis, at a distance ·7831 from it, and the curve of S_B/S_0 approaches the axis asymptotically.

The values of S_A/S_0, S_B/S_0 are given in the tables below.

TABLE of S_A/S_0.

	$\beta = \pi/6$.	$\beta = \pi/4$.
$\alpha = \pi/6$	1·1334	1·1741
$\alpha = \pi/3$	·8589	·9506
$\alpha = \pi/2$	·7796	·8246
$\alpha = 2\pi/3$	·7706	·7924
$\alpha = 3\pi/4$	·7724	
$\alpha = \infty$	·7831	

TABLE of S_B/S_0.

	$\beta = \pi/6$.	$\beta = \pi/4$.	$\beta = \pi/3$.
$\alpha = \pi/6$	1·2790	1·2168	1·1950
$\alpha = \pi/3$	1·1749	1·4494	1·7819
$\alpha = \pi/2$	·8521	1·2388	1·7773
$\alpha = 2\pi/3$	·5640	·9366	1·2928
$\alpha = 3\pi/4$	·4490		

§ 13. *Investigation of the Fail-Points other than the Points of Symmetry.*

The question now arises, are A and B really points of *maximum slide*, and therefore shear? Assuming that the fail-points do occur on the contour, are we sure that there are not other points on the boundary where greater maxima occur, and may we not be in presence of a case like that of SAINT-VENANT'S *section en double spatule.*

To see whether this is so, it is necessary to go at some length into the equations determining the maximum stress.

Consider first the sides of the section $\eta = \pm\beta$. It is clear that the resultant stress along this side will be simply the component parallel to the contour, since the normal component must vanish in virtue of the boundary conditions. Calling S this resultant stress, we find easily

$$S = \frac{\mu}{c\sqrt{J}}\left(\frac{dw}{d\xi} + \tfrac{1}{2}\tau c^2 \sin 2\eta\right),$$

where

$$J = \tfrac{1}{2}(\cosh 2\xi + \cos 2\eta).$$

Hence we have to make

$$\left(\frac{Sc}{\mu}\right)^2 = \frac{1}{J}\left[\left(\frac{dw}{d\xi}\right)_{\eta=\beta} + \tfrac{1}{2}\tau c^2 \sin 2\beta\right]^2,$$

a maximum with regard to ξ.

We have therefore

$$\frac{d}{d\xi}\left(\frac{1}{J}\left\{\frac{dw}{d\xi} + \tfrac{1}{2}\tau c^2 \sin 2\beta\right\}^2\right) = 0,$$

that is

$$\frac{1}{J}\left(\frac{dw}{d\xi} + \tfrac{1}{2}\tau c^2 \sin 2\beta\right)\left[2\frac{d^2w}{d\xi^2} - \frac{1}{J}\frac{dJ}{d\xi}\left(\frac{dw}{d\xi} + \tfrac{1}{2}\tau c^2 \sin 2\beta\right)\right] = 0,$$

Take now the second value of w as given by (17), and we get

$$2\frac{d^2w}{d\xi^2}-\frac{1}{J}\frac{dJ}{d\xi}\left(\frac{dw}{d\xi}+\tfrac{1}{2}\tau c^2\sin 2\beta\right)=\tau c^2\sinh 2\xi\tan 2\beta-8\tau c^2(\cosh 2\alpha+\cos 2\beta)$$

$$\times\sum_{n=0}^{n=\infty}\left\{\frac{(2n+1)\pi}{(2n+1)^2\pi^2-16\beta^2}\,\frac{\sinh\dfrac{\overparen{2n+1}\pi\xi}{2\beta}}{\cosh\dfrac{\overparen{2n+1}\pi\alpha}{2\beta}}\right.$$

$$\left.-\frac{2\beta\sinh 2\xi}{\cosh 2\xi+\cos 2\beta}\,\frac{\cosh\dfrac{\overparen{2n+1}\pi\xi}{2\beta}}{[\overline{2n+1}|^2\pi^2-16\beta^2]\cosh\dfrac{\overparen{2n+1}\pi\alpha}{2\beta}}\right\}.$$

Hence the equation giving the maxima and minima is

$$\tan 2\beta\sinh 2\xi$$

$$=4\frac{(\cosh 2\alpha+\cos 2\beta)(\cosh 2\xi+2\cos 2\beta)}{\cosh 2\xi+\cos 2\beta}\sum_{n=0}^{n=\infty}\frac{(2n+1)\pi}{(2n+1)^2\pi^2-16\beta^2}\,\frac{\sinh\dfrac{\overparen{2n+1}\pi\xi}{2\beta}}{\cosh\dfrac{\overparen{2n+1}\pi\alpha}{2\beta}}$$

$$+\frac{2(\cosh 2\alpha+\cos 2\beta)}{\cosh 2\xi+\cos 2\beta}\sum_{n=0}^{n=\infty}\left\{\frac{\sinh[\overparen{2n+1}\pi+4\beta]\dfrac{\xi}{2\beta}}{(2n+1)\pi+4\beta}+\frac{\sinh[\overparen{2n+1}\pi-4\beta]\dfrac{\xi}{\beta}}{(2n+1)\pi-4\beta}\right\}\operatorname{sech}\frac{\overparen{2n+1}\pi\alpha}{2\beta}.$$

One root of this equation is $\xi=0$. Now it is clear that since S^2 is zero, and therefore a minimum at $\xi=\alpha$, if it be not a maximum at $\xi=0$, then there must be a maximum somewhere between $\xi=0$ and $\xi=\alpha$.

To find whether $\xi=0$ is a maximum or not, we have to investigate the sign of

$$\left[\frac{d^2}{d\xi^2}\left(\frac{1}{J}\left\{\frac{dw}{d\xi}+\tfrac{1}{2}\tau c^2\sin 2\beta\right\}^2\right)\right]_{\substack{\xi=0\\ \eta=\beta}},$$

i.e., we have to investigate the sign of

$$\frac{1}{J}\left(\frac{dw}{d\xi}+\tfrac{1}{2}\tau c^2\sin 2\beta\right)\frac{d}{d\xi}\left[2\frac{d^2w}{d\xi^2}-\frac{1}{J}\frac{dJ}{d\xi}\left(\frac{dw}{d\xi}+\tfrac{1}{2}\tau c^2\sin 2\beta\right)\right],\ .$$

all the other terms vanishing when $\xi=0$.

Now $\frac{1}{J}\left(\frac{dw}{d\xi}+\tfrac{1}{2}\tau c^2\sin 2\beta\right)$ being always positive, we have to investigate the sign of

$$\frac{d}{d\xi}\left[2\frac{d^2w}{d\xi^2}-\frac{1}{J}\frac{dJ}{d\xi}\left(\frac{dw}{d\xi}+\tfrac{1}{2}\tau c^2\sin 2\beta\right)\right]\text{ when }\xi=0,$$

and therefore ultimately the sign of

$$\mathrm{E} = \beta \tan 2\beta - \frac{(\cosh 2\alpha + \cos 2\beta)(1 + 2\cos 2\beta)}{1 + \cos 2\beta} \sum_{n=0}^{n=\infty} \frac{(2n+1)^2\pi^2}{(2n+1)^2\pi^2 - 16\beta^2} \operatorname{sech} \frac{\overparen{2n+1}\pi\alpha}{2\beta}$$

$$- \frac{\cosh 2\alpha + \cos 2\beta}{1 + \cos 2\beta} \sum_{n=0}^{n=\infty} \operatorname{sech} \frac{\overparen{2n+1}\pi\alpha}{2\beta}.$$

I may remark here that all these differentiations are permissible, provided that ξ be less than α by a *finite amount*, however small, the series being then uniformly convergent. We may not, therefore, make α actually zero in the last expression. If, however, α is small but finite, then it is easy to see that E must be negative if $\beta < \pi/4$. Also if $\beta < \pi/4$, E algebraically increases continuously, until for a certain value α_0 of α it reaches the value zero. For all higher values of α it remains steadily positive.

§ 14. *Critical Values of α and β.*

It follows, therefore, that in all sections for which $\beta < \pi/4$, the maximum stress along the sides $\eta = \pm \beta$ occurs at the point B, until a certain critical value, $\alpha = \alpha_0$, is reached, when the stress at B becomes a minimum, and we now have two points of maximum stress on either side of B.

When $\beta = \pi/4$, E is apparently infinite, but it really tends to a finite limit. If we put $\beta = \pi/4 - \epsilon$ where ϵ will ultimately be made very small, and neglect terms in ϵ, ϵ^2, &c., we find that the expression becomes

$$\frac{\pi}{8\epsilon} - \tfrac{1}{2} - 2\cosh 2\alpha \sum_{n=0}^{n=\infty} \operatorname{sech} \overparen{2n+1}\, 2\alpha - \cosh 2\alpha \sum_{n=1}^{n=\infty} \operatorname{sech}(2n+1)\, 2\alpha + \tfrac{1}{4}$$

$$- (\cosh 2\alpha + 2\epsilon)(1 + 2\epsilon)\left(\frac{\pi}{2} - 2\epsilon\right)\frac{1}{4\epsilon} \operatorname{sech}\left\{2\alpha\left(1 + \frac{4\epsilon}{\pi}\right)\right\}$$

$$= -\frac{\pi}{4} - \frac{\pi}{4} \operatorname{sech} 2\alpha + \alpha \tanh 2\alpha - \frac{7}{\pi} - 3\cosh 2\alpha \sum_{n=1}^{n=\infty} \operatorname{sech}(2n+1)\, 2\alpha.$$

When α is very great, this will ultimately be positive. Hence, here also we shall have a critical value α_0. This value, however, is easily seen to lie outside the values of α taken in this paper.

I have not investigated so carefully the cases when $\beta > \pi/4$, as, for the particular sections selected, the maximum stress certainly occurs at B.

One point, however, is clear. When α is very large, the second and third terms settle the sign of E. The sign of these terms, again, is settled by the sign of the leading terms. E is negative if

$$1 + (1 + 2\cos 2\,\beta)\frac{\pi^2}{\pi^2 - 16\beta^2}$$

is positive, *i.e.*, if

$$\frac{\cos^2\beta}{4\beta^2} + \frac{1 + 2\cos 2\beta}{\pi^2 - 16\beta^2} > 0.$$

This will always be the case, provided β be greater than the root of the equation

$$\frac{2\beta}{\pi} = \cos\beta.$$

This root is 53° 31′ nearly.

Hence, for values of $\beta > 53° \, 31'$ the maximum stress always occurs at the thinnest point of the section. This is the case for the sections of fig. 4 with broad keyways, for which $\beta = 60°$.

Returning to the case of sections for which $\beta < \pi/4$ we have α_0 given by the equation

$$\beta \tan 2\beta = (\cosh 2\alpha_0 + \cos 2\beta)\left[2\sum_{n=0}^{n=\infty} \operatorname{sech} \frac{\widehat{2n+1}\pi\alpha_0}{2\beta} + \frac{16\beta^2(1+2\cos 2\beta)}{1+\cos 2\beta} \sum_{n=0}^{n=\infty} \frac{\operatorname{sech} \frac{\widehat{2n+1}\pi\alpha_0}{2\beta}}{\overline{2n+1}^2\pi^2 - 16\beta 2}\right].$$

Putting $\beta = \pi/6$ in this we have for the transcendental equation giving α_0 in this case

$$\frac{\pi}{2\sqrt{3}} = (\cosh 2\alpha_0 + \tfrac{1}{2})\left[2\sum_{n=0}^{n=\infty} \operatorname{sech}\,[2n+1]\,3\alpha_0 + \frac{16}{3}\sum_{n=0}^{n=\infty} \frac{\operatorname{sech}[2n+1]\,3\alpha_0}{(6n+1)(6n+5)}\right].$$

Now, for fairly large values of α_0, we may neglect all terms in this summation except the first. We then have

$$\frac{\pi}{2\sqrt{3}} = \frac{23}{15}\,\frac{2\cosh 2\alpha_0 + 1}{\cosh 3\alpha_0} = \frac{23}{15}(\operatorname{sech}\alpha_0 + 2\operatorname{sech} 3\alpha_0)$$

whence I find $\alpha_0 = 1{\cdot}225$.

Hence for values of α greater than this the maximum occurs at certain points on the contour, given by previously obtained equation in ξ.

§ 15. *Calculation of the Position of Fail-Points and the Magnitude of the Maximum Stress for the Sections $\beta = \pi/6$, $\alpha = \pi/2$ and $\beta = \pi/6$, $\alpha = 2\pi/3$.*

We have, therefore, if we wish to find the maximum stress for the sections $\alpha = \pi/2$ and $\alpha = 2\pi/3$ when $\beta = \pi/6$, to solve this transcendental equation:

$$\frac{\pi\sqrt{3}}{6}\,\frac{\sinh 2\xi\,(2\cosh 2\xi + 1)}{2\cosh 2\alpha + 1} = 6\,(\cosh 2\xi + 1)\sum_{n=0}^{n=\infty} \frac{2n+1}{(6n+1)(6n+5)}\,\frac{\sinh(6n+3)\,\xi}{\cosh(6n+3)\,\alpha}$$
$$+ \sum_{n=0}^{n=\infty}\left\{\frac{\sinh(6n+5)\,\xi}{6n+5} + \frac{\sinh(6n+1)\,\xi}{6n+1}\right\}\frac{1}{\cosh(6n+3)\,\alpha}$$

which may be put into the slightly simpler form

$$\frac{\pi\sqrt{3}}{6\,(\cosh 2\alpha + \tfrac{1}{2})} = \frac{12\cosh\xi}{\sinh 3\xi}\sum_{n=0}^{n=\infty} \frac{(2n+1)}{(6n+1)(6n+5)}\,\frac{\sinh(6n+3)\,\xi}{\cosh(6n+3)\,\alpha}$$
$$+ \frac{1}{\cosh\xi\,\sinh 3\xi}\sum_{n=0}^{n=\infty}\left\{\frac{\sinh(6n+5)\,\xi}{(6n+5)\cosh(6n+3)\alpha} + \frac{\sinh(6n+1)\,\xi}{(6n+1)\cosh(6n+3)\alpha}\right\}.$$

I find the roots of this to be approximately given by

$$\xi = 1{\cdot}475 \text{ when } \alpha = 2\pi/3,$$

$$\xi = {\cdot}821 \text{ when } \alpha = \pi/2.$$

The corresponding values of $S/\mu\tau c$ are found to be $2{\cdot}1084$ and $1{\cdot}4041$ respectively. Comparing these with the values of $S_A/\mu\tau c$ given on p. 331, we see that the values of the stresses corresponding to the maximum on the broad side are the greater, and hence we have really a case absolutely analogous to SAINT-VENANT'S *section en double spatule*, with four fail-points symmetrically distributed, all of them lying on the broad sides of the contour.

§ 16. *Case where α is made very great.*

It is interesting to see to what limit the fail-points tend, when α is made very great. We have

$$\frac{S}{\mu\tau c} = \frac{1}{\sqrt{J}}\left[\begin{aligned}&\tfrac{1}{2}\tan 2\beta\,(\cosh 2\xi + \cos 2\beta) \\ &- 8\beta\,(\cosh 2\alpha + \cos 2\beta)\sum_{n=0}^{n=\infty}\frac{\cosh\dfrac{\overline{2n+1}\pi\xi}{2\beta}}{\left[(2n+1)^2\pi^2 - 16\beta^2\right]\cosh\dfrac{\overline{2n+1}\pi\alpha}{2\beta}}\end{aligned}\right].$$

Replace now $\frac{1}{2}\tan 2\beta$ by its equivalent $8\beta\sum\limits_{n=0}^{n=\infty}\dfrac{1}{(2n+1)^2\pi^2 - 16\beta^2}$.

$$\frac{S}{\mu\tau c} = \frac{8\beta}{\sqrt{J}}\sum_{n=0}^{\infty}\frac{\left\{(\cosh 2\xi + \cos 2\beta) - (\cosh 2\alpha + \cos 2\beta)\dfrac{\cosh\dfrac{\overline{2n+1}\pi\xi}{2\beta}}{\cosh\dfrac{\overline{2n+1}\pi\alpha}{2\beta}}\right\}}{(2n+1)^2\pi^2 - 16\beta^2}.$$

Now if ξ, α be great, we have $\cosh = \frac{1}{2}$ (exp.) approximately. Hence, if we suppose $\xi = \alpha - \theta$ where θ is finite, so that we are dealing with points whose distances from the centre bear a finite ratio to the dimensions of the section, we find

$$\frac{S}{\mu\tau c} = 8\beta e^{\alpha}\sum_{n=0}^{\infty}\frac{e^{-\theta} - e^{\theta\left(1-\frac{\overline{2n+1}\pi}{2\beta}\right)}}{\overline{2n+1}|^2\pi^2 - 16\beta^2}$$

+ terms negligible in comparison and ultimately vanishingly small when α is made infinite.

Hence we have to make

$$\sum_{n=0}^{n=\infty}\frac{e^{-\theta} - e^{\theta\left(1-\frac{(2n+1)\pi}{2\beta}\right)}}{(2n+1)^2\pi^2 - 16\beta^2}$$

a maximum.

Differentiating, the series being absolutely and uniformly convergent, we get

$$e^{-\theta}\sum_{n=0}^{n=\infty}\frac{1}{(2n+1)^2\pi^2-16\beta^2}-\sum_{n=0}^{n=\infty}\frac{\left(\frac{\overline{2n+1}\,\pi}{2\beta}-1\right)e^{\theta\left(1-\frac{(2n+1)\pi}{2\beta}\right)}}{(2n+1)^2\pi^2-16\beta^2}=0,$$

i.e.,

$$\frac{\tan 2\beta}{8}=\sum_{n=0}^{n=\infty}\frac{(2n+1)\pi-2\beta}{(2n+1)^2\pi^2-16\beta^2}\,e^{\theta\left(2-\frac{(2n+1)\pi}{2\beta}\right)}.$$

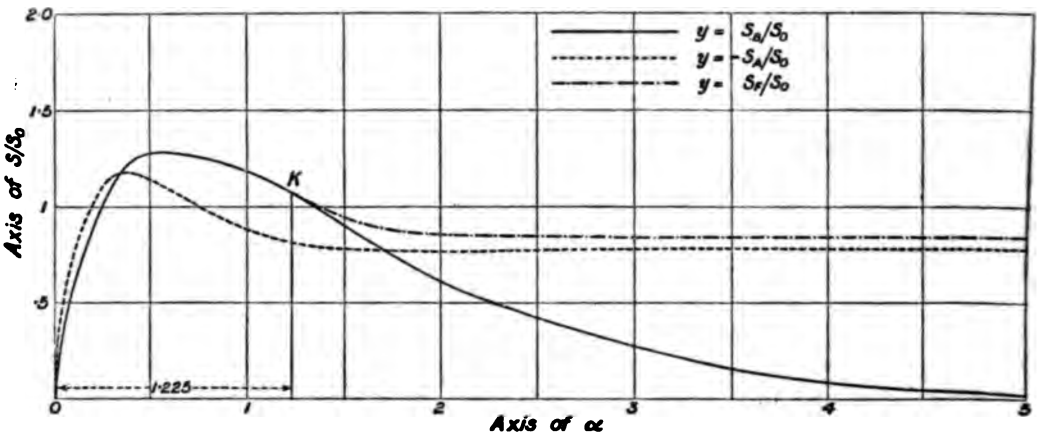
This is the equation giving θ. When we put in it $\beta=\pi/6$

$$\frac{\pi}{16\sqrt{3}}=\sum_{n=0}^{n=\infty}\frac{3n+1}{(6n+1)(6n+5)}\,e^{-(6n+1)\theta}.$$

I find from this $\theta = \cdot577$ approximately, and the corresponding value of $S/\mu\tau c = \cdot48984\cosh\alpha$. Referring to the value of $S_A/\mu\tau c$ given on p. 333, we see that the real fail-point is on the broader side.

The curve of true maximum stress $\widehat{S}/S_0$ is shown also in fig. 9. It joins on to the curve of S_B/S_0 at the point corresponding to the value $\alpha = 1\cdot225$. It then remains above the curve of S_A/S_0, tending ultimately to a straight line parallel to the axis, at a height $\cdot8484$ above it.

Fig. 9.



Showing the variation in the value of S/S_0 as α increases, at the three points A, B, F, of Fig. 2. β = const. = $\pi/6$.

§ 17. *Proof that there can be no other Maxima.*

It remains to show that the point $\eta = 0$ corresponds to a maximum along the side $\xi = \alpha$, and also that there is no maximum of the resultant stress inside the section.

The stress S along $\xi = \alpha$ is given by

$$\left(\frac{Sc}{\mu}\right)^2=\frac{1}{J}\left(\frac{dw}{d\eta}-\tfrac{1}{2}\tau c^2\sinh 2\alpha\right)^2_{\xi=\alpha}.$$

The equation giving the maxima or minima is found as before. It is

$$\left[2\frac{d^2w}{d\eta^2} - \frac{1}{J}\frac{dJ}{d\eta}\left(\frac{dw}{d\eta} - \tfrac{1}{2}\tau c^2 \sinh 2\alpha\right)\right]_{\xi=\alpha} = 0,$$

or, using the *first* of expressions (17) for w

$$\sin 2\eta \tanh 2\alpha + 8(\cosh 2\alpha + \cos 2\beta)\sum_{n=0}^{n=\infty}\frac{\sinh\dfrac{(2n+1)\pi\eta}{2\alpha}}{\cosh\dfrac{(2n+1)\pi\beta}{2\alpha}}\frac{(2n+1)\pi}{(2n+1)^2\pi^2+16\alpha^2}$$

$$+\frac{16\alpha\sin 2\eta\,(\cosh 2\alpha+\cos 2\beta)}{\cosh 2\alpha+\cos 2\eta}\sum_{n=0}^{n=\infty}\frac{\cosh\dfrac{\overparen{2n+1}\pi\eta}{2\alpha}}{(2n+1)^2\pi^2+16\alpha}\,\frac{1}{\cosh\dfrac{\overparen{2n+1}\pi\beta}{2\alpha}} = 0.$$

The left-hand side is always positive. Hence there is no root except $\eta = 0$, and that corresponds to a maximum.

That no absolute *maximum* can exist inside the section can be proved as follows:—

We have

$$\widehat{xz}/\mu = dw/dx - \tau y, \qquad \widehat{yz}/\mu = dw/dy + \tau x.$$

Suppose at any point P inside the section $S^2 = \widehat{xz}^2 + \widehat{yz}^2$ is a maximum. The above forms for $\widehat{xz} + \widehat{yz}$ being independent of axes, let us take for axes of x and y the direction of the resultant stress across the section at P and the perpendicular to it. So that $\widehat{yz} = 0$ at P.

Consider a near point P′. Let $\widehat{xz}'$, $\widehat{yz}'$ be the stresses at P′. Then, since S^2 is a maximum at P,

$$\widehat{xz}^2 > \widehat{xz}'^2 + \widehat{yz}'^2,$$

but

$$yz'^2 > 0.$$

Therefore

$$xz^2 > xz'^2,$$

or $\widehat{xz}$ is a numerical maximum.

But since

$$d^2w/dx^2 + d^2w/dy^2 = 0,$$

therefore also

$$\frac{d^2(\widehat{xz})}{dx^2} + \frac{d^2(\widehat{xz})}{dy^2} = 0;$$

and it is well known that no function can have an absolute maximum or an absolute minimum inside a region where it satisfies LAPLACE's equation. This is in fact a particular case of the general theorem that a potential cannot have a maximum in free space.

Hence we have proved that the only fail-points are those which we have already investigated.

§ 18. *Deductions from the above and Criticism of* BOUSSINESQ'S *Proof of the Position of Fail-Points.*

Thus we see the study of these symmetrical sections is extremely instructive as regards the position of the fail-points. They show us the connection between the rectangular section and the section with a neck, and they give us the limiting cases, when the four fail-points coalesce into two, and *vice versâ*. The four fail-points begin to occur after the ratio of the long to the short axis of the section exceeds a certain value, which depends upon the angle of the bounding hyperbolas. As the indented appearance of the section becomes more obvious, that is, as β increases, this limiting value becomes greater and greater until, when the angle between the asymptotes is less than 73° the fail-point is always at the vertex of the hyperbolas. But in no case are the fail-points on the convex sides of the sections, unless the ellipses are so flat that the points A are nearer to the centre than the points B.

M. BOUSSINESQ has given ('Journal de Mathématiques,' Série II., vol. 16, p. 200) a sketch of a proof that the fail-points must be sought for "sur les petits diamètres des sections." As this statement is in opposition with the results of the present paper and with DE SAINT-VENANT'S results for the rail sections already mentioned, I venture to suggest that M. BOUSSINESQ'S reasoning hardly holds in the case of sections part of whose contour is convex and part concave, for the following reason. The problem of torsion is mathematically analogous to that of a cylindrical vortex of uniform strength, whose cross-section is that of the shaft considered. The motion being in two dimensions we have a stream function ψ, and the resultant stress at any point in the torsion problem is the same as $d\psi/dn$ in the hydrodynamical analogy, dn being an element of the normal to any stream-line.

Fig. 10.

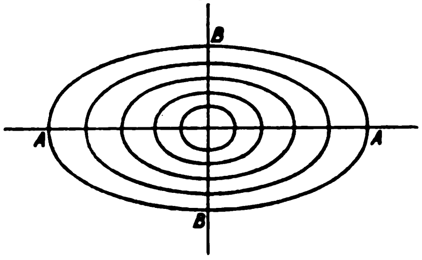
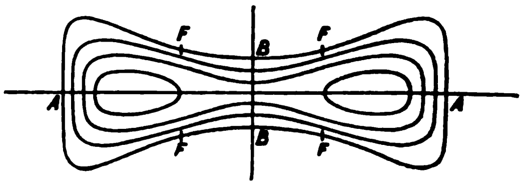


Fig. 11.



Now if we draw the stream-lines for equidistant values of ψ, they will, says M. BOUSSINESQ, "reproduce the irregularities of the contour, but more and more faintly, so that the curves are spaced at greater intervals along the large, then along the small diameters." In consequence, $d\psi/dn$, or the stress, is greater in the latter case. The above argument assumes that the same number of stream-lines cross each diameter. This will undoubtedly be the case, and M. BOUSSINESQ's proof will hold, if the boundary of the section be everywhere convex, for then it is evident that each stream-line will consist of a single simple oval (fig. 10). But when we deal with sections like those of the present paper it is by no means so clear that this will be the case. There may be stream-lines which consist of two separate ovals (see fig. 11), and it then becomes very much open to question towards which part of the section the lines will be most crowded. In fact in some cases, as we have seen, this will occur at intermediate points, F (fig. 11), and in other cases the narrowness of the neck or the sharpness of the bend will counterbalance the limited number of the stream-lines through the neck, and the fail-points will be at B, B.

§ 19. *Comparison with* SAINT-VENANT'S *Results.* "*Efficiency*" *of the various Sections.*

When we compare the results given by these sections bounded by confocal ellipses and hyperbolas with those given by DE SAINT-VENANT in his edition of the 'Leçons de NAVIER' for the "sections en double spatule," we find very good agreement. Thus the critical value of the ratio of breadth of neck to length for which the fail-points split up each into two is given by SAINT-VENANT as ·3247. For the sections of this paper, when $\beta = \pi/6$, I find this critical value to be ·3215.

The following are the principal numerical points (see 'Leçons de NAVIER,' p. 365) :—

	Section $\alpha = \pi/2$, $\beta = \pi/6$.	SAINT-VENANT'S section, $c/b =$ ·20.
Ratio of breadth of neck to length	·2173	·20
M/M_0	·4443	·3921
S/S_0 (S being maximum stress)	·9144	·8668
$E = (M/M_0)/(S/S_0)$	·4859	·4524
Ratio of distance of fail-points from short axis to length of longer semi-axis	·3449	·4561

	Section $\alpha = 2\pi/3$, $\beta = \pi/6$.	SAINT-VENANT'S section, $c/b =$ ·14.
Ratio of breadth of neck to length	·1250	·14
M/M_0	·3955	·3465
S/S_0	·8557	·8226
E	·4622	·4212
Ratio of distance of fail-points from short axis to length of longer semi-axis	·4486	·523

On the whole the sections of the present paper appear the more useful, M/M_0 and the quantity E being greater than for DE SAINT-VENANT'S rail sections. This quantity E I propose to call the "efficiency" or "usefulness" of the shaft. It is equal to $(S_0/M_0)/(S/M)$, and gives us the ratio of maximum stress to torsional rigidity, compared with the same ratio for the circle of equal area, that is, it gives us a measure of how much torsion we may put into the shaft without impairing its elasticity. Thus if $\bar{M}$ be the maximum torsion moment which the given shaft will bear without failure of elasticity, $\bar{M}_0$ the maximum torsion moment which the shaft of equal circular section (made of the same material) will stand, then the corresponding stresses, $\bar{S}$, S_0 are each equal to the limiting elastic stress of the material and

$$\bar{M}/\bar{M}_0 = \text{efficiency} = E,$$

or the limiting torsion moment of a shaft of any section is obtained from that of the circular section of the same area by merely multiplying by the "efficiency" as thus defined.

TABLE of "Efficiency" of the sixteen given sections.

	$\beta = \pi/6$.	$\beta = \pi/4$.	$\beta = \pi/3$.	$\beta = \pi/2$.
$\alpha = \pi/6$	·6846	·7104	·6869	0
$\alpha = \pi/3$	·5206	·5022	·4376	0
$\alpha = \pi/2$	·4859	·4218	·3211	0
$\alpha = 2\pi/3$	·4622	·4643	·3550	0

A glance at the above suffices to show that the efficiency of these sections is, in general, about one-half, that is, on the whole, this form of shaft is about half as useful as the shaft of circular section. The zero efficiency in the case of the slit $(\beta = \pi/2)$ is due to the infinite stress in the keyway.

§ 20. *Analysis for the Sections bounded by one Elliptic and one Hyperbolic Arc.*

I now pass on to the consideration of cross-sections, bounded by an ellipse and a single branch of a confocal hyperbola. Such sections are shown in figs. 6–8.

In this case we shall find it more convenient to define ξ and η by the equations

$$\left.\begin{aligned} x &= c \cosh \xi \cos \eta \\ y &= c \sinh \xi \sin \eta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where $-\alpha < \xi < +\alpha$, $0 < \eta < \beta$.

We find in the same way as before

$$d^2w/d\xi^2 + d^2w/d\eta^2 = 0.$$

$$\left.\begin{array}{ll} dw/d\xi + \frac{1}{2}\tau c^2 \sin 2\eta = 0 & \xi = \pm\alpha, \quad 0 < \eta < \beta \\ dw/d\eta + \frac{1}{2}\tau c^2 \sinh 2\xi = 0 & \eta = \beta, \quad -\alpha < \xi < +\alpha \end{array}\right\} \quad . \quad . \quad (24).$$

There is, however, in the present case, a further condition. For, referring to figs. 6–8, when we cross the line SB_1B_2, S being that focus of the conics which is inside the section, η is continuous, passing through the value 0, but ξ is discontinuous and changes from positive to negative. We have to ensure that the function of ξ and η, which shall represent w, shall be continuous in crossing $\eta = 0$, and that its space-differential coefficients shall also be continuous. The latter condition implies that $dw/d\eta$ must change sign: this being so, all the conditions will be satisfied.

Let us write

$$w = -\tfrac{1}{4}\tau c^2 \frac{\sinh 2\xi \sin 2\eta}{\cosh 2\alpha} + w_1,$$

conditions (24) then become

$$\left.\begin{array}{l} dw_1/d\xi = 0 \quad \text{when} \quad \xi = \pm\alpha, \quad 0 < \eta < \beta \\ \dfrac{dw_1}{d\eta} + \tfrac{1}{2}\tau c^2\left(1 - \dfrac{\cos 2\beta}{\cosh 2\alpha}\right)\sinh 2\xi = 0 \\ \qquad \text{when} \quad \eta = \beta, \quad -\alpha < \xi < +\alpha \end{array}\right\} \quad . \quad . \quad . \quad . \quad (25).$$

If we now assume

$$w_1 = \sum_{n=0}^{n=\infty} A_n \sinh \frac{\overline{2n+1}\pi\eta}{2\alpha} \cdot \sin \frac{\overline{2n+1}\pi\xi}{2\alpha},$$

then the conditions of continuity and the first of (25) are identically satisfied.

A_n is found from the second equation of (25) in the same way as in § 4.

$$A_n = -\frac{16\tau c^2\alpha^2(-1)^n(\cosh 2\alpha - \cos 2\beta)}{\pi(2n+1)[\pi^2(2n+1)^2 + 16\alpha^2]\cosh \frac{\overline{2n+1}\pi\beta}{2\alpha}}.$$

And thus

$$w = -\tfrac{1}{4}\tau c^2 \frac{\sinh 2\xi \sin 2\eta}{\cosh 2\alpha} - 16\tau c^2\alpha^2(\cosh 2\alpha - \cos 2\beta)$$

$$. \quad . \quad \times \sum_{n=0}^{n=\infty} \frac{(-1)^n \sinh \frac{\overline{2n+1}\pi\eta}{2\alpha} \sin \frac{\overline{2n+1}\pi\xi}{2\alpha}}{(2n+1)\pi[\pi^2(2n+1)^2 + 16\alpha^2]\cosh \frac{\overline{2n+1}\pi\beta}{2\alpha}} \quad . \quad . \quad . \quad . \quad (26).$$

The shears are given by the equations

$$\left.\begin{aligned}\frac{\widehat{xz}}{\mu} &= \frac{1}{cJ}\left(\sinh\xi\cos\eta\frac{dw_1}{d\xi} - \cosh\xi\sin\eta\frac{dw_1}{d\eta}\right) - \tau c\sinh\xi\sin\eta\,(1+\operatorname{sech}2\alpha)\\ \frac{\widehat{yz}}{\mu} &= \frac{1}{cJ}\left(\cosh\xi\sin\eta\frac{dw_1}{d\xi} + \sinh\xi\cos\eta\frac{dw_1}{d\eta}\right) + \tau c\cosh\xi\cos\eta\,(1-\operatorname{sech}2\alpha)\end{aligned}\right\}\quad(27),$$

where J now stands for the quantity

$$\cosh^2\xi\sin^2\eta + \sinh^2\xi\cos^2\eta.$$

The torsion moment M

$$= \iint(x\widehat{yz} - y\widehat{xz})\,dx\,dy$$

$$= \mu\tau c^4\int_0^\beta d\eta\int_{-\alpha}^{\alpha} d\xi\,\{\cosh^2\xi\cos^2\eta\,(1-\operatorname{sech}2\alpha) + \sinh^2\xi\sin^2\eta\,(1+\operatorname{sech}2\alpha)\}\,\mathrm{J}$$

$$+ \frac{\mu c}{2}\int_0^\beta d\eta\int_{-\alpha}^{\alpha} d\xi\left(\sin 2\eta\frac{dw_1}{d\xi} + \sinh 2\xi\frac{dw_1}{d\eta}\right)$$

$$= \frac{\mu\tau c^4}{8}\int_0^\beta d\eta\int_{-\alpha}^{\alpha} d\xi\left[\begin{aligned}&(\cosh 4\xi - \cos 4\eta) - \operatorname{sech}2\alpha\,(\cosh 2\xi - \cos 2\eta)\\ &\qquad - \operatorname{sech}2\alpha\,(\cosh 4\xi\cos 2\eta - \cosh 2\xi\cos 4\eta)\end{aligned}\right]$$

$$+ \frac{\mu c^2}{2}\int_0^\beta\Big[w_1\Big]_{-\alpha}^{\alpha}\sin 2\eta\,d\eta + \frac{\mu c^2}{2}\int_{-\alpha}^{\alpha}\Big[w_1\Big]_0^\beta\sinh 2\xi\,d\xi.$$

When this is integrated out and reduced as before, it is found that

$$\mathrm{M} = \frac{\mu\tau c^4}{16}\left\{\begin{aligned}&(\beta\sinh 4\alpha - \alpha\sin 4\beta)\tanh^2 2\alpha\\ &\quad + 2\sin 2\beta\,(2\alpha\operatorname{sech}^2 2\alpha - \tanh 2\alpha)(\cosh 2\alpha - \cos 2\beta)\\ &\quad - 2048\alpha^4(\cosh 2\alpha - \cos 2\beta)^2\sum_{n=0}^{n=\infty}\frac{1}{\pi(2n+1)}\frac{\tanh\dfrac{\widehat{2n+1}\,\pi\beta}{2\alpha}}{[\pi^2(2n+1)^2+16\alpha^2]^2}\end{aligned}\right\}\quad(28).$$

§ 21. *Alternative Solution for these Sections.*

For this type of section also we can find an alternative solution.

Suppose we assume

$$w = -\tfrac{1}{4}\tau c^2\frac{\sinh 2\xi\sin 2\eta}{\cos 2\beta} + w_1.$$

These conditions (24) reduce to:

$$\left.\begin{aligned}&\frac{dw_1}{d\xi} + \tfrac{1}{2}\tau c^2\sin 2\eta\left(1 - \frac{\cosh 2\alpha}{\cos 2\beta}\right) = 0,\ \xi = \pm\alpha,\ 0 < \eta < \beta\\ &dw_1/d\eta = 0\quad \eta = \beta\quad -\alpha < \xi < \alpha\end{aligned}\right\}\ .\ .\ (29).$$

Also the condition of continuity requires that w_1 must be odd in η.

We assume, therefore,

$$w_1 = \sum_{n=0}^{n=\infty} \mathrm{A}_n \sin \frac{\overset{\frown}{2n+1}\pi\eta}{2\beta} \sinh \frac{\overset{\frown}{2n+1}\pi\xi}{2\beta}.$$

But since

$$\sin 2\theta = a_1 \sin \frac{\pi\theta}{2\beta} + \ldots + a_n \sin \frac{\overset{\frown}{2n+1}\pi\theta}{2\beta} + \ldots,$$

where

$$a_n = \frac{(-1)^n 16\beta \cos 2\beta}{(2n+1)^2\pi^2 - 16\beta^2},$$

we find easily from the first equation of (29), comparing coefficients

$$\mathrm{A}_n = \frac{(-1)^n 16\tau c^2\beta^2 (\cosh 2\alpha - \cos 2\beta)}{(2n+1)\pi[\overline{2n+1}^2\pi^2 - 16\beta^2] \cosh \frac{\overset{\frown}{2n+1}\pi\alpha}{2\beta}},$$

and

$$w = -\tfrac{1}{4}\tau c^2 \frac{\sinh 2\xi \sin 2\eta}{\cos 2\beta}$$

$$+ 16\tau c^2\beta^2(\cosh 2\alpha - \cos 2\beta) \sum_{n=0}^{n=\infty} (-1)^n \frac{\sin \frac{\overset{\frown}{2n+1}\pi\eta}{2\beta} \sinh \frac{\overset{\frown}{2n+1}\pi\xi}{2\beta}}{(2n+1)\pi[\overline{2n+1}|^2\pi^2 - 16\beta^2] \cosh \frac{\overset{\frown}{2n+1}\pi\alpha}{2\beta}} \quad (30).$$

The stresses and torsion moment are deduced without difficulty. They are :

$$\frac{\widehat{xz}}{\mu} = -\tau c \sinh \xi \sin \eta (1 + \sec 2\beta) + 32\tau c\beta^2 \frac{(\cosh 2\alpha - \cos 2\beta)}{(\cosh 2\xi - \cos 2\eta)}$$

$$\times \sum_{n=0}^{n=\infty} \frac{\left(\sinh \xi \cos \eta \frac{d}{d\xi} - \cosh \xi \sin \eta \frac{d}{d\eta}\right)(-1)^n \sin \frac{\overset{\frown}{2n+1}\pi\eta}{2\beta} \sinh \frac{\overset{\frown}{2n+1}\pi\xi}{2\beta}}{\overline{2n+1}|\pi[\overline{2n+1}|^2\pi^2 - 16\beta^2] \cosh \frac{\overset{\frown}{2n+1}\pi\alpha}{2\beta}}. \quad (31).$$

$$\frac{\widehat{yz}}{\mu} = \tau c \cosh \xi \cos \eta (1 - \sec 2\beta) + 32\tau c\beta^2 \frac{(\cosh 2\alpha - \cos 2\beta)}{(\cosh 2\xi - \cos 2\eta)}$$

$$\times \sum_{n=0}^{n=\infty} \frac{\left(\cosh \xi \sin \eta \frac{d}{d\xi} + \sinh \xi \cos \eta \frac{d}{d\eta}\right)(-1)^n \sin \frac{\overset{\frown}{2n+1}\pi\eta}{2\beta} \sinh \frac{\overset{\frown}{2n+1}\pi\xi}{2\beta}}{\overline{2n+1}|\pi[\overline{2n+1}|^2\pi^2 - 16\beta^2] \cosh \frac{\overset{\frown}{2n+1}\pi\alpha}{2\beta}}. \quad (32).$$

$$\mathrm{M} = \frac{\mu\tau c^4}{16}\left[\begin{array}{l} -(\beta \sinh 4\alpha - \alpha \sin 4\beta)\tan^2 2\beta \\ \quad + 2\sinh 2\alpha(\cosh 2\alpha - \cos 2\beta)(2\beta \sec^2 2\beta - \tan 2\beta) \\ \quad - 2048\beta^4(\cosh 2\alpha - \cos 2\beta)^2 \displaystyle\sum_{n=0}^{n=\infty} \frac{\tanh \frac{\overset{\frown}{2n+1}\pi\alpha}{2\beta}}{\overline{2n+1}\,\pi[\overline{2n+1}^2\pi^2 - 16\beta^2]^2} \end{array}\right] \quad (33).$$

The same remarks which were made concerning the solution in § 5 apply here. The critical values are $\beta = \pi/4$, $\beta = 3\pi/4$. The limits to which the values of w, M and the stresses, given in (30)-(33), tend, are easily obtained if required. In practice, however, the other solution would probably be used.

§ 22. *Numerical Results. Effects of Keyways upon Torsional Rigidity.*

I have worked out numerically six cases of this type of section. The sections selected are shown in figs. 6–8. The bounding ellipses are $\alpha = \pi/6$ and $\alpha = \pi/2$, the bounding hyperbolas are $\beta = \pi/2$, $3\pi/4$ and π, giving respectively (*a*) the half ellipse, (*b*) the ellipse with a rectangular hyperbolic keyway, (*c*) the ellipse with a single thin keyway or slit.

The values of the torsion moment and of its ratio to the torsion moment of the equal circular section, are shown in the tables below.

TABLE of $M/\mu\tau c^4$.

	$\beta = \pi/2$.	$\beta = 3\pi/4$.	$\beta = \pi$.
$\alpha = \pi/6$	·1365	·3944	·4731
$\alpha = \pi/2$	10·354	26·319	40·142

TABLE of M/M_0.

	$\beta = \pi/2$.	$\beta = 3\pi/4$.	$\beta = \pi$.
$\alpha = \pi/6$	·8907	·8244	·7718
$\alpha = \pi/2$	·7907	·7985	·7664

When we look at these results we see that the torsional rigidity of these sections is always less than that of the circular section. The sections consisting of a complete ellipse with one fine slit up to the focus are weaker than the half-ellipse or the sections with a broad keyway. This is more particularly shown in the case of the more elongated ellipse, $\alpha = \pi/6$.

With regard to the effects of slits, or thin keyways, it is interesting to compare the values of M/M_0 for the ellipses $\alpha = \pi/6$, $\alpha = \pi/2$, when (i.) there are no slits, (ii.) there is one slit, (iii.) there are two slits

We find

	$\alpha = \pi/6$.	$\alpha = \pi/2$.
(i.) M/M_0 = . . .	·7807	·9963
(ii.)	·7718	·7664
(iii.)	·7628	·5711

It follows that, in the first case, the cutting of one thin keyway lowers the rigidity of the section by 1·14 per cent., and of two keyways by 2·29 per cent. Hence the effect of two such keyways is slightly greater, if anything, than twice the effect of a single keyway, in the case of the more elongated ellipse. The difference is, however, practically negligible.

In the other case the result is different. The reduction of the torsional rigidity is very great: it amounts to 23·08 per cent. for one keyway and to 42·68 per cent. for two keyways. Here we see the effect of two keyways is rather less than twice the effect of one.

We may infer, however, from these two results that we may in practice, without very large error, if we have a number of keyways cut symmetrically into a section, and we know the effect on the torsional rigidity of any single keyway, assume that the effect of all the keyways is the sum of their separate effects.

Another important point which is brought out by the above results is that the effect of such a keyway upon the torsional rigidity is by no means simply proportional to the depth of the keyway, but increases according to some much more rapid law.

Thus, for the ellipse $\alpha = \pi/6$, the depth of the keyway $=$ ·123 (semi-major axis). For the ellipse $\alpha = \pi/2$, the depth $=$ ·601 (semi-major axis). Thus, when the depth of the keyway is only decreased to one-fifth of what it was before, the reduction of torsional rigidity falls from 23 per cent. to 1 per cent., or nearly in the ratio of the *squares* of the depths of the keyways.

This result may explain the fact that, when keyways of only moderate depth are cut into shafts, the decrease of torsional rigidity is by no means so great as would have been inferred from DE SAINT-VENANT'S results for a circular section, with a thin keyway or slit extending right up to the centre.

If we suppose, which appears reasonable, that the effect of such a slit upon an ellipse, which is not very elongated, does not differ much from the effect on a circle, we see that a keyway, whose depth equals about one-eighth of the radius, will decrease the torsional rigidity by only about 1 per cent.

Now, when we make $\alpha = \infty$, we get the case of the circle with a keyway going right up to the centre. The reduction of torsional rigidity is then 44 per cent. about.

Hence we have, roughly,

Depth of keyway in terms of radius.	Reduction of torsional rigidity.
	per cent.
1·00	44
·60	23
·12	1

If the reduction of torsional rigidity were simply proportional to the depth of the keyway, the last two results ought to be 26·4 per cent. and 5·3 per cent. respectively.

It may be objected to these deductions that in the above sections, the fact that the stress at the vertex of the keyway is infinite, violates the physical conditions assumed by the theory of elasticity and renders our results untrustworthy.

My answer to this is that these cases should really be considered as limiting cases. If, instead of considering a section where the keyway is actually a straight line, we consider a hyperbola with a very sharp bend, we can easily ensure, provided the angle of torsion be not too great, that the physical conditions shall *not* be violated, and, on the other hand, the values of the torsional rigidity (since they tend to a finite limit) will differ but little from the values obtained above. The very fact that the torsional rigidity tends to a finite limit shows that, even in the extreme case, the area where the physical considerations are violated is infinitesimal.

Finally, in drawing conclusions from such results we shall only be following the example of SAINT-VENANT and of THOMSON and TAIT, who have not hesitated to use results found in a precisely similar way for such keyways cut into a circle.

§ 23. *Values of the Stresses.*

I have also determined the stresses at three points on the boundary of the section, in order to find where the stress was greatest. The points selected are denoted in figs. 6, 7, and 8 by the letters A, B, C.

A is the vertex of the hyperbola, B is the point opposite to A, and C is the point corresponding to $\eta = \pi/2$, $\xi = \pm \alpha$.

A is given by $\xi = 0$, $\eta = \beta$, and B by $\xi = \pm \alpha$, $\eta = 0$.

If S_A, S_B, S_C denote the corresponding stresses, I find

$$\frac{S_A}{\mu\tau c} = \cos\beta\,(1 - \operatorname{sech} 2\alpha) - \frac{8\alpha(\cosh 2\alpha - \cos 2\beta)}{\sin\beta} \sum_{n=0}^{n=\infty} \frac{(-1)^n \tanh \dfrac{\overparen{2n+1}\,\pi\beta}{2\alpha}}{\pi^2(2n+1)^2 + 16\alpha^2}$$

$$= -\tan 2\beta \sin\beta + \frac{8\beta(\cosh 2\alpha - \cos 2\beta)}{\sin\beta} \sum_{n=0}^{n=\infty} \frac{\operatorname{sech} \dfrac{\overparen{2n+1}\,\pi\alpha}{2\beta}}{(2n+1)^2\pi^2 - 16\beta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34),$$

$$\frac{S_B}{\mu\tau c} = \tanh 2\alpha \sinh \alpha$$

$$- \frac{8\alpha\,(\cosh 2\alpha - \cos 2\beta)}{\sinh \alpha} \sum_{n=0}^{n=\infty} \frac{\operatorname{sech} \dfrac{\widehat{2n+1}\pi\beta}{2\alpha}}{(2n+1)^2\pi^2 + 16\alpha^2}$$

$$= -\cosh \alpha\,(\sec 2\beta - 1)$$

$$+ \frac{8\beta\,(\cosh 2\alpha - \cos 2\beta)}{\sinh \alpha} \sum_{n=0}^{n=\infty} \frac{(-1)^n \tanh \dfrac{\widehat{2n+1}\pi\alpha}{2\beta}}{\overline{2n+1}^2\pi^2 - 16\beta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35),$$

$$\frac{S_C}{\mu\tau c} = -\sinh \alpha\,(1 + \operatorname{sech} 2\alpha)$$

$$+ \frac{8\alpha\,(\cosh 2\alpha - \cos 2\beta)}{\cosh \alpha} \sum_{n=0}^{n=\infty} \frac{\cosh \dfrac{\widehat{2n+1}\pi^2}{4\alpha} \Big/ \cosh \dfrac{\widehat{2n+1}\pi\beta}{2\alpha}}{\overline{2n+1}|^2\pi^2 + 16\alpha^2}$$

$$= -\sinh \alpha\,(1 + \sec 2\beta)$$

$$- \frac{8\beta\,(\cosh 2\alpha - \cos 2\beta)}{\cosh \alpha} \sum_{n=0}^{n=\infty} \frac{(-1)^n \sin \dfrac{\widehat{2n+1}\pi^2}{4\beta} \tanh \dfrac{\widehat{2n+1}\pi\alpha}{2\beta}}{\overline{2n+1}|^2\pi^2 - 16\beta^2} \quad . \quad . \quad . \quad (36),$$

whence I find—

TABLE of Stresses.

	$\beta=\pi/2$.	$\beta=3\pi/4$.	$\beta=\pi$.	
$\alpha=\pi/6$	- ·67631	- ·85397	$-\infty$	} $S_A/\mu\tau c$
$\alpha=\pi/2$	- 2·0716	- 3·6144	$-\infty$	
$\alpha=\pi/6$	·40265	·42625	...	} $S_B/\mu\tau c$
$\alpha=\pi/2$	1·7276	2·0509	...	
$\alpha=\pi/6$	0	·85106	...	} $S_C/\mu\tau c$
$\alpha=\pi/2$	0	2·2794	...	

We see that the greatest of the three stresses occurs at A, *i.e.*, at the vertex of the hyperbola, and it is probable that A is the true fail-point.

The following are the values of S_A/S_0, S_0 having the same meaning as in § 12 :—

TABLE of S_A/S_0.

	$\beta=\pi/2$.	$\beta=3\pi/4$.	$\beta=\pi$.
$\alpha=\pi/6$	1·2101	1·1496	∞
$\alpha=\pi/2$	1·2192	1·6888	∞

The values of the "efficiency" are easily obtained.

TABLE of E.

	$\beta=\pi/2$.	$\beta=3\pi/4$.	$\beta=\pi$.
$\alpha=\pi/6$	·7361	·7171	0
$\alpha=\pi/2$	·6485	·4728	0

The above results do not need any detailed discussion. We see that in all cases the maximum stress is greater than the maximum stress for the circular section. Also the efficiency is always less than unity. If we compare these values of the efficiency with those in § 19, we see that, on the whole, the rule holds that the more compact the section the higher its efficiency. On the other hand, by indenting a section we render it less efficient.

§ 24. *Conclusion and Summary.*

Looking back upon the results of the paper, we see that the study of these special forms of cross-section sheds new light upon several little-explored parts of the theory of elasticity.

It confirms to a great extent DE SAINT-VENANT'S investigations concerning the behaviour under torsion of a rail, or of shafts of similar section.

Owing to the great generality of the forms treated, it enables us to correlate the results previously obtained for sections of various shapes, especially with regard to the maximum stress. It shows us what type of cross-section will give us four fail-points not at the points of the contour closest to the centre; within what limits we may expect to find this exception to the ordinary rule; and in what manner this case passes into others.

Again, with regard to the effect of keyways upon the torsional rigidity, the results of the paper tell us that, without risk of sensible error, we may in practice, in order to get the joint effect of two indefinitely thin keyways or slits, add the known effects of each keyway, taken separately.

Further, these results bring the theory of elasticity into better accordance with observed facts, by showing that the effects of keyways of moderate depth are comparatively much smaller than would have been expected from the results for the circle.

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XI. BAKERIAN LECTURE.—*The Crystalline Structure of Metals.*

By J. A. EWING, *F.R.S., Professor of Mechanism and Applied Mechanics in the University of Cambridge, and* WALTER ROSENHAIN, *St. John's College, Cambridge,* 1851 *Exhibition Research Scholar, University of Melbourne.*

Received and Read May 18—Revised June 30, 1899.

[PLATES 15–28.]

THE microscopic study of metals was initiated by SORBY,[*] and has been pursued by ARNOLD, ANDREWS, BEHRENS, CHARPY, CHERNOFF, HOWE, MARTENS, OSMOND, ROBERTS-AUSTEN, SAUVEUR, STEAD, WEDDING, WERTH and others.[†] The work of these authors has demonstrated the value of the microscope in metallurgy, not only as an aid to analysis, but as a means of observing structure. The structure of pure metals, of metals containing small quantities of foreign matter, and of alloys, has been made the subject of microscopic examination, and important conclusions have been reached. The work to be described in this paper proceeds on the same general lines. A large part of it deals with a branch of the subject which has not hitherto received much notice, namely, the effects of strain. The writers believe that they have established the fact that the structure of metals is crystalline even under conditions which might be supposed to destroy crystalline structure. They have found that the plastic yielding of metals when severely strained occurs in such a manner that the crystalline structure is preserved. The observations to be described show how crystalline aggregates exhibit plasticity, and how, after straining, a metal continues to be a crystalline aggregate. The distinction which is often drawn between crystalline and non-crystalline states in metals appears to be unfounded.

Except for a few simple innovations, the methods of experiment used in this research, especially as regards the preparation of specimens, do not differ materially from those of earlier workers. The specimens were first polished on commercial emery-paper which had been previously rubbed on a piece of hard steel in order to remove the coarser particles. They were finished on a rapidly revolving disc coated with fine wash-leather and charged with a thin paste of rouge and water. For most purposes the finest jeweller's rouge is suitable, but in special cases we resorted to the use of peroxide of iron obtained by precipitation from a solution of pure ferric chloride.

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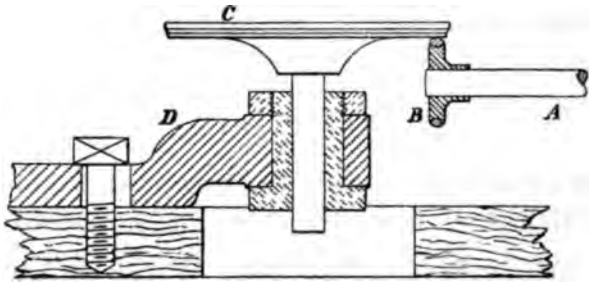
[*] 'Roy. Soc. Proc.,' vol. 13, p. 333.

[†] For the bibliography see Sir W. ROBERTS-AUSTEN'S book on 'Metallurgy' (Edition of 1895); also a paper by Mr. J. E. STEAD on the "Crystalline Structure of Iron and Steel," 'Journal of the Iron and Steel Institute,' 1898.

The polishing machine is shown in diagram-section in fig. 1. *A* is the spindle of an electro-motor carrying a small driving disc *B*, fitted with a rubber ring to increase the driving friction. The polishing disc *C* is horizontal and has a vertical axis running in a bearing in the casting *D*. The under side of the polishing disc bears upon the driving wheel *B* and takes motion from it. When it is desired to stop temporarily the polishing disc is raised out of contact with the driving wheel. The casting *D* is held to the base-board by a bolt passing through a slot which allows the disc to be brought nearer to the motor, so that it may bear upon the driving wheel at any desired distance from its axis, thus giving a means of varying the speed. More usually, however, the speed was varied by regulating the current in the motor.

Fig. 1.



With metals which are very liable to tarnish, wet polishing has to be avoided ; in some cases dry rouge, and in others rouge moistened with a little paraffin, may be used—the latter we found particularly useful in polishing copper. But in some metals, especially the more fusible ones, such as lead, zinc, and tin, it is difficult to obtain a satisfactory surface by any method of polishing. In most of these cases, however, a method of obtaining a good surface entirely without polishing becomes available. This consists in pouring the molten metal on a smooth body, such as glass or polished steel, in contact with which it is allowed to solidify. We have, as a rule, used glass for this purpose, and in spite of frequent failures, due to fracture of the glass or to less obvious causes, we find that it is generally practicable to get a good specimen in this way with much less trouble than is required to produce a specimen of iron or steel by the ordinary process of polishing. This method of "glass casting" cannot well be applied to metals which oxidize at temperatures below their melting points, and for metals which have a very high melting point a smooth body other than glass must be used. The metals most readily treated by casting against a smooth surface are gold, bismuth, cadmium, lead, tin, zinc, and fusible alloys.

In the examination of lead another method of obtaining a good surface, without either melting or polishing, was also used. A face of the specimen was freshly cut to remove the tarnish, and was then pressed against a smooth surface of plate-glass. Whenever a sufficient pressure could be reached without breaking the glass a very beautiful surface was obtained. In some cases polished steel was used as the smooth object against which the metal was pressed.

The polished surfaces were, in general, slightly etched before microscopic examination, in most instances by dilute nitric acid. Frequently, however, no etching was resorted to, especially in observations dealing with the effects of strain. It has been pointed out by CHARPY that straining, by the relative displacement which it brings about among the crystalline grains, serves to reveal the structure. Our observations confirm this, and in some cases we found that a better investigation of the effects of strain could be made when the surface was not etched.

Most of the microscopic observations were made with ZEISS' apochromatic lenses, a 2-millim. homogeneous immersion lens of 1·40 aperture being employed for high power work, with compensating eye-pieces magnifying from 4 to 18 times, giving in direct vision a total magnification up to 3000 diameters. "Vertical" illumination was generally used, the objective serving as condenser, but in some cases the specimens were examined under oblique light. Photographic records of the most interesting features were obtained, some of which are reproduced in this paper. The source of light was an arc lamp, the beam from which was condensed on the illuminator through a "Gifford" screen which allowed only a very limited portion of the spectrum to pass. Most of the high power photographs were taken with a magnification of 1000 diameters; in a few instances it was 4000 diameters or more.

It is well known that when the polished surface of a metal, such as gold or iron, is lightly etched, and is then examined by means of normally reflected ("vertical") light, the surface appears divided up into a number of areas separated by more or less polygonal boundaries. These areas are the sections of the crystalline grains which constitute the mass of the metal; the boundaries between them have been made evident by the differential action of the acid which has produced differences of level by attacking one grain more energetically than its neighbour. Fig. 2 (Plate 15) illustrates this appearance in ordinary iron. There the black patches are due to the presence of slag, and the black lines forming the boundaries are due to differences of level between the grains. Each of the short sloping surfaces which connects one grain with another appears black because it does not reflect the normally incident light back into the tube.

It is also well known that a further differential action of the acid is to reveal a difference of texture between the grains. This is visible in fig. 2, but is much more pronounced when the surface is examined under oblique light. When the light is uni-directional or nearly so, the various grains differ very much in brightness and colour—some are almost black, while others shine out brightly; if, however, the incidence of the light or the orientation of the specimen be changed, other grains shine out strongly, while those previously bright become dark. This effect was first observed in gold by ARNOLD ('Engineering,' February 7, 1896), who accounted for it by considering that each crystalline grain is built up of a very large number of what he (following ANDREWS) calls secondary crystals, which have different orientations in different grains. Substantially the same view is expressed in other terms by OSMOND and ROBERTS-AUSTEN ('Phil. Trans.,' A, vol. 187, pp. 424–5), who speak of little crystals, "the general orientation of which remains constant in the area of each grain." Many beautiful illustrations of the same effect in iron have been given by STEAD, along with a clear discussion of the cause to which this appearance is to be ascribed ('Journal of the Iron and Steel Institute,' 1898). A striking instance is shown in his photographs of steel containing about 4½ per cent. of silicon. The fractured ingot of this material exhibits large crystals, and by deeply etching a polished surface he has obtained a fine development of the regularly oriented elements of which the crystalline grains are built up, on a scale so large as to be clear with little or no magnification.

These observations have made it plain that each of the grains which appear on the polished and etched surface of a metal is simply a crystal, the growth of which has been arrested by its meeting with neighbouring grains. The irregular boundaries of the grains are determined by these meetings. We may imagine the grains to grow from as many centres or nuclei as there are grains. Each grain is built up of similarly oriented parts, but the orientation changes from grain to grain. Etching a polished surface develops a multitude of facets which have the same orientation over the surface of any one grain, but different orientations in different grains. Seen under oblique illumination these facets show themselves to be similarly oriented in each grain by the uniform manner in which the grain reflects light, and by the disappearance of brightness over the whole surface of the grain when the incidence of the light is changed. The mass of the grain consists of similarly oriented elements; as to the size of the elements no assumption need be made. The facets which are developed by etching do not, in general, appear of constant size; it is to the constancy of their orientation that the effect is due.

A striking illustration of how a metal crystallises by the simultaneous building up of groups of elements from a number of centres, the elements in each group being similarly oriented, while the orientation varies from group to group, may be seen in a cake of solidifying bismuth from which the still molten metal has been poured away. The operation then goes on upon a scale so large that no magnification is required to make it apparent. Fig. 3 is a photograph of a specimen of bismuth in the Cambridge University Museum of Mineralogy, for the loan of which the authors are indebted to Professor LEWIS and Mr. A. HUTCHINSON. The scale of the photograph is only about two-fifths natural size, but it shows well how the cake is made up of crystalline grains, each composed of elements with a definite orientation, the boundaries between the grains being due to the casual meeting of the several groups in the process of growth.

The references given above will show that there is nothing novel in this view of the structure of metals. Several of the authors' observations may, however, be mentioned as affording additional evidence that the structure of metals in general consists of crystalline grains built up in the manner which has been described. One class of evidence has been obtained by an examination of specimens of iron which had been deeply etched under very high magnification, *i.e.*, 2000 to 3000 diameters. Under favourable circumstances it is possible to resolve the minute structure to which the peculiar reflection of oblique light is due. The face of each grain is then seen to be covered with minute projections resembling scales, more or less square or oblong in shape and similar and similarly oriented over the entire face of one grain. Fig. 4 is a photograph of this appearance as seen in nearly pure wrought iron: it may be compared (as Mr. STEAD compares a like appearance occurring on a large scale in silicon steel) to the arrangement of slates on the roof of a house. In other cases the action of the acid is different; the general surface of a grain cannot be resolved under the highest powers, but here and there the acid has etched out minute pits showing a distinct geometric form. All these pits found over the face of one grain are similar and similarly situated figures, but the shape and orientation of the pits changes as soon as a boundary is crossed. This is shown in a very striking way where a comparatively large pit crosses the boundary so that a portion appears on each side. Each portion preserves its proper shape and orientation, and there is consequently a marked angle in the sides of the pit where the sides cross the boundary. The shape of all these pits in iron is consistent with the assumption that they are plane sections of cubes or octahedra.

Fig. 5 (Plate 16) is a photograph of such etched pits in Swedish iron.

A good development of geometrically etched pits is not very readily obtained: in some specimens of iron they occur with much greater readiness than in others, and this occurrence is to some extent an accident of etching. Possibly the presence of a minute quantity of impurity in the iron is an essential factor, but we have no evidence on the subject. Geometrical etched pits are a well known phenomenon in non-metallic mineral crystals. BAUMHAUER* finds that they have a definite relation to the crystallographic nature of the crystal upon which they occur—but the facets developed by etching often lie in planes which are not parallel to the natural faces of the crystal. He finds that these etched pits, though generally truly geömetrical, frequently show curved or irregular outlines which he attributes to local concentration of the acid. We find that curved or irregular outlines often occur in the larger etched pits in iron, and in view of BAUMHAUER's observations it is clear that they cannot be taken to affect the evidence for the strictly crystalline nature of the metal, since similar appearances are to be found in bodies that are characteristically crystalline.

When metals are cast against glass or other smooth bodies, to get a surface fit for microscopic examination, evidences of crystalline structure appear apart from anything that is shown by etching. The cast surface generally shows clearly, without any etching, the boundaries between the crystalline grains. Good examples of this are found in cadmium, lead, tin, and zinc. In some instances the boundaries between the grains are emphasised in a remarkable way by the accumulation there of air or of gas given off by the metal during solidification. The boundaries then appear as more or less deep and wide channels. As the growth of the crystalline grains proceeds most of the air or gas which has been entrapped between the glass and the metal, and most of the gas given out from solution in the metal itself, tends to accumulate at the boundaries, which are the last parts of the surface to undergo solidification. A network of channels consequently appears on the surface next the glass; in general these channels coincide with boundaries, but occasionally a channel forms a *cul-de-sac* or terminates in a large cavity which is obviously a bubble or blow-hole. Fig. 6 shows the appearance of a surface of cadmium cast against glass under conditions of temperature favourable to the formation of these channels. By etching or straining the specimen it is easy to prove that each channel (except when it is a *cul-de-sac* formed by excess of gas) coincides in general with a real boundary between the crystalline grains. The true boundary is merely the trace of a surface on the plane of casting, but it may be broadened out in this way, by the presence of gas, into a shallow channel of considerable width. It is only in certain conditions of temperature, on the part of the metal and of the body against which it is cast, that any marked development of such channels is obtained. Under most conditions, however, it is easy to distinguish the intergranular boundaries in the cast surface, either by the presence of some gas there or by slight differences of level between one grain and the next.

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* BAUMHAUER, 'Resultate der Aetzmethode in der Krystallographischen Forschung.' WILHELM ENGELMANN, Leipzig, 1894.

Occasionally the faces of some of the grains in the surface cast against glass are covered with a number of very minute pits, whose true character appears only under a magnification of about 1000 diameters. They are then seen to be pits of definite geometrical form both as to outline on the surface and as to inner facets. Figs. 7, 8, and 9 are photographs of these pits in glass-cast cadmium. It will be seen that they are systems of geometrical figures which remain similar and similarly situated over the entire surface of a single grain, but change in character from one grain to the next. These pits appear to be excessively small bubbles or blow-holes which have taken a geometrical shape, forming, in fact, negative crystals. During the crystallisation the crystalline elements have built themselves around the gas bubbles in regular orientation, finally leaving the pits as we see them. In support of this view the appearance seen on fig. 8 may be cited; the pits are seen to be surrounded by a halo or circular patch of bright surface—due apparently to the absorption by these larger geometrical bubbles of the numerous tiny bubbles that dot the surface elsewhere.

The photographs show these "air-pits" in cadmium cast against glass under a magnification in most cases of 1000 diameters, and in one case (fig. 9) of 4200 diameters. The characteristic of the pits is that they are similar and similarly oriented over any one grain, but on passing from one grain to another, across a boundary, the orientation of the pit is found to have changed. Good examples of this are seen in fig. 7, where the boundaries are widened out into channels through the presence there of air or gas in the manner described above. This characteristic of the air-pits is, of course, in complete conformity with the view already stated of the crystalline structure of metals. Additional photographs of air-pits in cadmium are shown in another connection in figs. 26, 27, and 28 (Plates 22 and 23).

Examination of the air-pits in cadmium shows that the forms may be accounted for as sections of hexagonal prisms. It may be concluded that each crystalline element of which the grains of cadmium are built up is a hexagonal prism with plane base.*

These air-pits are not very readily developed, and the precise conditions which determine their appearance are not easily specified. Many specimens of the metal may be cast without obtaining them. That they are not, however, peculiar to cadmium is certain, for we have found them also in tin and in zinc. Fig. 10 shows them in a glass-cast surface of tin.

The same photograph illustrates another interesting feature. The surface of the tin is seen to be covered with a multitude of small dark crystals irregularly disposed. These are evidently inclusions of some foreign matter, which we conjecture to have been sulphide, because it was observed that they appeared in large numbers after the metal had, during melting, been directly exposed to a gas flame. The foreign crystals have no definite orientation, and are quite independent of the orientation of the metal within the grains in which they are embedded. The crystallisation of the metal has proceeded around them without check or disturbance, just as in iron containing slag the growth of each crystalline grain ignores the presence of that impurity. It is well known that a slag band is often seen running right through a crystalline grain without affecting the uniformity of orientation of the elements of which the grain is built up.

Although it must be admitted that a really good development of geometrical bubbles such as those shown in the photographs is exceptional and cannot as yet be reproduced at will, the authors have observed that nearly all bubbles or blow-holes found on surfaces prepared in this way show a distinct tendency to geometrical shapes. A truly round bubble is rarely or never found, and even the larger bubbles often show a multitude of distinct facets reflecting light at different angles.

The occurrence of such geometrical pits in surfaces of metals that have never been polished or etched may be taken as very strong evidence in support of the view that the crystalline grains of metals are built up of crystalline elements which are similarly oriented throughout the mass of each grain.

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* It may be added in this connection that BEHRENS remarks on the frequency of six-sided forms in the polygonal boundaries of the crystalline grains of cadmium. The form of boundary is, however, of little service in determining the character of the crystallisations within the grain. (See his work 'Das Microscopische Gefüge der Metalle und Legierungen,' Leipzig, 1894.)

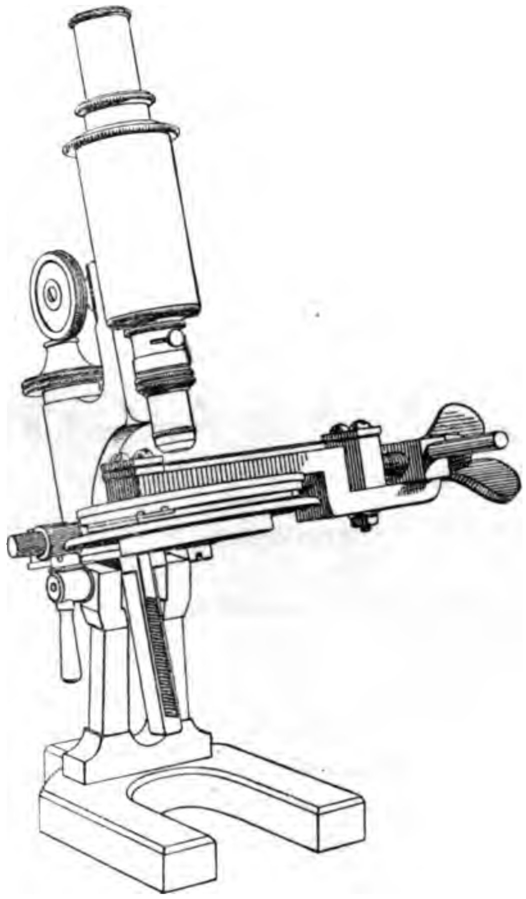
The experiments now to be described were intended to throw light on the nature of plastic strain in metals, and the results obtained are interpreted by the aid of the theory of the crystalline structure of metals, of which an outline has been stated at the beginning of this paper. Their complete agreement with that theory affords further confirmation of its truth.

It has long been known that when a specimen of iron or steel with a bright smooth surface is strained sufficiently to give it permanent set the smoothness of the surface is destroyed. A microscopic examination of the surface shows, as CHARPY has pointed out, that the crystalline grains become visibly differentiated from one another by straining, and the effect is in this respect not unlike that which etching would have produced. There is, however, a further effect of straining, which will be described immediately.

It is also well known that when a piece of severely-strained metal is polished and etched, as for instance a bar which has had its section reduced by hammering or rolling in the cold state, the crystalline grains are found to have changed their form. They are lengthened in the direction in which the piece was extended and shortened in the direction in which the section has contracted. Accordingly, a severely-strained piece is readily recognised on polishing and etching it, on account of the shape of its crystalline grains. In the strained specimen these are found to have a direction of predominating length according to the direction of strain, while in the unstrained specimen there is no direction of predominating length. Further, it is well known that in a strained specimen elongated grains are not found after the strained state has been relieved by annealing. The effect of rolling or hammering is often spoken of as a conversion of the metal from a crystalline to a "fibrous" state. In the present writers' view this language is misleading, and the physical conception underlying it is a mistaken one.

In the first experiments on the effects of strain we aimed at keeping a particular place on the polished and etched surface of the specimen under continuous observation while the specimen was being strained. This was accomplished by constructing a small straining machine which could be attached to the stage of the microscope, and by which strips of sheet metal could be gradually extended until they broke. Fig. 11 shows the arrangement which was used. The stress was applied by a screw which could be turned by hand while the specimen was under observation, and any displacement of the particular grains on which the microscope was focussed could be made good by means of the traversing screws in the mechanical stage. With this apparatus it was easy to keep the same group of crystalline grains under observation from the first application of stress until the specimen was broken, and to obtain photographs of the same group at any number of stages during the strain. The first specimens observed in this way were strips of sheet iron, of the nearly pure kind supplied for use in electrical transformers. The following account of what we have observed may be read as applying not only to iron but to other metals. Within the limit of elasticity no effects of strain are detected, but when the yield point is reached a remarkable change is seen on the surface of the crystalline grains. As soon as plastic deformation begins the faces of the grains show fine black lines, and as the strain increases these lines increase in number; they are more or less straight and parallel in each grain, but are differently directed in different grains. The first lines to appear are those approximately transverse to the pull, but as the strain increases systems of inclined lines appear on other grains. With further straining some of the grains begin to show more than one system of such lines, and eventually two, three, and even four systems of intersecting lines on a single grain may be seen.

Fig. 11.



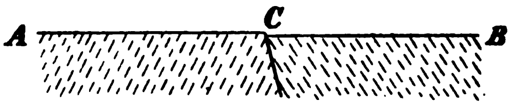
A characteristic example of these lines as exhibited by iron is shown in fig. 12, which is a photograph of a piece of Swedish iron strained by pull. Figs. 13 and 14 are two views of the same group of crystalline grains in another piece of Swedish iron, fig. 13 being taken before straining, and fig. 14 after a considerable amount of straining. When the piece is much strained the surface becomes so rough as to make it difficult to secure a satisfactory photograph.

The appearance of the grains after straining so closely resembles that of a crevassed glacier that the black lines might be taken for cracks. But from the outset it was clear that they could not be actual fissures. A piece of strained iron, when it has been allowed to rest for a time, or has been heated to 100° C.,* recovers its original elasticity and more than its original strength, yet the dark lines do not disappear under such treatment. Further, if a specimen showing these lines be re-polished the lines disappear, and even light etching does not reproduce lines of the same nature.

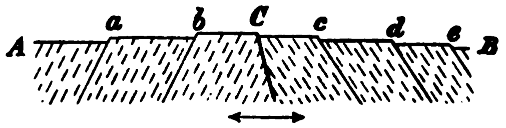
The real character of the lines is apparent when the crystalline constitution of each grain is considered. They are not cracks but steps in the surface. These steps are due to slips along the cleavage or gliding planes of the crystals.†

The diagram, fig. 15, is intended to represent a section through the upper part of two contiguous surface grains, having cleavage or gliding places as indicated by the dotted lines, A B being a portion of the polished surface, C being the junction between the two grains.

Fig. 15.



Before straining.



After straining.

When the metal is strained beyond its elastic limit, as say by a pull in the direction of the arrows, yielding takes place by finite amounts of slips at a limited number of places, in the manner shown at *a*, *b*, *c*, *d*, *e*. This exposes short portions of inclined cleavage or gliding surfaces, and when viewed in the microscope under normally incident light these surfaces appear black because they return no light to the microscope. They consequently show as dark lines or narrow bands extending over the polished surface in directions which depend on the intersection of the polished surface with the surfaces of slip.

The correctness of this view is demonstrated when these bands are examined under oblique light. When the light is incident at only a small angle to the polished surface, that surface appears for the most part dark ; but here and there a system of the parallel bands shines out brilliantly in consequence of the short cleavage surfaces which constitute the bands having the proper inclination for reflecting light into the microscope. The groups of bright bands which are seen under oblique light are readily observed under the microscope to be exactly coincident with the black bands seen under vertical illumination. Figs. 16 and 17 are photographs of the same strained crystalline grains of iron under vertical light and under oblique light.

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* See J. MUIR "On the Recovery of Iron from Overstrain," 'Phil. Trans.,' A, 1899.

† See 'Roy. Soc. Proc.,' March 16, 1899 (vol. 65, p. 85), where the authors have published a preliminary account of some of these observations.

Rotation of the stage upon which the strained specimen is fixed makes the bands on one or another of the grains flash out successively with kaleidoscopic effect. When the specimen is rotated through 180° from the position in which the lines show brightly on one particular grain, the lines on that grain do not shine out again, though they may be visible as black lines on a very faintly luminous background; this is important as proving that the lines are not due to either furrows or ridges, but to steps in the surface. In this respect there is a striking contrast between the slip-bands and any accidental furrows or scratches which may have been left on the specimen through imperfect polishing.

Incidentally, fig. 17 illustrates the fact that oblique light picks out the boundaries of the crystalline grains, showing that these boundaries are marked by inclined surfaces connecting grains whose faces are at different levels. This is observed also in the etched surface of the metal before straining. The boundaries, which appear dark under vertical light, are bright on one side of each crystalline grain when the light falls with grazing incidence from one side; but the sloping surfaces which mark the boundaries between the grains have not the sharply-defined inclination of the slip-surfaces. The lines due to slip-bands on one or more grains will shine out brightly when the light has a particular angle of incidence, and will vanish when the incidence is slightly changed. The boundaries are generally not so bright, but remain visible under a considerable range of incidence.

Figs. 18, 19, and 20 present a striking example of the kaleidoscopic effect under oblique light just referred to. Here the metal is lead, the surface has been obtained by allowing molten lead to solidify on a smooth glass plate, and the metal has been strained by bending. The figures show the appearance under vertical light and also under two different incidences of oblique light. Figs. 21 and 22 show another example of strained lead under vertical and oblique light.

In their preliminary notice* of some of the present results (communicated to the Royal Society on March 16, 1899) the authors have applied the name "slip-bands" to the lines developed on metallic surfaces by plastic strain, and in what follows they will be referred to under that name.†

So far as the writers' observations go it appears that slip-bands occur in all metals as soon as plastic deformation takes place. They have been observed in specimens of platinum, gold, silver, copper, lead, zinc, tin, cadmium, bismuth, aluminium, iron, and nickel, as well as in steel, brass, bronze, and other alloys. In the case of iron it was proved definitely that under tensile stress the bands appear as soon as the yield point is passed. For this experiment a flat was polished on the side of a test-piece, which was strained in a 50-ton testing-machine, and the surface was kept under observation, during the application of the stress, by means of a microscope hung from the bar itself.

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* "Experiments in Micro-metallurgy: Effects of Strain," 'Proc. Roy. Soc.,' vol. 65, p. 85. Since writing that paper the authors' attention has been directed to the following remark by Mr. STEAD:—"It would appear that there is a tendency for iron to etch out into thin plates, and when such etched specimens are distorted or pulled out for these etched plates to slide over one another" ("On Brittleness Produced in Soft Steel by Annealing," 'Journal of the Iron and Steel Institute,' 1898). It will, however, be shown presently that the development of slip-bands is independent of any previous etching.

† "Experiments in Micro-metallurgy: Effects of Strain," EWING and ROSENHAIN, 'Roy. Soc. Proc.,' 1899, vol. 65, p. 85.

Slip-bands are developed by all kinds of strain involving permanent deformation of the piece. A microscopic inspection of the surface after straining does not enable us to detect whether the deformation has been caused by tension, compression, bending, or torsion, but the appearance depends very much on the amount of strain that has taken place. The more severely the specimen has been strained the greater is the number of slip-bands developed. After slight straining there is generally only one system of bands to be seen on each crystalline grain, but as many as four such systems intersecting one another may come into view after severe straining. Fig. 23 (Plate 21) is a photograph of strained lead, showing four systems of slip-bands.

The general appearance of the slip-bands is different in different metals. In lead they are particularly straight, even under extreme magnification, as is illustrated in fig. 24. In silver there is a tendency to more wavy outlines, as appears in fig. 25, Plate 22, but in gold the lines are as straight as those in lead. In iron the lines tend, as a rule, to be rather wavy, and to fork and branch. Examples of slips in this metal have already been given in figs. 12, 14, and 16; others have been given in the preliminary notice referred to above.*

In gold, lead, and other metals showing straight slip-bands it is easy to distinguish well-marked steps where intersecting systems cross. Fig. 24 is a characteristic example.

In several specimens of lead, prepared by casting against glass, a peculiarity was noticed which forms an apparent exception to the statement made above that slip-bands which shine out under oblique light at one particular incidence do not reappear when the incidence is changed (*i.e.*, the specimen rotated) by 180°. The specimens in question were examined under oblique light with an objective of 16 millims. focus, giving a magnification of 100 diameters. On rotating the stage carrying the specimen the slip-bands on one crystalline grain were found to shine out strongly in two positions, very nearly 180° apart. Under this low magnification it seemed that identical bands were visible in both positions; but on applying a power of 2000 diameters with a combination of "vertical" light and an oblique beam of grazing incidence from an arc lamp it was seen that there were really two systems of parallel slip-bands on the same crystalline grain, and that only one of them was picked out by the oblique light. In the other system of slip-bands the slope of the cleavage surfaces exposed by the slips was inclined away from the source of light, and consequently remained dark. The explanation, then, is that in this grain the crystalline elements are oriented in such a way that the intersection of two sets of cleavage or gliding planes is parallel to the surface, and consequently their traces on the surface are parallel to one another. This seems to be an accidental occurrence, possibly favoured by the condition under which the specimen crystallised, namely in contact with cold glass. In many other instances a more or less close approximation to such parallelism has been observed, resulting in systems of slip-bands intersecting at a very small angle; in that case the two positions where slip-bands appear under oblique light are nearly, but not exactly, 180° apart. Seen under vertical light, with low magnification, such intersecting systems produce the appearance which is exemplified in the central grain in fig. 18, Plate 19.

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* 'Roy. Soc. Proc.,' vol. 65, Plates 1–5.

The relation of slip-bands to the geometrical air-pits in cadmium is illustrated by the photographs, figs. 26, 27, and 28. In these examples the metal was cast against glass in such a way as to produce air-pits, and was then strained sufficiently to develop slip-bands. It appears that the slip-bands are always parallel to one side of the geometrical pits—the two phenomena thus confirming the views which have been advanced above as to the crystalline structure of the metal. Figs. 29 and 30 show the relation of slip-bands to geometrical etched pits in iron. It will be observed that the slips are not generally parallel to a side of the etched figures, but in specimens that have been more severely strained than those here illustrated, one set of slip-bands in each grain is generally found to be parallel to one side of the etched pits, while the other systems intersect these sides diagonally. The observations point to the conclusion that in iron slipping occurs most readily along the octahedral planes, although slips parallel to the sides of the cubical crystals are also found.

The development of slip-bands in strained metal throws what appears to us to be a new light on the character of plastic strain. Plasticity is due to the occurrence of these slips. When metals are strained beyond their limit of elasticity the deformation occurs through sliding over one another of the elementary portions of which each crystalline grain is built up.

The sliding which gives rise to slip-bands is of finite amount, and occurs at a limited number of places. "Flow" or plastic strain in metals is not a homogeneous shear such as occurs in the flow of a viscous fluid, but is the result of a limited number of separate slips.

The conception that metals adapt themselves to the new shapes imposed upon them when they undergo plastic deformation by means of slips along cleavage or gliding planes within each crystalline grain, leads naturally to the supposition that the crystalline elements themselves undergo no deformation in this process. The portions of the metal between one surface of slip and the next may remain undeformed, except elastically, under all stresses. The effect of a stress sufficient to produce plastic strain is analogous to that of a tractive force overcoming the static friction between two surfaces.

If we assume that plastic strain takes place solely by these slips, it follows that the ultimate crystalline elements should always remain parallel to themselves. The orientation of the elements would remain uniform throughout the mass of one grain, however much the outline of that grain were distorted by slips occurring within it. In other words, the crystalline structure of a metal should survive even the severest strain.

This conclusion is borne out by the fact that in metals which have been much strained we find evidence of crystalline structure similar to that which is found in unstrained metal.

Taking this evidence in the same order as before, we would refer first to the appearance which is presented in severely strained metal, by a polished and etched section when seen under oblique light. We have examined a bar of fine Swedish iron (kindly supplied by Messrs. EDGAR ALLEN and Co.), which had been for the purpose of these experiments rolled in the cold state from a diameter of three-quarters of an inch to a diameter of half an inch,.and had not been subsequently heated. A longitudinal section of this cold-rolled bar showed a great lengthening of the crystalline grains in the direction of rolling, and even in the transverse section it was obvious that the grains had been much distorted, though there was no direction of predominant length. Under oblique light both sections (longitudinal and transverse) exhibit the effect described above for unstrained metal—the grains are distinguished from one another by differences of brightness, which vary when the incidence of the light is altered, and this brightness is uniform over the entire surface of each grain. As in the case of unstrained metal, we regard this as evidence of the uniform orientation of the crystalline elements throughout each grain. Fig. 31, Plate 23, is a photograph under oblique light of a specimen of this bar cut transversely, and polished and etched. It illustrates the uniform brightness of each grain, due to uniform orientation. Similar characteristics have been observed in many other specimens of severely strained metal.

Another line of evidence in proof of the persistence of crystalline structure after severe straining is afforded when a polished specimen of, say, cold-rolled iron is subjected to a slight further strain. The slip-bands appear as they would have done had the specimen never been strained before. The general features are much the same as in annealed metal, but they show on the whole a greater tendency towards sudden steps and branches. This difference in the character of the slips may be connected with the well-known fact that such strained material is considerably harder, in the sense of having a higher yield-point and less capability of plastic deformation than it shows in the virgin or annealed state, and also with our own observation that the slip-bands are much more straight and regular in very plastic metals, such as lead, gold, and copper, than in harder metals like iron and nickel. The mere fact that finite slips occur at intervals throughout the grain implies that it is easier for sliding to take place over certain surfaces than it is over others. The surface on which uniform slip occurs is not necessarily plane; it may be the trace of a straight line moving parallel to itself; a line which is in the direction of slip and always lies parallel to one of the cleavage planes. For convenience in argument we may for the moment assume that the surfaces of easiest sliding are determined by some accident, such as the presence there of minute layers of some impurity, such as occluded gas. If, as the straightness of the slip-bands seems to indicate, these surfaces are true planes in plastic metals like lead or gold (so far as each individual grain is concerned), sliding might take place in two directions on each of these planes. But when sliding has once taken place, the intersecting layers of impurity would no longer be distributed over planes, and further sliding on transverse planes would necessitate the starting of fresh slips in surfaces that had no special tendency to facilitate sliding. This suggests how such a metal may be hardened by previous straining; and how, also, the slip-bands formed on re-straining a piece of metal hardened in this way would be less straight and more liable to sudden steps and branches than those that are found in the virgin material; the slips would, as far as possible, follow the old surfaces of easy sliding, but these would now be stepped instead of plane as before.

A striking proof of the persistence of crystalline structure in metals which have been submitted to severe distortion is found in the existence of geometrical etched pits. These are readily developed in sections cut from cold-rolled iron, and they differ in no way from the etched pits developed in the virgin material; like these, they appear as similar and similarly oriented geometrical figures over the face of each grain. Fig. 32 is the photograph of a group of crystalline grains in the specimen of cold-rolled Swedish iron referred to above (cold-rolled from three-quarters of an inch to half an inch diameter). Among them is a large grain (showing light in the figure) with an outline so unlike those found in unstrained metal that its form is evidently due to violent distortion in the process of rolling. The face of this grain is covered with minute etched pits, and an examination of these under high powers shows that they have preserved their similarity of shape and orientation in spite of the violent distortion which the grain, as a whole, has undergone. Fig. 33 is a photograph under 800 diameters of a portion of the large grain in question, which appears near the middle of fig. 32.

From the various lines of evidence here indicated we conclude that the characteristic crystalline structure of metals is not destroyed by strain, no matter how severe, and that plastic deformation occurs by means of slips along the cleavage or gliding planes of the crystalline grains, the crystalline elements which build up each grain remaining unaltered both as to shape and orientation. This statement, however, is subject to the following qualification. We have found in certain metals, notably copper, gold, silver, lead, cadmium, tin, zinc, and nickel, that twin crystals are liable to be developed by straining. Hence in such cases it is not exact to say that straining produces no change in the orientation of the crystalline elements, for twinning implies a rotation of one group of elements with respect to the rest through a definite angle. The twinning which we have found in many strained metals corresponds to the twinning observed in calcite by BAUMHAUER and REUSCH, and subsequently produced in isolated crystals of antimony and bismuth by MUGGE.*

BAUMHAUER found that by forcing a knife blade into a crystal of calcite at the proper angle a portion of the crystal could be made to swing over into the twinned position. This implies a corresponding change of orientation in the crystalline elements. The experiment may be said to afford an example of plasticity in a crystal, but it is not entirely analogous to the plasticity by pure sliding which is found in iron and in most other metallic crystals. In the process of twinning by strain there is both slip and rotation of the elements.

The existence of twin crystals in certain metals became apparent when systems of slip-bands were found like those shown in figs. 34, 35, and 36. These are photographs from specimens of copper. They show that certain of the crystalline grains are crossed by twin lamellæ, the twin planes being defined by a sudden change in the direction of the slip-bands. Where several such twin lamellæ occur in one crystalline grain, we have a periodic structure with alternate systems of parallel slip-bands. The change of orientation in passing from one lamella to another is constant. It is clear that in these examples we have true cases of twin crystallisation. An example of twinning in gold, as seen under vertical light, is given in fig. 37, and fig. 38 is a photograph of twins in gold, seen under oblique light.

The question arose whether these twin crystals were a feature in the primitive crystallisation of the metal, or whether they were subsequently produced as a consequence of strain. They were first seen in wrought copper (namely in a piece of rolled plate), which had been raised to a bright red heat before polishing. We next examined specimens of copper, gold, silver, and lead, each in the cast state, and in none of these found any appearance of twinning. For the purposes of this examination only a slight strain was applied. The same pieces were then wrought, that is to say they were severely strained (namely, by cold hammering) and they were again examined both before and after annealing at a red heat. The result showed that the violent strain produced by working the metal had developed twins in specimens where none could be seen before, and that the twins were still found in the wrought specimens after annealing. Fig. 39 is a photograph of a "twin" in cold-hammered copper (not heated after hammering); incidentally it illustrates the persistence of crystalline structure after violent deformation. Still more striking in this respect is the appearance shown in fig. 40. The specimen in this instance was an ordinary piece of plumber's sheet lead; the surface was scraped bright with a knife, and was then squeezed against a piece of plate-glass in a vice, thus producing a beautiful surface. The specimen was then very slightly bent in the fingers to develop slip-bands, and on examination under a high power it showed the appearance reproduced in the photograph.

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* See P. GROTH'S 'Physikalische Krystallographie.' VOSS, Leipzig.

Twinning, as revealed by slip-bands, has been observed in nickel; in this case the specimen was a virgin casting, but the twin (shown in the photograph, fig. 41) was observed only after very severe straining, and was, in all probability, produced by that straining.

In zinc, tin, and cadmium twins either occur very freely in the cast metal, or else are very readily produced by the slight strain which is applied to develop slip-bands. Surfaces of these metals, produced by casting against glass, show twin bands even under fairly low powers when the specimen is slightly bent; the twin bands then appear as shaded bands running across the crystalline grains. Very frequently the twin bands run on continuously across two or more grains, with more or less change in direction when they cross a boundary.

A specimen of cadmium showing this feature is photographed in fig. 42. This particular specimen presents another peculiarity; it was prepared by casting against glass, and in this instance the glass surface was intentionally given a considerable slope, with the result that the metal solidified in a long strip while it was running down the glass. On examining the under face of this strip two modes of crystallisation were observed. Part of the surface there showed a very small structure with no direction of greatest length. On another part there were large grains very considerably longer in the direction of the length of the strip than in a transverse direction. The photograph, fig. 42, is taken from an area showing these long grains. When the piece was strained by bending, twin lamellæ appeared in a more or less transverse direction, passing across from one elongated grain to the next with only a slight change in direction. The twin band in one grain is associated with a twin band in a neighbouring grain, the bands being continuous except for a change of direction as they pass from grain to grain.

We have observed twinning in gold, silver, copper, lead, nickel, zinc, tin, and cadmium. It does not appear to occur in iron.

The facility with which most metals undergo twinning as a consequence of strain shows that there are in general two modes by which plastic yielding takes place in an aggregate of crystals. One is by simple slips, where the movements of the crystalline elements are purely translatory, and their orientation is preserved unchanged. The other is by twinning, when rotation occurs through an angle which is the same for each molecule in the twinned group. Both modes are often found in a single specimen of metal, and even in a single crystalline grain. Thus, in gold or copper, it is very usual to find, on examining a strained specimen that one portion of a grain is covered with simple slip lines, while another portion of the same grain shows one or more lamellæ which are twinned with respect to the rest of the grain.

On surfaces prepared by casting against glass, particularly with cadmium, but also with zinc and tin, a curious feature often occurs which is closely associated with the facility these metals show in developing twins. The appearance in question is that of an apparent duplication of the inter-granular boundaries, as seen in the cadmium casting, fig. 43. The second system of boundaries consists, like the first, in polygonal markings, and has such an obvious relation to the first that it almost appears as though the upper layer of crystalline grains were transparent, and that we were seeing their lower edges. To decide as to which of the markings on the surface constituted the true surface boundaries, two methods were available: etching with an acid and slightly straining the specimen so as to develop the slip-bands. The latter method is much the more instructive, and its results are confirmed by the etching process. When a specimen having this characteristic was strained, it was seen at once that some of the apparent boundaries were consistently ignored by the slip-bands, the others being the real junction lines of the grains. But the true nature of the pseudo-boundaries comes out on examining them under a high power. Although under low powers there is no obvious difference in definition between the genuine and pseudo-boundaries, under greater magnifications it becomes impossible to focus the pseudo-boundaries at all—they are seen to be more or less ill-defined slopes or changes of level, whereas the real boundaries are sharply defined. In general the real boundaries show some accumulation of gas bubbles along them, and they are never crossed by slip-bands. The pseudo-boundaries are found to consist in small variations of level in the surfaces of the grains in which they occur. Fig. 44 is a high-power photograph of a set of real and pseudo-boundaries showing slip-bands.

It will be noticed that on the two sides of a real boundary the slip-bands are independent of one another, whereas the slip-bands cross a pseudo-boundary with only a slight change of inclination, which is to be ascribed to the fact that the surface under examination is not a true plane. There is a slight slope on each side of the pseudo-boundary, and the lines are consequently more or less inclined to one another. Again, as we have noticed in other examples, the slope is not as a rule constant and hence the lines are slightly curved.

An explanation of this appearance of pseudo-boundaries is, we think, to be found in the strains set up by contraction on cooling. If we suppose the outer layer of crystals to cool more rapidly than the inner ones, the resulting contraction will drive the projecting edges of the lower layer into the outside grains and thus cause slight local deformation, which will project itself on the surface, probably by means of twin bands running through the grains and appearing on the surface. The effect resembles that of a Japanese "magic" mirror, in which slight inequalities of the surface, corresponding to a pattern behind, cause light reflected from the mirror to produce an image in which a ghost of the pattern may be traced.

The foregoing conclusions refer to experiments on pure or nearly pure metals. We have also examined the effects of strain on various alloys. The micro-structure of alloys has received attention at the hands of most of the workers already named, especially BEHRENS, CHARPY, GUILLEMIN, OSMOND, ROBERTS-AUSTEN, and STEAD. Our observations have been directed towards supplementing theirs, in respect particularly of the effects of strain.

The experiments on iron were extended to certain steels. In very mild steel slip-bands can be readily observed in what are generally called the "ferrite" areas, which remain white after light etching. This is shown in fig. 45. The first effect of strain is to develop the inter-granular junctions in these white areas, then more severe straining makes the slip-bands appear. In steels containing larger proportions of carbon, the scale of the granular structure of the "ferrite" diminishes and the slip-bands become correspondingly minute, requiring the highest powers of the microscope for their observation. We have not been able to observe anything of the nature of slip-bands in the dark or "pearlite" areas of steel, but the correspondence which has been recognised by OSMOND to exist between the structure of "pearlite" and that of typical eutectic alloys, taken with facts to be described below, points to the possibility that "pearlite" may also yield plastically by slipping.

Slip-bands have also been observed in various specimens of brass and bronze.

The behaviour of eutectic alloys under plastic strain is of special interest, because these bodies apparently differ so widely in structure from pure metals. Our observations have been made on the eutectics of lead-tin, copper-silver, and lead-bismuth. The micro-structure of such eutectics has been described by OSMOND.

Fig. 46 is from a specimen of lead-tin eutectic kindly prepared for us by Messrs. HEYCOCK and NEVILLE; the surface was obtained by casting against glass, and was lightly etched with a 1 per cent. solution of nitric acid. Figs. 47 and 48 illustrate the most obvious effect of strain on such structures; the surfaces have not been etched, the differentiation of the two constituents by differences of level being here entirely due to strain. It will be observed that the scale of this structure is similar to that of the slip-bands seen in pure metals, and examination of strained specimens shows that plastic yielding is associated with slips occurring between layers of the two constituents. A close examination of strained specimens has enabled us to detect slip-bands in the light-coloured constituent. By adopting the device of slow cooling, which has led to such excellent results in the hands of Messrs. HEYCOCK and NEVILLE, we have succeeded in producing specimens of eutectics in which the characteristic structure is developed upon a much larger scale. Fig. 49 exemplifies this in the eutectic of bismuth and lead, and shows slips which occur in the white constituent as a consequence of straining. This photograph illustrates a feature very characteristic of eutectic alloys; a parallel system of slip-bands extends over many patches of the white constituent, thus pointing to the fact that the crystalline elements are similarly oriented throughout considerable areas of at least one of the two constituents of the alloy. This suggests that the alloy as a whole has comparatively coarse granular structure, and the same conclusion is borne out by observing the general character of a surface under lower magnifications (such as 100 diameters), when its structure is revealed either by straining or etching. The surface is then seen to be divided into rather large more or less polygonal areas, each covered with a system of ribs radiating from one point, giving an appearance which resembles roughly the ribs of an umbrella. Fig. 47 incidentally shows the boundary of two such areas.

In the course of experiments on the lead-bismuth eutectic, specimens were obtained showing comparatively large isolated crystallites. When the piece was strained these crystallites were found to exhibit slip-bands. Examples are given in figs. 50 and 51. These are interesting as showing the development of slip-bands in bodies which are evidently fully developed crystals, even as to external form.

A study of the micro-structure of alloys suggests a possible explanation of the peculiarities they present in regard to variation of electrical conductivity with temperature. The two constituents may behave individually as pure metals in this respect, but if their coefficients of expansion are different the closeness of the joints between them will depend on the temperature. Thus, if the more expansible metal exists as plates or separate pieces of any form within the other, the effect of heating will be to make the joints between the two conduct more readily, with the result of reducing the increase of resistance to which heating would otherwise give rise, and in extreme cases with the effect even of producing a negative temperature coefficient.

Reviewing the general results of the experiments, we consider that they establish the view that the structure of metals in general is crystalline, and remains crystalline when the form of the metal is altered by strain, plastic yielding being due to slips on cleavage or gliding planes within each individual crystalline grain, and partly (in some metals) to the production of twin crystals. In a pure metal, when straining is carried far enough to produce fracture, the crystalline grains suffer cleavage, and the cleavage surfaces thus developed give to the fracture its characteristically crystalline appearance. In impure metals fracture may occur through the parting of grains from one another at their boundaries. In both cases, however, the plastic yielding which precedes fracture takes place by slips in the manner we have described.

In conclusion we should like to express our indebtedness to Sir W. ROBERTS-AUSTEN, Mr. T. ANDREWS, and Professor ARNOLD, for giving us at the outset of our work the benefit of their large experience in preparing specimens of metals for microscopic examination. Messrs. HEYCOCK and NEVILLE and Mr. A. HUTCHINSON have assisted us materially by various suggestions, and by supplying specimens for examination. We have also to thank Mr. ANDREWS, Mr. STEAD, Mr. HADFIELD, Professor HICKS, and Messrs. EDGAR ALLEN and Co. for special specimens of iron.

The work described in this paper was carried out in the Engineering Laboratory at Cambridge.

To facilitate reference to the illustrations an index is added in which brief particulars are given of the subject of each photograph.

Explanation of Figures on the Plates.

Plates 15–28.

PLATE 15.

Fig. 2. Nearly pure commercial iron (transformer plate), polished and etched. Magnified 200 diameters, vertical illumination.

Fig. 3. Photograph of bismuth crystals, from a specimen in the Cambridge University Mineralogical Museum. $\frac{4}{5}$ths of full size.

Fig. 4. Facets on a crystalline grain of iron, produced by etching. These facets give rise to differences of brightness when the grains are seen under oblique illumination. Magnified 1500 diameters, vertical light.

PLATE 16.

Fig. 5. Etched pits in Swedish iron. 1000 diameters, vertical light.

Fig. 6. Surface of cadmium cast against glass, showing crystalline boundaries emphasized by air-channels. 100 diameters, vertical light.

Fig. 7. Air-pits on a glass-cast surface of cadmium. 1000 diameters, vertical light. Pits are shown on three crystals, having a different shape and orientation on each.

PLATE 17.

Fig. 8. Air-pits on a glass-cast surface of cadmium. 1000 diameters, vertical light. The pits are surrounded by halos due to the absorption of smaller bubbles by the pits.

Fig. 9. Similar pits to those in figs. 7 and 8, but more highly magnified—4200 diameters. The photograph shows that the pits have geometrical inner faces.

Fig. 10. Air-pits in glass-cast tin. Magnified 1000 diameters, vertical light. The irregularly oriented black patches are crystals of an impurity.

Fig. 12. Slip-bands in Swedish iron strained by tension. 400 diameters, vertical light. The photograph shows a feature that is frequently observed, viz., a tendency in the lines to become curved near the inter-crystalline boundaries, suggesting the existence of keyed steps at the boundaries.

PLATE 18.

Figs. 13 and 14. Two views of the same crystalline grains in iron before and after straining. Magnified 200 diameters, vertical light. Close comparison of the two will show the amount by which the crystalline grains have been extended.

Fig. 16. Strained Swedish iron. Magnified 300 diameters, vertical light.

Fig. 17. The same field as in fig. 16, with the same magnification, but seen under oblique light. The slip-bands on a few grains only are picked out as bright bands.

PLATE 19.

Fig. 18. Slip-bands developed on a glass-cast surface of lead when strained by bending. Magnified 100 diameters, vertical light.

Fig. 19. The same field as in fig. 18, seen under oblique light.

PLATES 20 AND 21.

Fig. 20. The same field as in figs. 18 and 19, seen under oblique light after the stage carrying the specimen had been turned through about 15°.

Fig. 21. Slip-bands in lead. 100 diameters, vertical light.

Fig. 22. The same field as in fig. 21, seen under oblique light.

Fig. 23. Slip-bands in lead, showing four intersecting systems. 600 diameters, vertical light.

Fig. 24. Slip-bands in lead. 1000 diameters, vertical light. The photograph shows the straight slips and stepped intersections characteristic of this metal.

PLATE 22.

Fig. 25. Slip-bands in silver. Magnified 750 diameters, vertical light.

Fig. 26. Geometrical air-pits in glass-cast cadmium, slightly strained to show slip-bands. 1000 diameters, vertical light.

Fig. 27. Ditto.

PLATE 23.

Fig. 28. Ditto. See also remarks on Figs. 7, 8, and 9.

Fig. 29. Geometrical etched pits and slip-bands, produced by slight straining, in iron. 750 diameters, vertical light.

Fig. 30. Etched pits and slip-bands in iron. 1000 diameters, vertical light.

Fig. 31. Polished and etched section of cold-rolled Swedish iron. 45 diameters, oblique light, showing the differences of brightness on various grains and the uniform brightness over each individual grain.

PLATES 24 AND 25.

Fig. 32. Distorted crystalline grains in a transverse section of cold-rolled Swedish iron, also showing geometrical etched pits. 200 diameters, vertical light.

Fig. 33. A portion of the large distorted grain in Fig. 32, more highly magnified; the enlarged portion is at the angle of the distorted grain. 800 diameters, vertical light.

Figs. 34, 35, and 36. Slip-bands in twin crystals of copper. 1000 diameters, vertical light.

Fig. 37. Slip-bands in twin crystals of gold. 200 diameters, vertical light.

Fig. 38. Slip-bands in twin crystals in gold. 45 diameters, oblique light.

Fig. 39. Slip-bands in twin crystals of cold-hammered copper. 1000 diameters, vertical light.

PLATE 26.

Fig. 40. Slip-bands in twin crystals observed in a specimen of plumbers' sheet-lead. 1000 diameters, vertical light.

Fig. 41. Slip-bands and twinning in nickel. 1000 diameters, vertical light.

Fig. 42. Elongated crystalline grains in cadmium cast on a sloping surface of glass. Transverse twin bands developed by slight straining are seen to run across a number of adjacent grains, with slight changes of direction on crossing a boundary. 100 diameters, vertical light.

Fig. 43. Glass-cast surface of cadmium, showing double system of boundaries. 100 diameters, vertical light.

PLATES 27 AND 28.

Fig. 44. High-power appearance of real and pseudo-boundaries as indicated by slip-bands. 1000 diameters, vertical light.

Fig. 45. Strained mild-steel, showing "pearlite" patches and slips in "ferrite" areas. 1000 diameters, vertical light.

Fig. 46. Glass-cast and etched surface of lead-tin eutectic. 750 diameters, vertical light.

Figs. 47 and 48. Glass-cast and strained (but unetched) lead-tin eutectic. 750 diameters, vertical light.

Fig. 49. Slowly-cooled lead-bismuth eutectic, etched and strained, showing slip-bands. 1000 diameters, vertical light.

Figs. 50 and 51. Slip-bands in crystallites found in lead-bismuth alloy. 1000 diameters, vertical light.

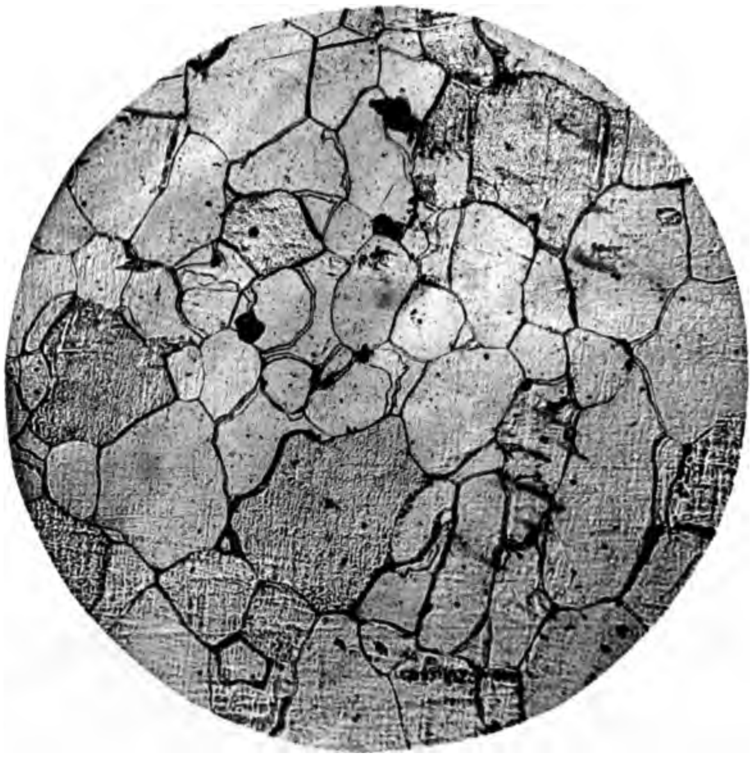


Fig. 2. Iron × 200.

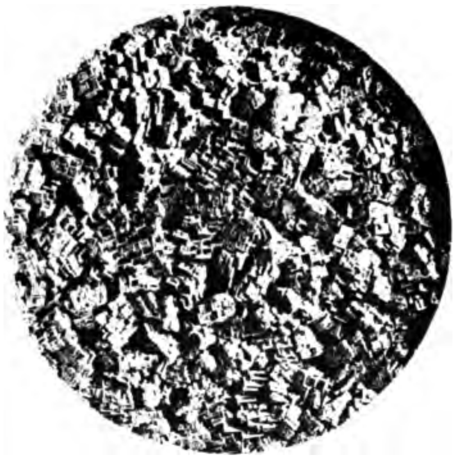


Fig. 3. Bismuth × $\frac{2}{3}$.

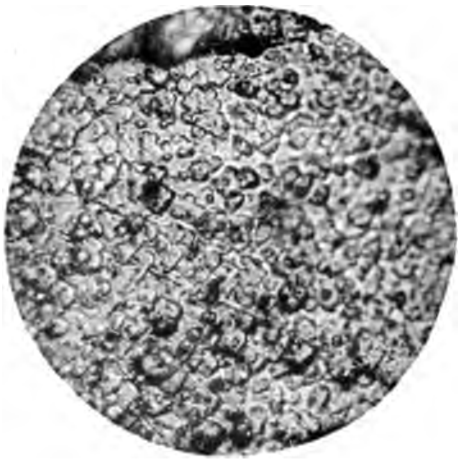


Fig. 4 Iron × 1500.

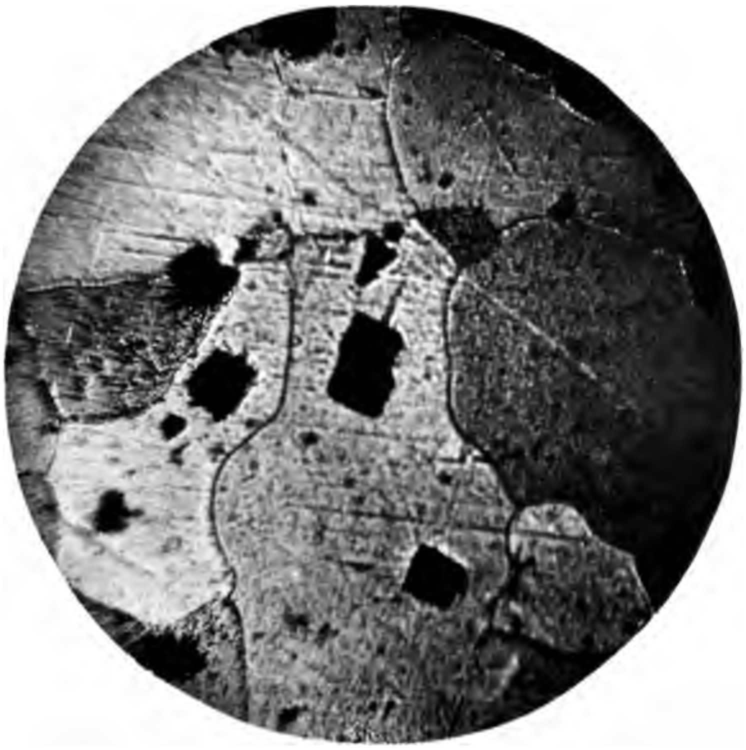


Fig. 5. · IRON × 1000.

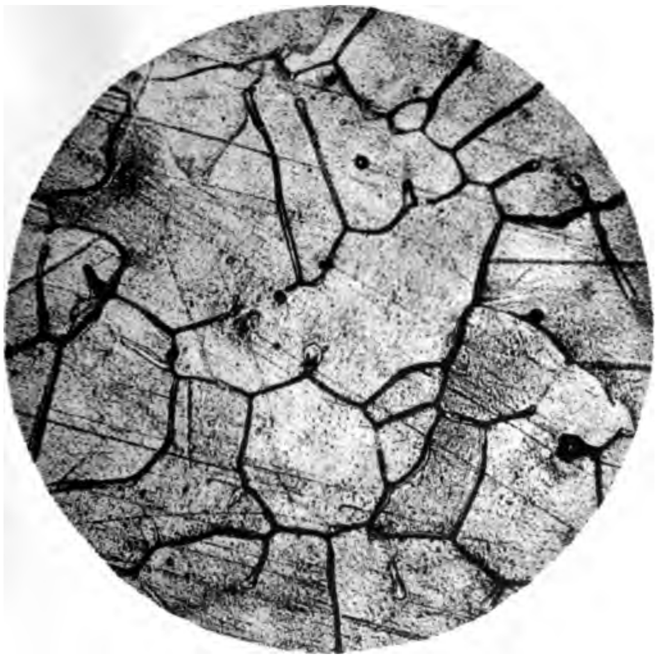


Fig. 6. CADMIUM × 100.

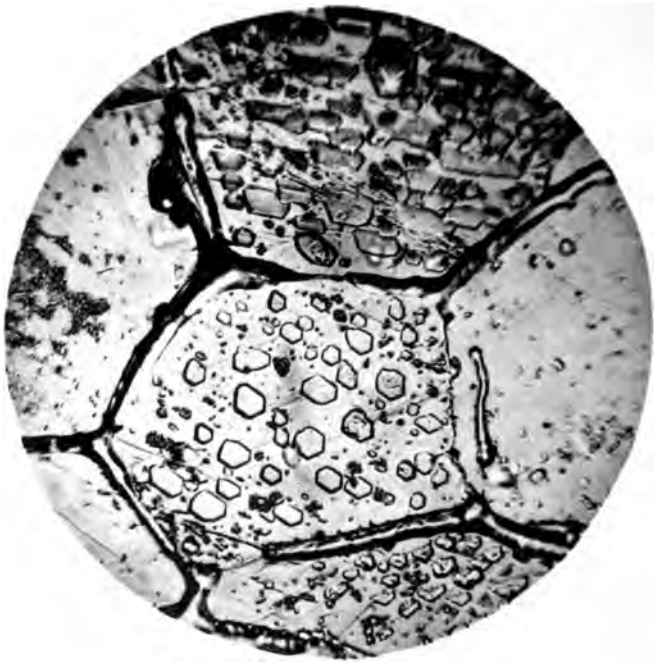
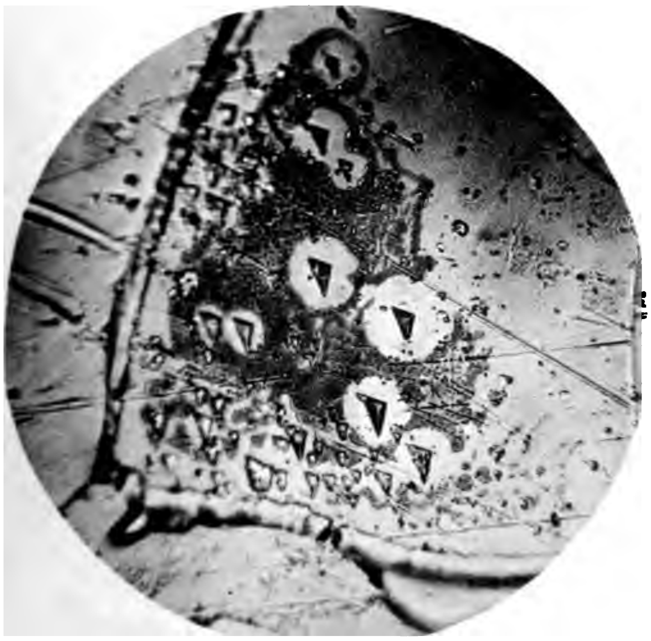


Fig. 7. CADMIUM × 1000



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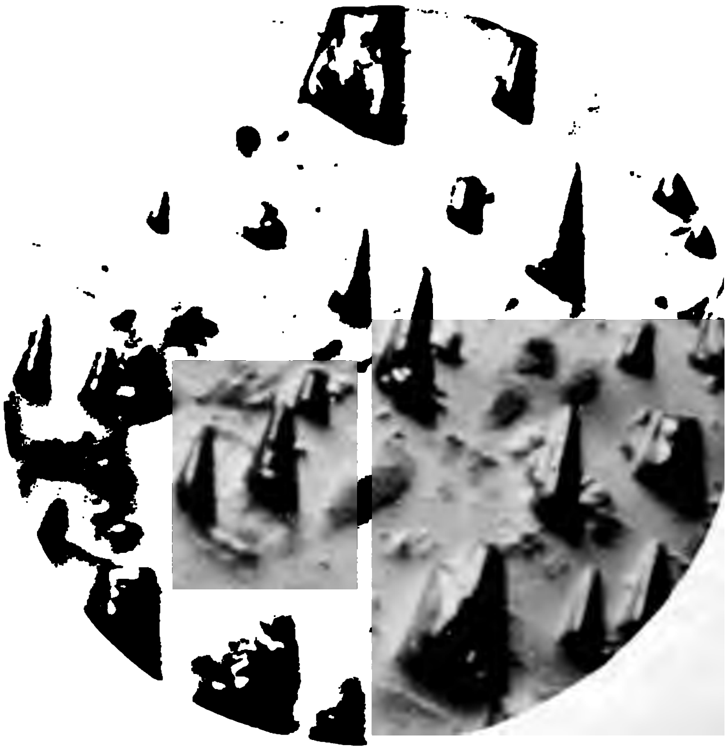

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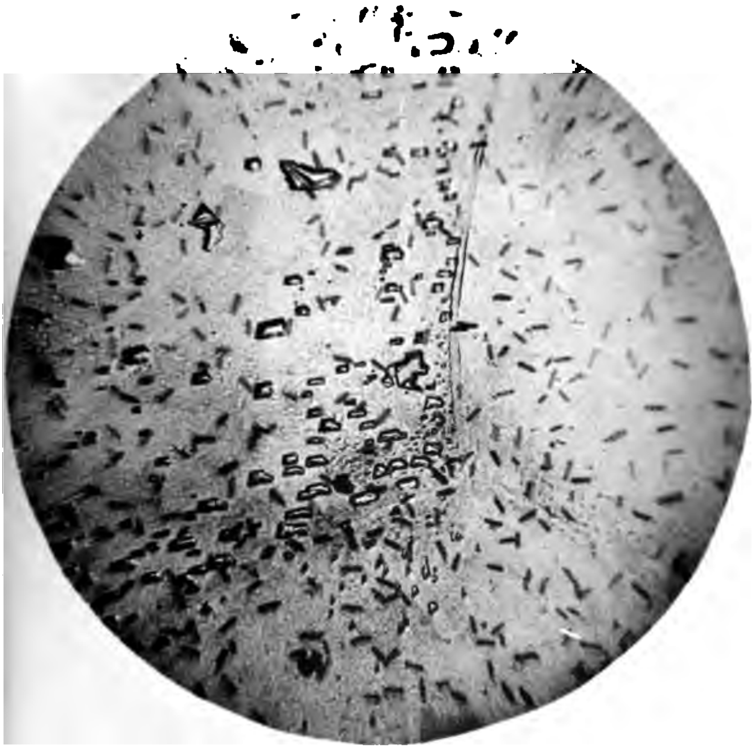

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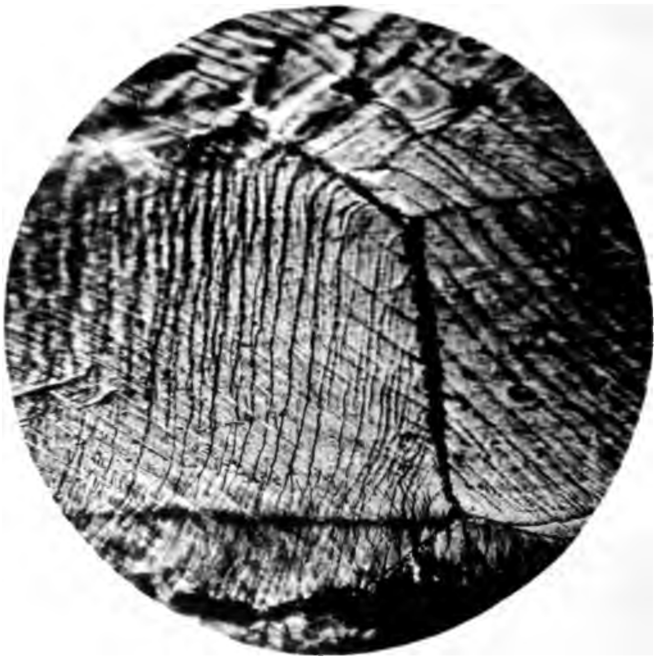

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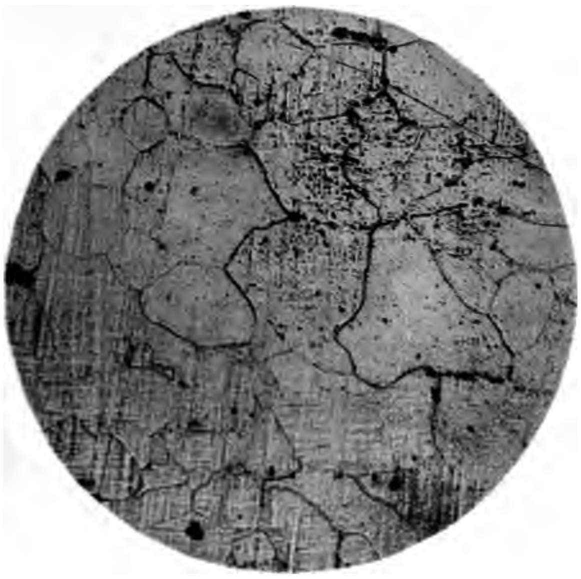
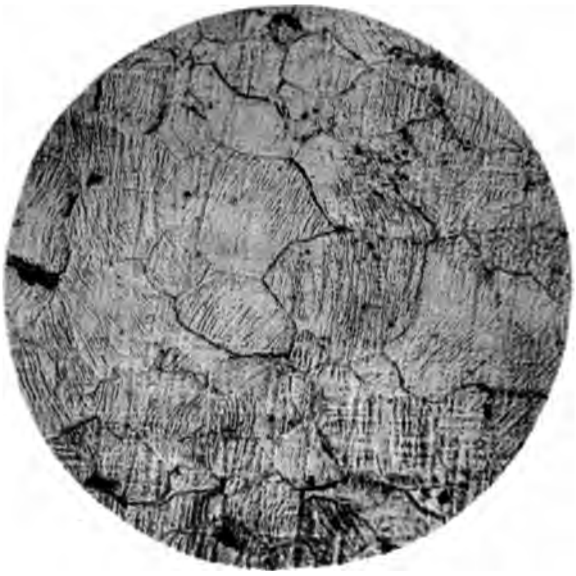


Fig. 13. IRON × 200.



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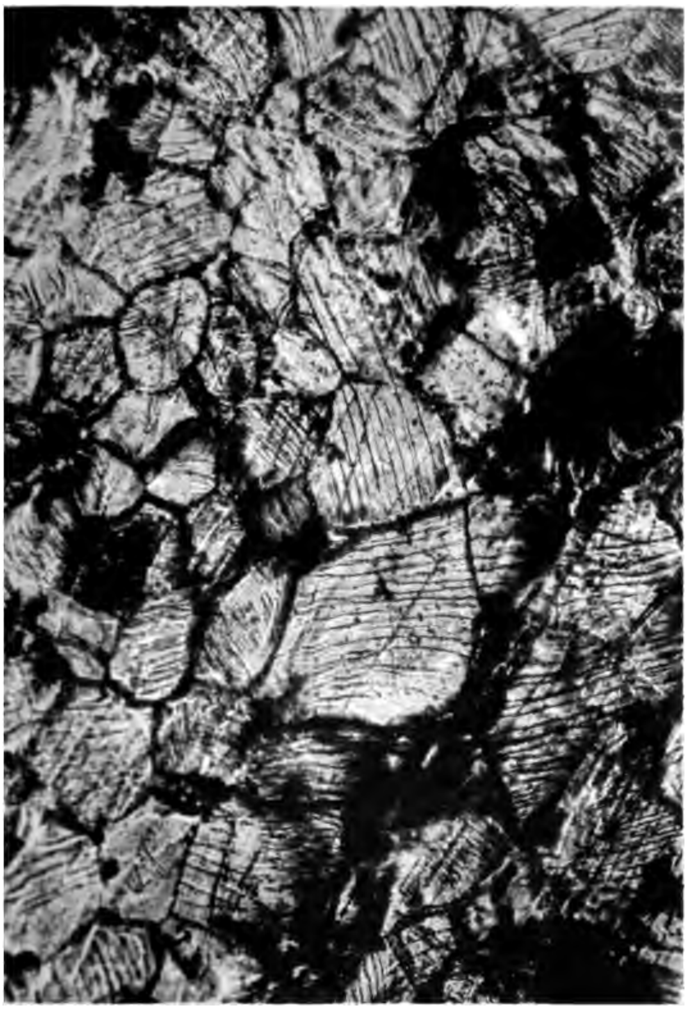

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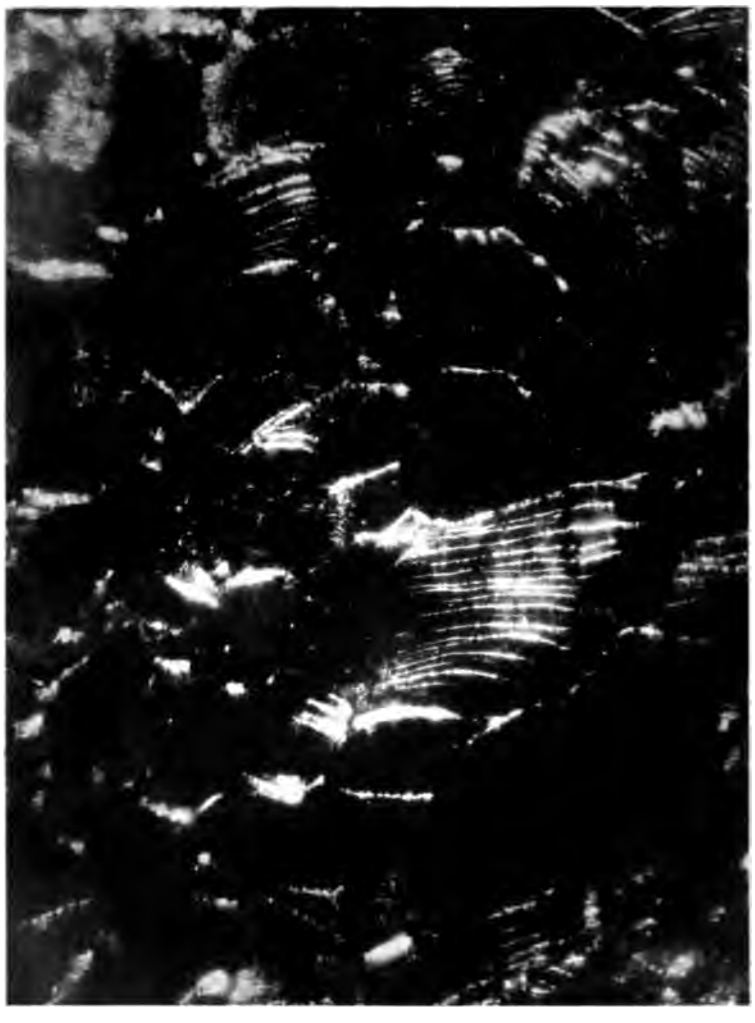

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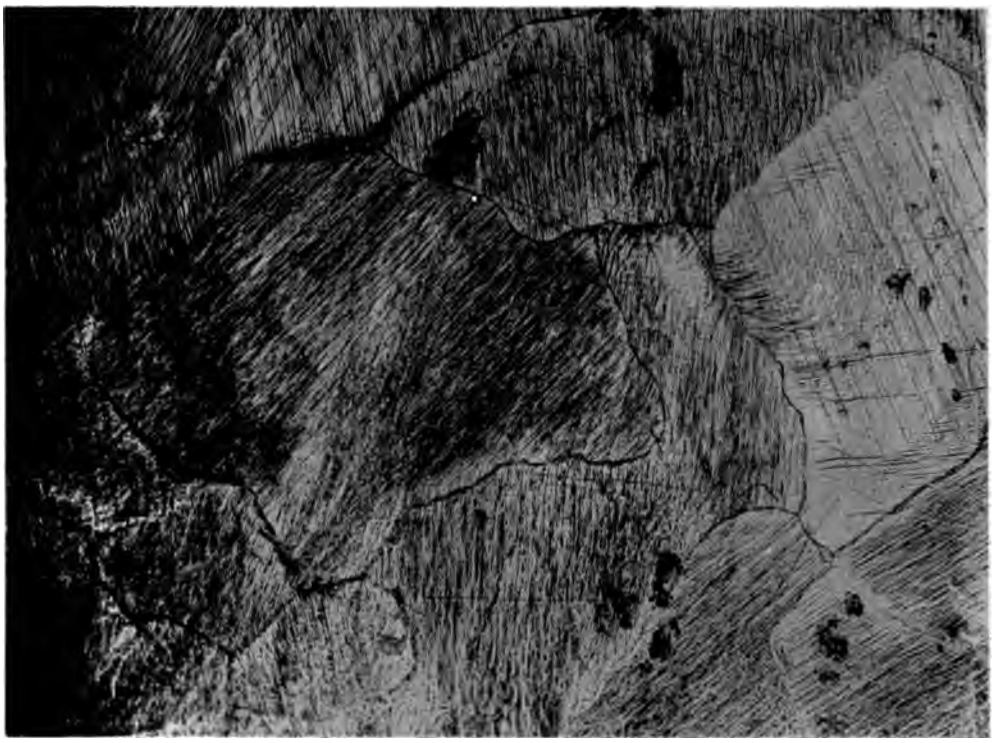
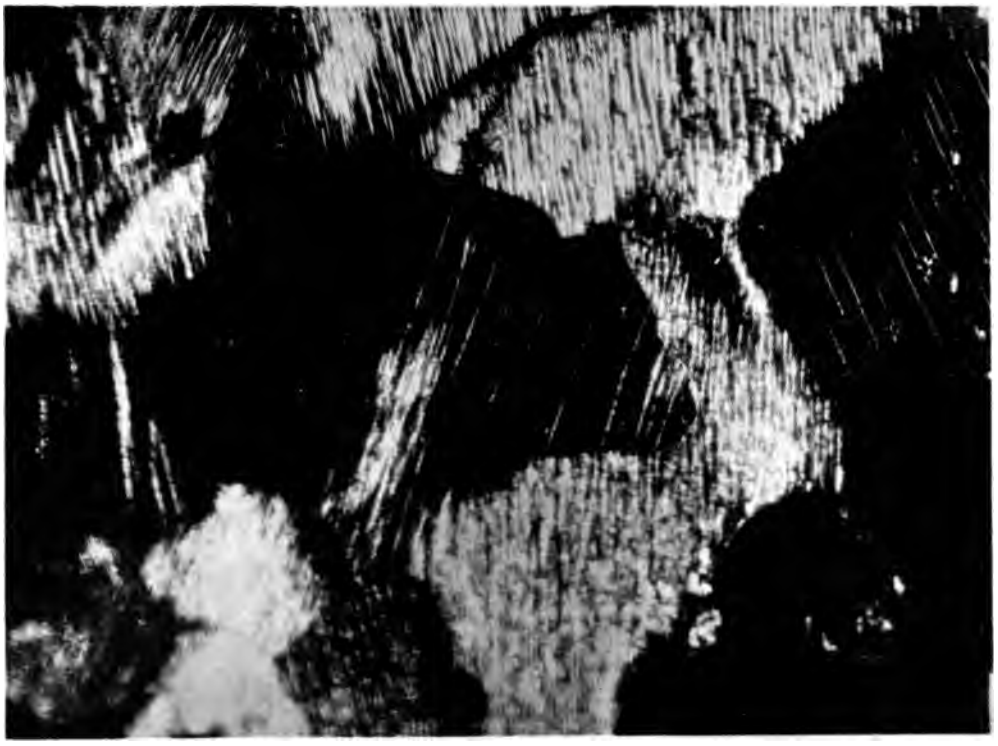


Fig. 18. Lead × 100.



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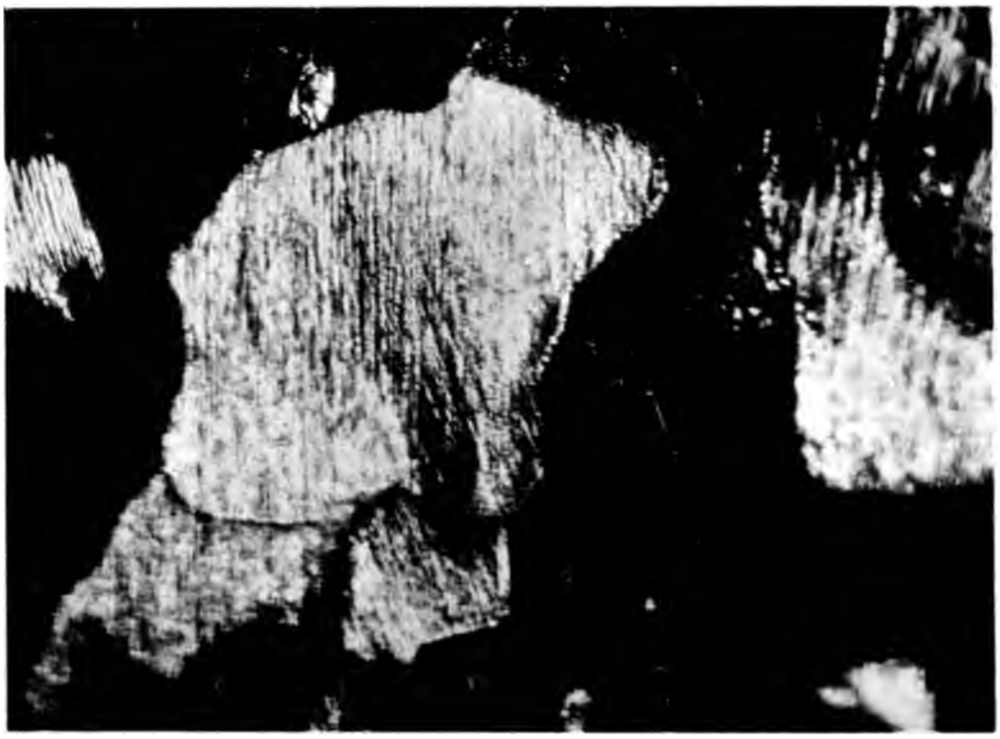

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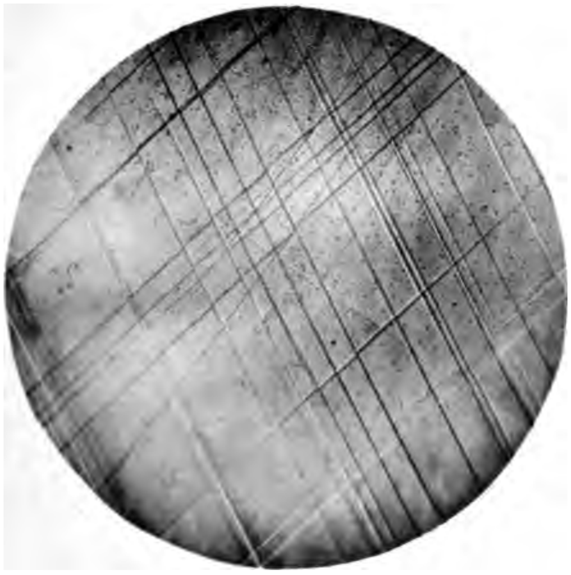

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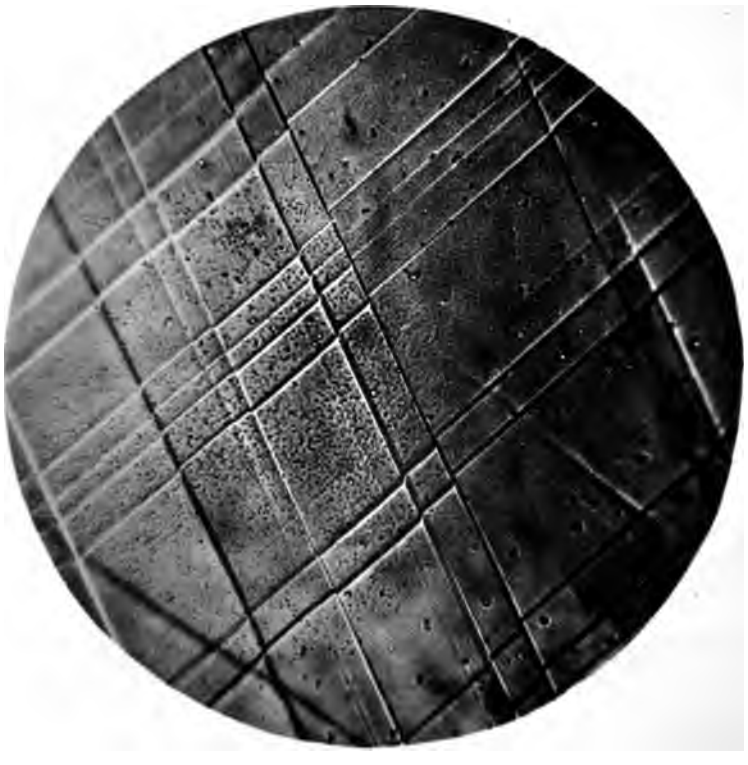

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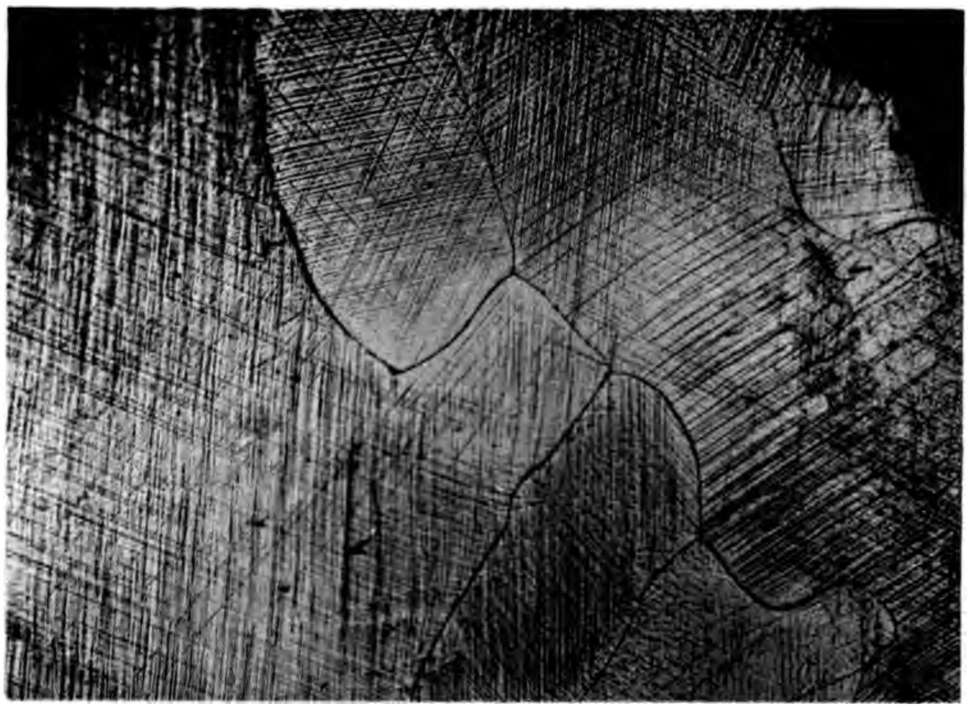

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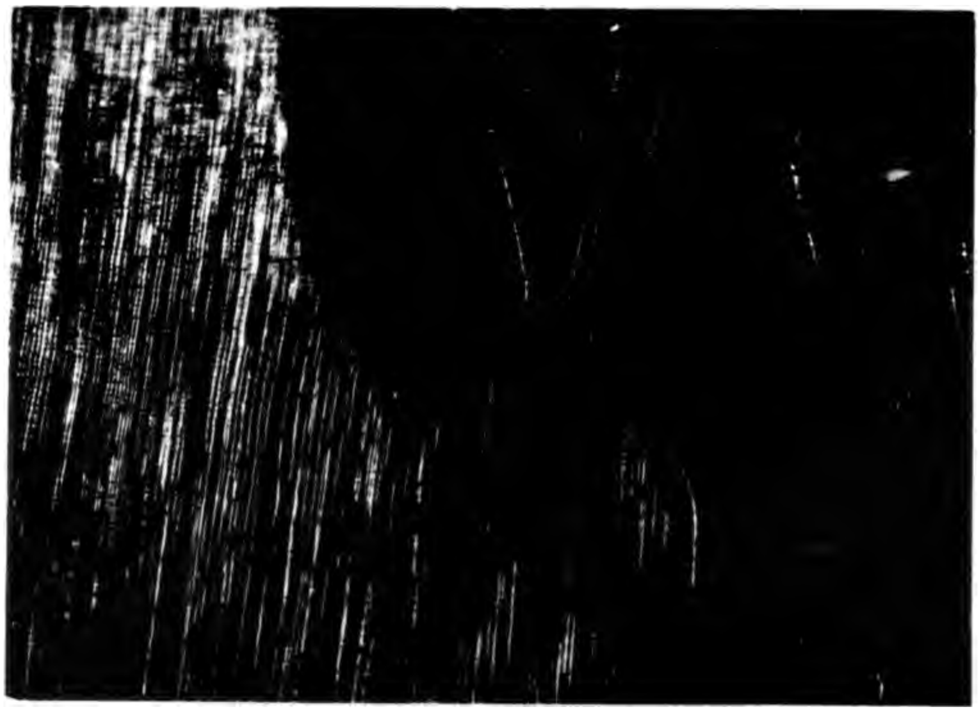


Fig. 22. LEAD × 100

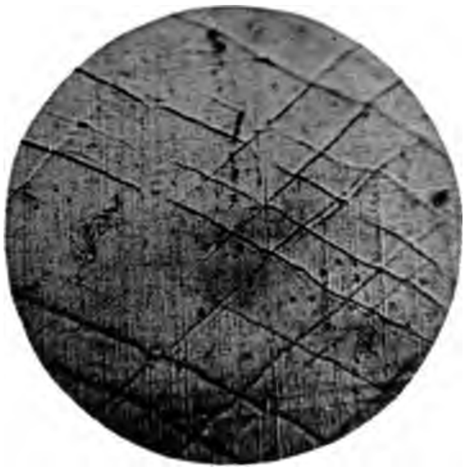


Fig. 25. Silver × 750.

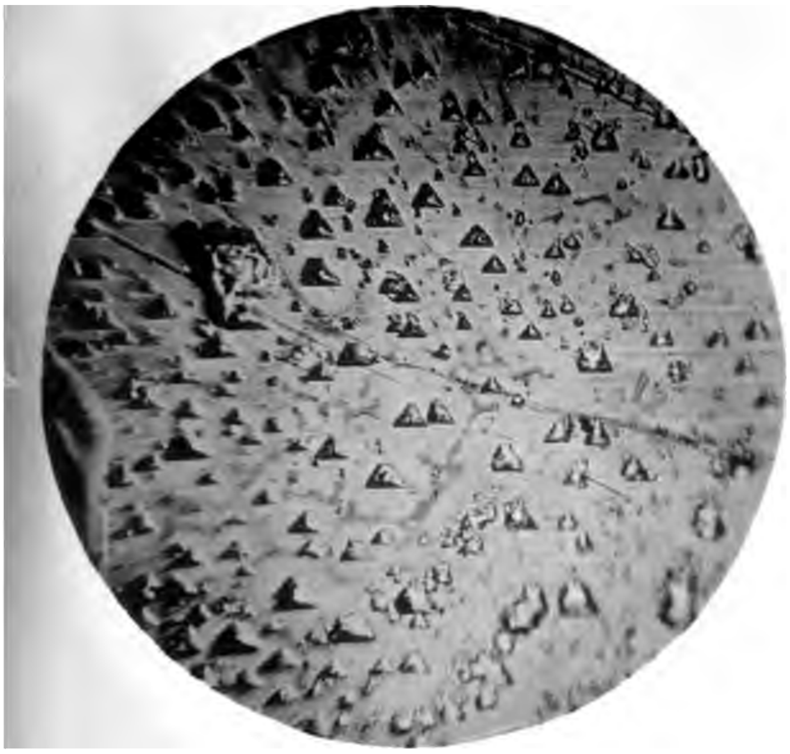


Fig. 26. Cadmium × 1000.

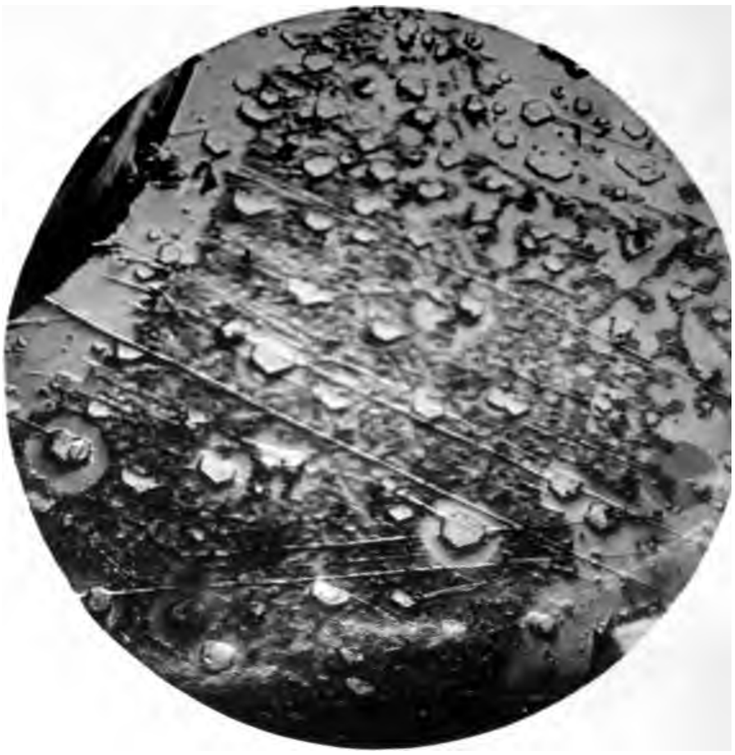


Fig. 27. Cadmium × 1000.

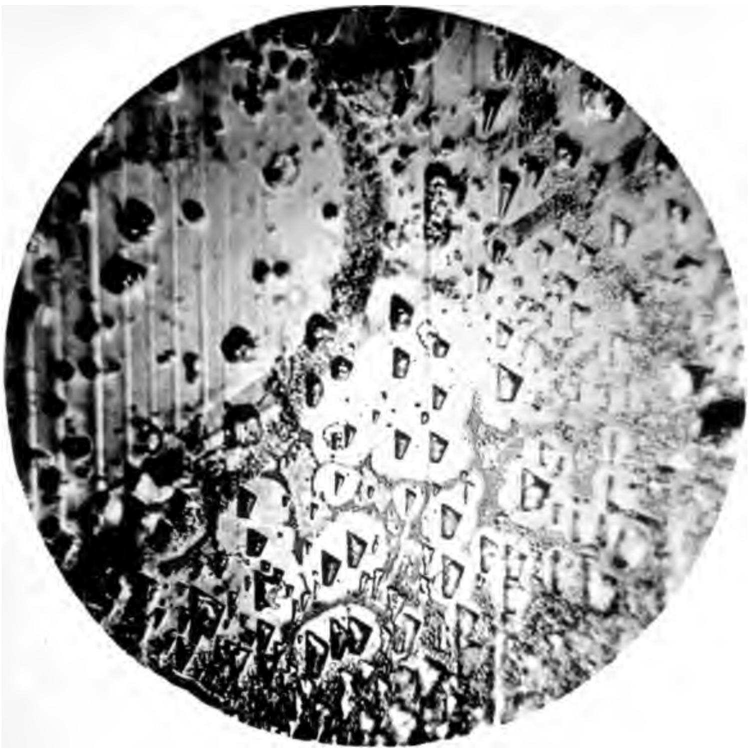


Fig. 28. Cadmium × 1000.

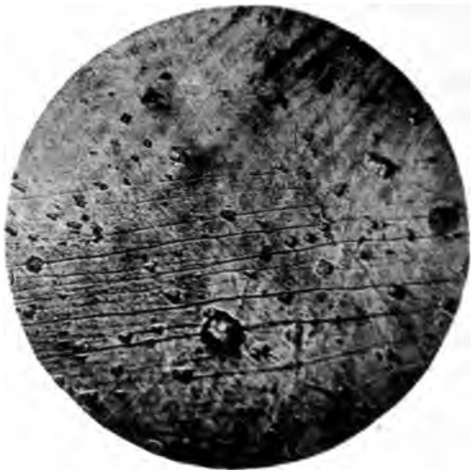


Fig. 29. Iron × 750.

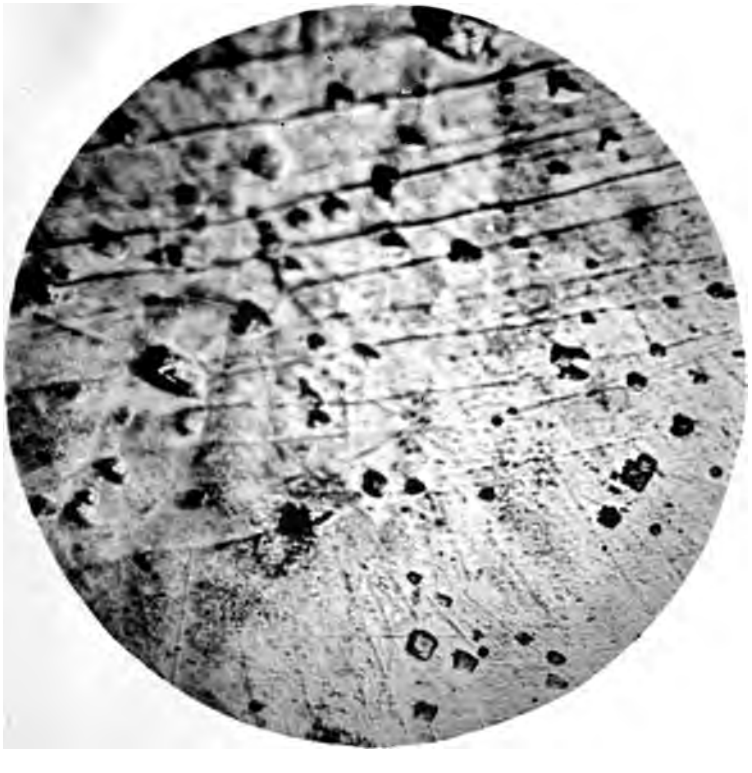


Fig. 30. Iron × 1000.

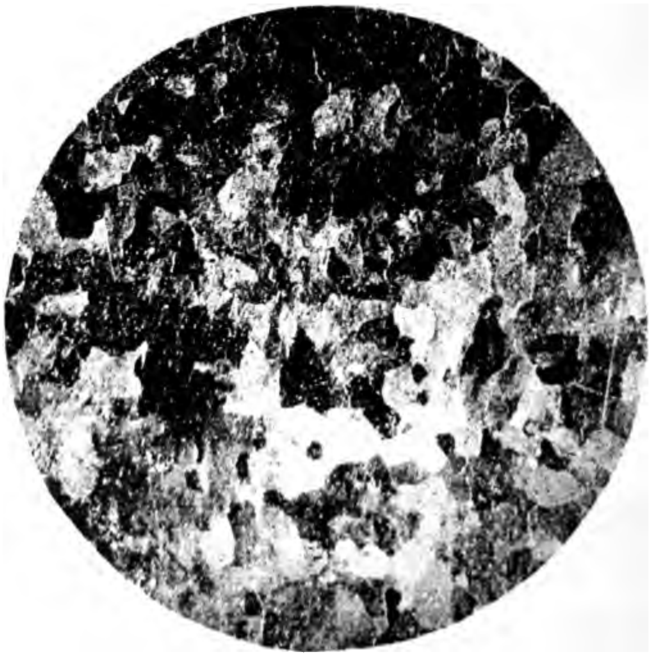


Fig. 31. Iron × 45.

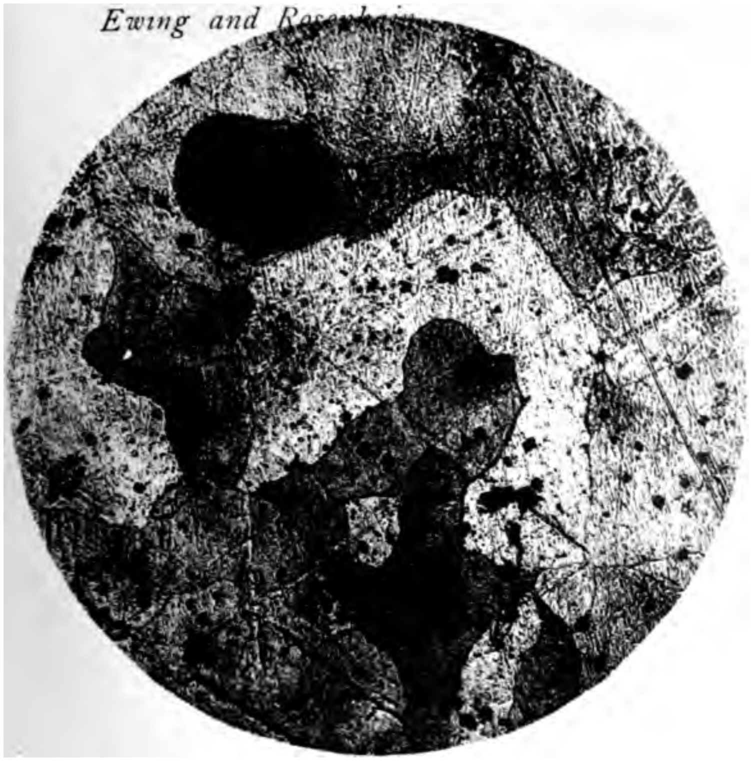
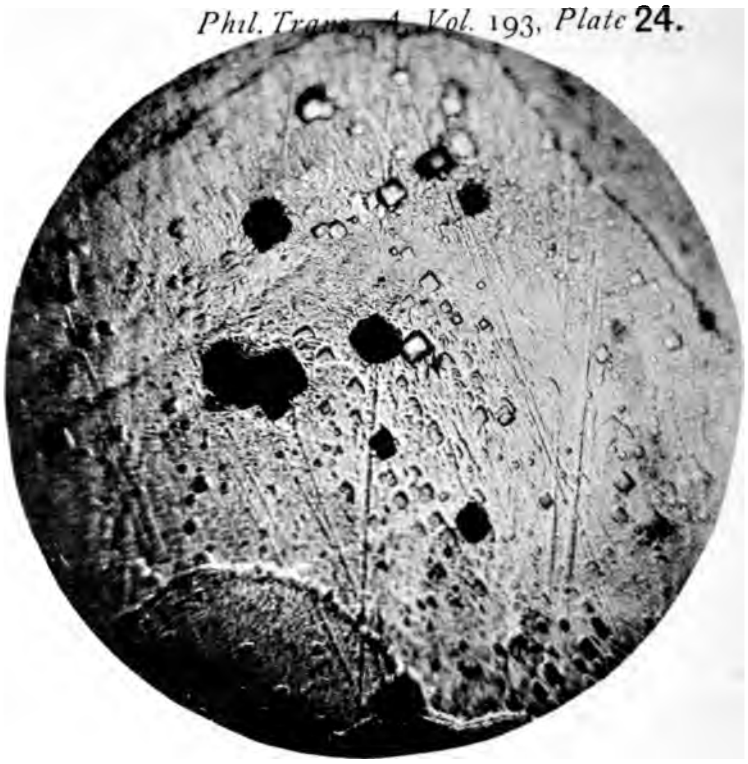


Fig. 32. IRON × 200.



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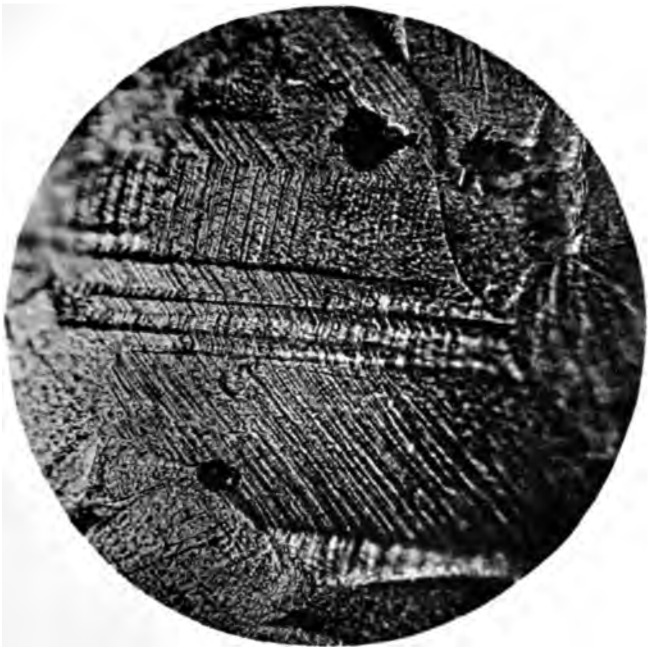
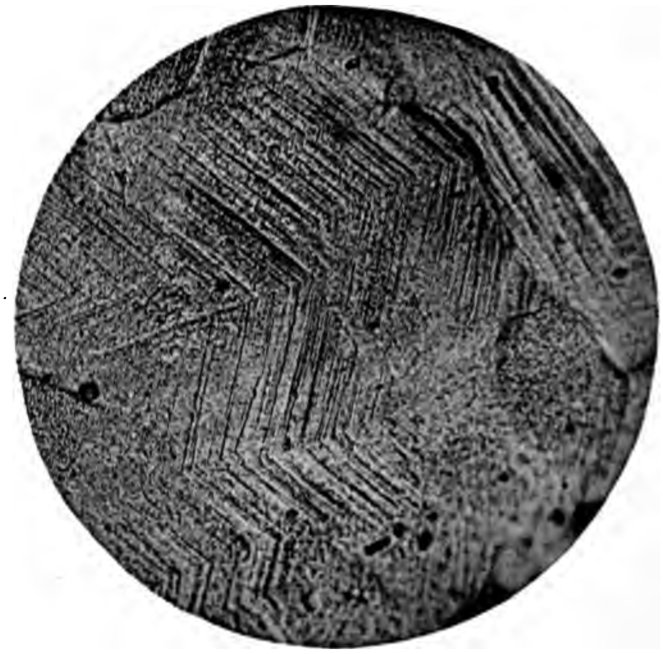


Fig. 34. COPPER × 1000.



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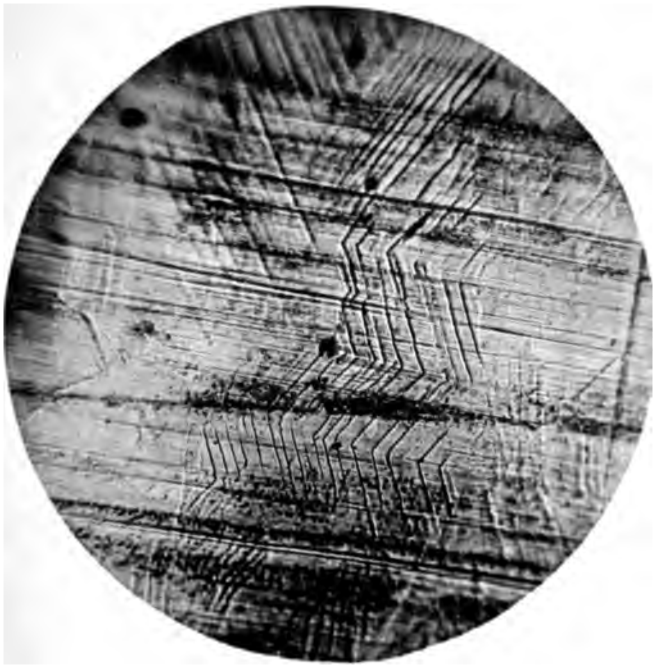

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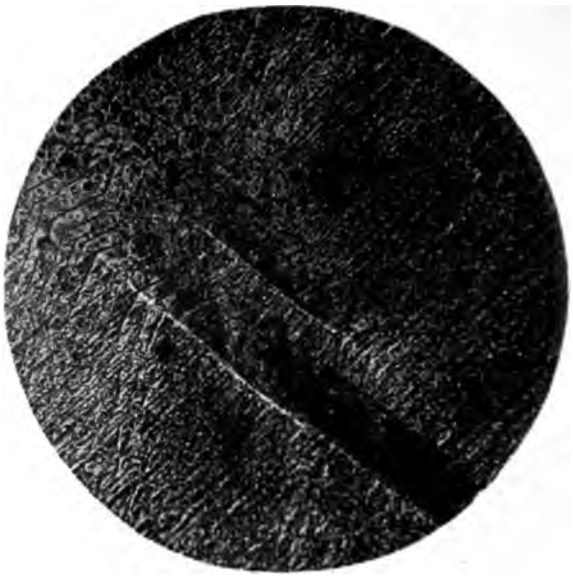


Fig. 37. GOLD × 200.

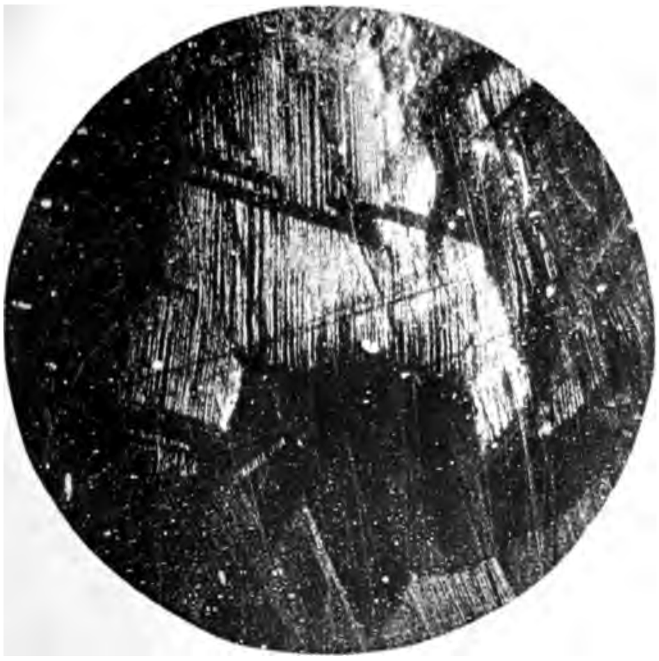
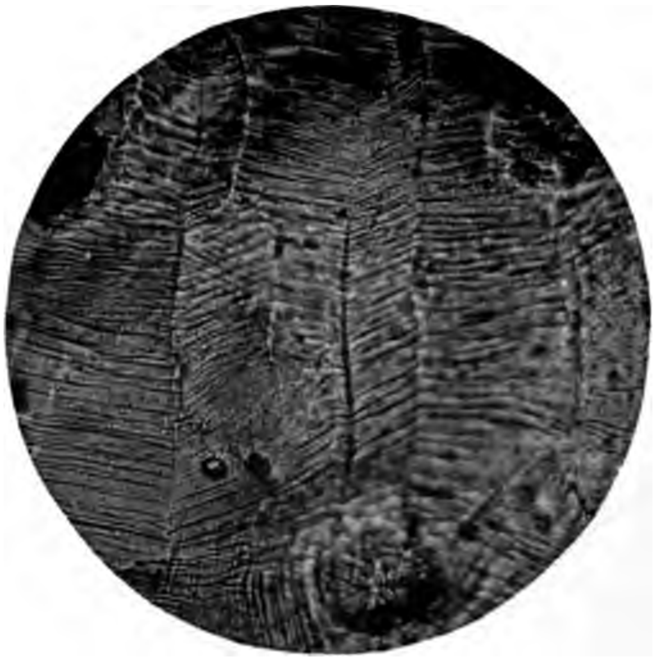


Fig. 38. GOLD × 45.



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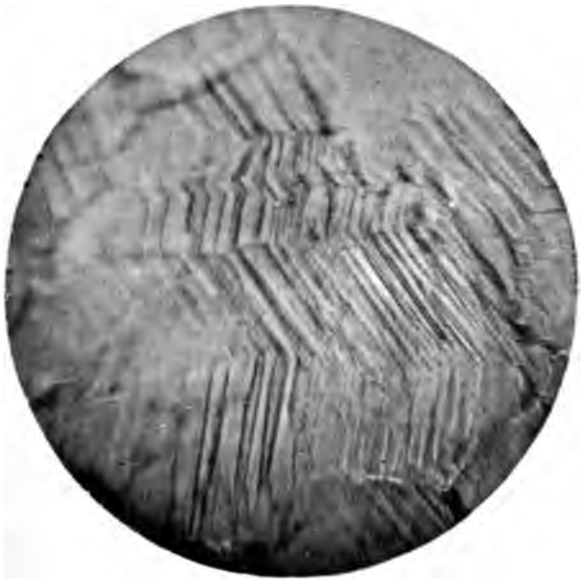


Fig. 40. LEAD × 1000.

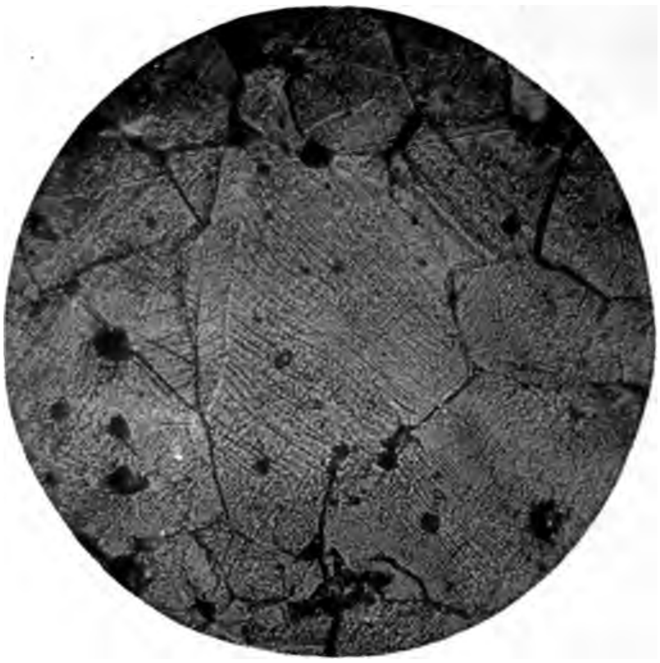


Fig. 41. NICKEL × 1000.

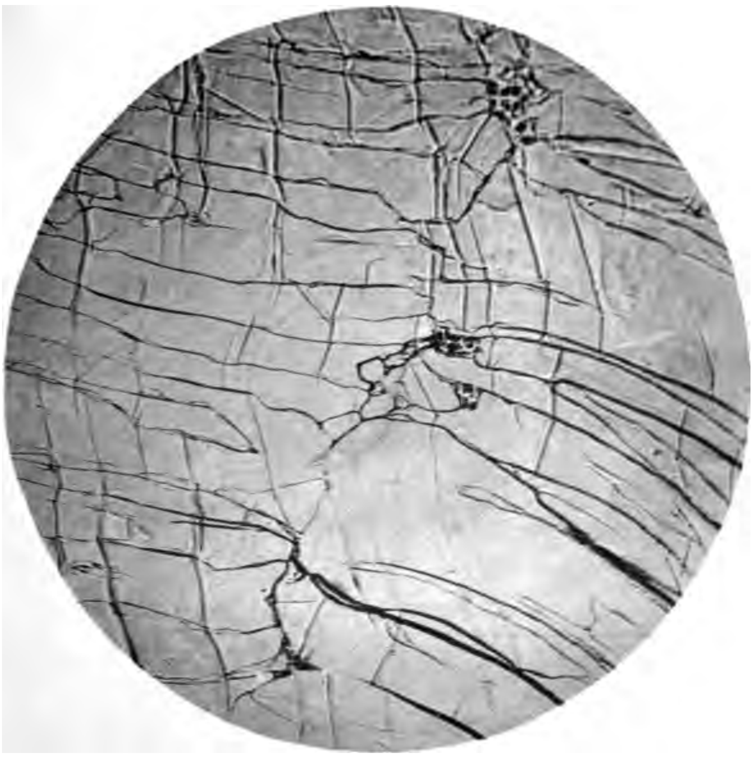
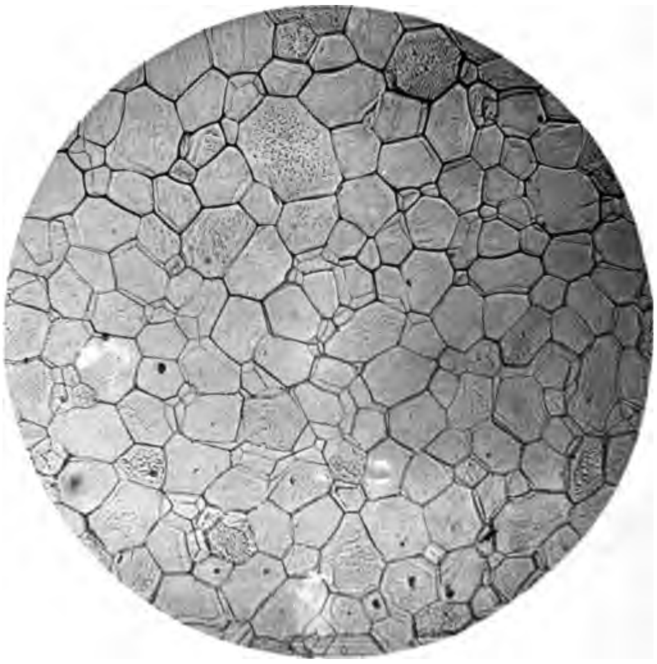


Fig. 42. CADMIUM × 100.



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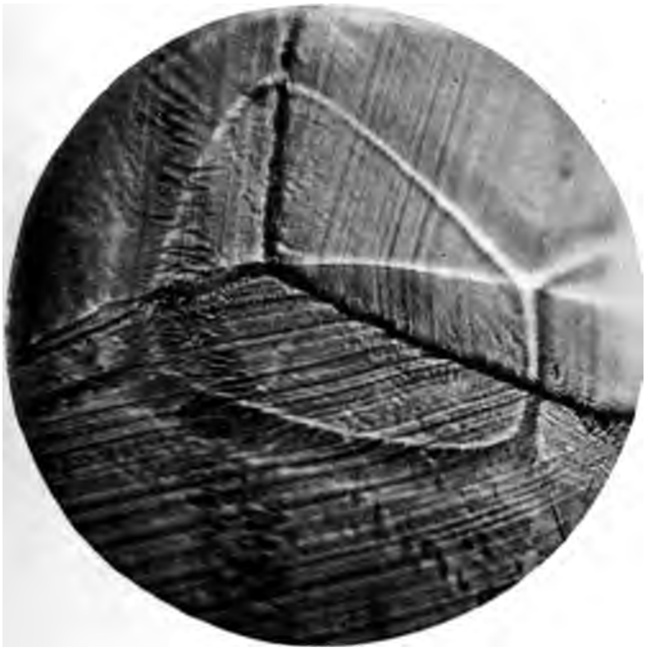

Fig. 44. CADMIUM × 1000.

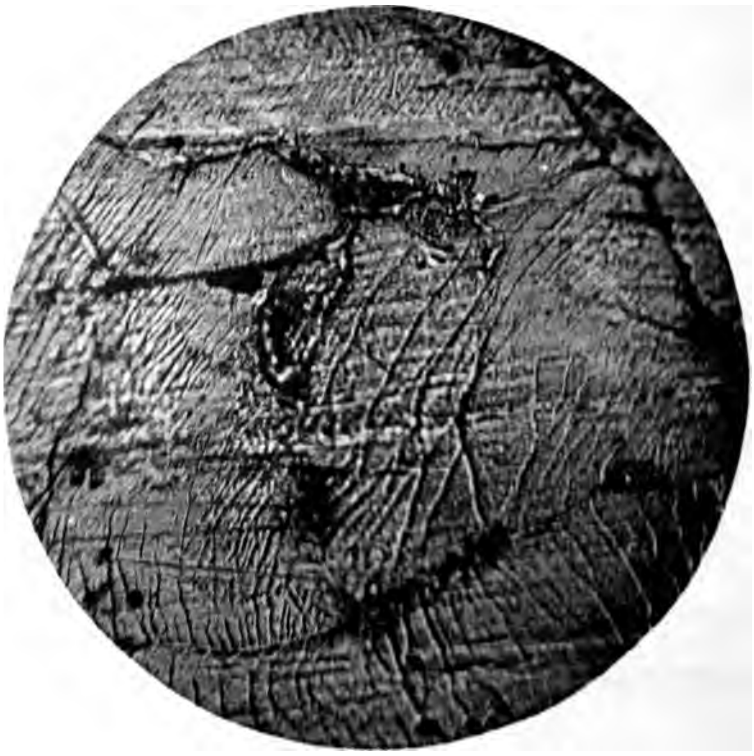

Fig. 45. MILD STEEL × 1000.

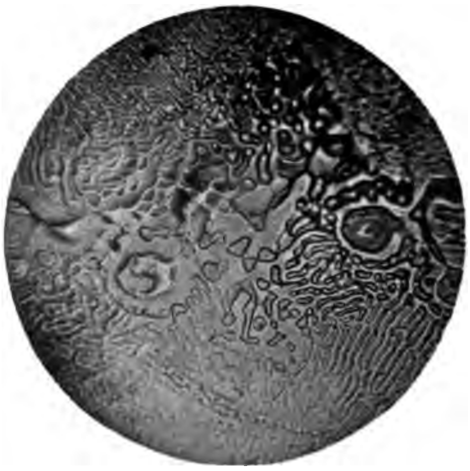

Fig. 46. LEAD-TIN EUTECTIC × 750.

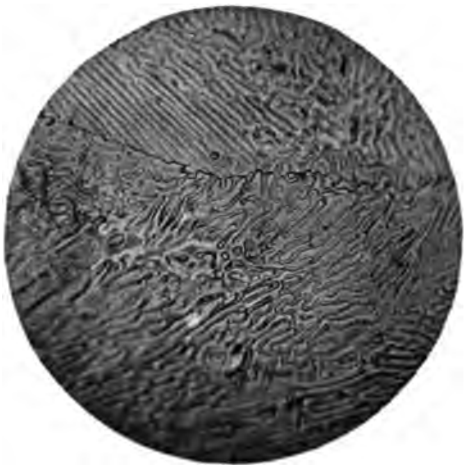

Fig. 47. LEAD-TIN EUTECTIC × 750.

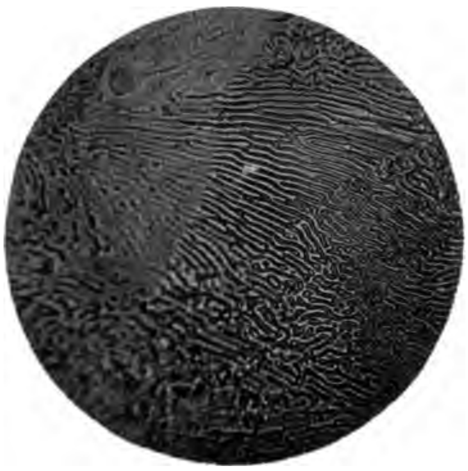

Fig. 48. Lead-Tin Eutectic × 750.

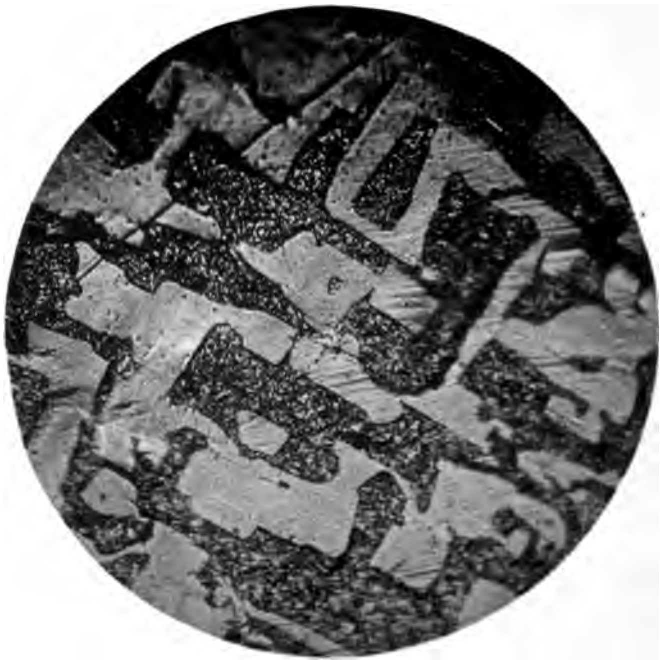

Fig. 49. Lead-Bismuth Eutectic × 1000.

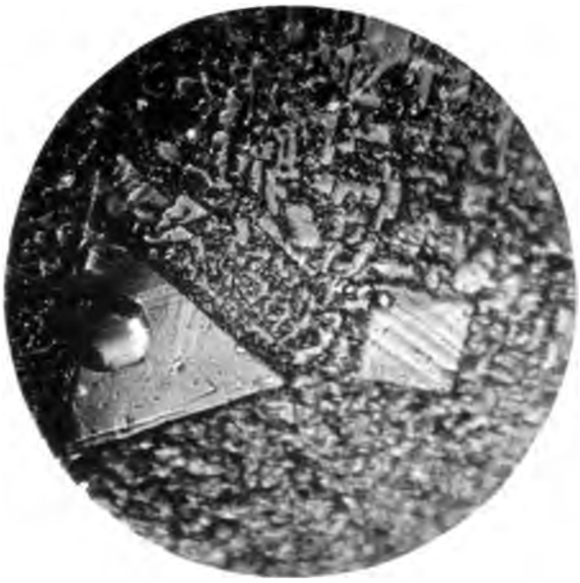

Fig. 50. Lead-Bismuth Alloy × 1000.

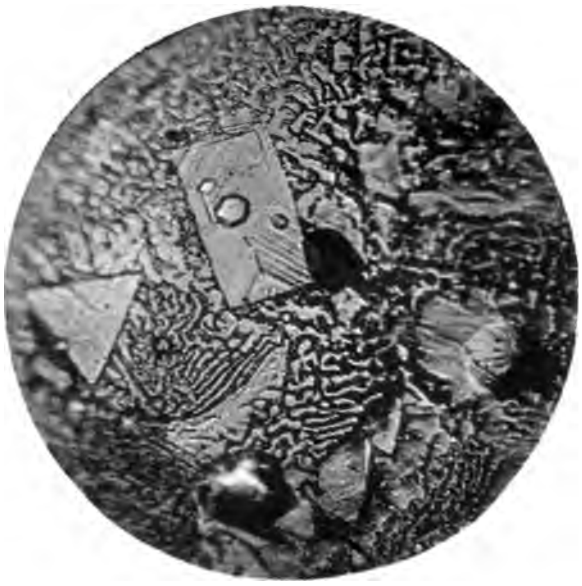

Fig. 51. Lead-Bismuth Alloy × 1000.

Discharge through

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XII. *On the Least Potential Difference Required to Produce Discharge through Various Gases.*

By the Hon. R. J. Strutt, *B.A., Scholar of Trinity College, Cambridge.*

Communicated by Lord Rayleigh, *F.R.S.*

Received October 17,—Read November 16, 1899.

Introduction.

Many experimenters have made observations on the potential difference necessary to produce electric sparks through gases. The field is a very wide one, since the number of circumstances which may be varied is large. The nature and the pressure of the gas, the shape of the electrodes, the distance between them, and the pressure of the gas, may each be altered. The investigation of which I wish to give an account in this paper deals with the potential difference required to produce sparks (or *striking potential* as I shall call it for brevity) in various gases, between large parallel planes, at a fixed distance apart, and at various pressures.

It was found by Mr. Peace ('Roy. Soc. Proc.,' vol. 52, p. 99) that the striking potential between two parallel plates in air diminished as the pressure diminished till a certain point was reached, and then began to rise very rapidly. The pressure at which the striking potential was a minimum depended on the distance between the plates, and increased as the distance was lessened. The minimum potential itself, however, varied very little with the distance between the plates.

This minimum potential was of the same order as the cathode fall of potential in air, as has been pointed out by Professor J. J. Thomson ('Recent Researches in Electricity and Magnetism,' p. 158). The following explanation may be offered of the fact that this is a minimum striking potential, and that it is approximately constant.

The negative glow in any gas [as has been shown by Warburg ('Wied. Ann.,' 31, p. 579)] requires for its production a definite potential difference (about 340 volts in the case of air), independent of the pressure, and constant, so long as the glow is not crushed into a smaller space than that which it would naturally occupy. If the glow is crushed, the potential fall is more. Let us now suppose that the discharge takes place between two parallel plates; a part of the space between these plates is occupied by the negative glow, a part by the positive column. So long as any of the positive column remains it is clear that the negative glow is not constricted, and consequently it only requires 340 volts to produce it. The greater the length of the positive column the greater the corresponding potential difference; so that the striking potential will be the least possible when the pressure is low enough to make the negative glow occupy the whole space between the plates, but not low enough to make it require more space.

Although Mr. PEACE found the minimum striking potential to vary very little with the distance between the plates, and consequently very little with the pressure, yet the variation with the pressure was much greater than that observed by WARBURG for the cathode fall. WARBURG gives a table showing that a tenfold diminution in the pressure does not alter the cathode fall by 1 volt, *i.e.*, by ½ per cent.

Mr. PEACE gives data showing a rise of 67 volts (20 per cent.) in the minimum striking potential.

It will thus be seen that, though there are theoretical reasons for thinking that the minimum striking potential should be equal to the cathode fall, the experimental evidence hitherto produced is scarcely sufficient to establish this relation. My experiments have been made with a view to determining the relation. The results will be discussed after the experimental arrangements have been described.

Description of Experimental Arrangements.

It was necessary to design the apparatus so as to require as little of the gas as possible, since it was intended to make some experiments on helium as well as on the common gases. The two brass plates of 1½ inches diameter used as electrodes were embedded quite flush in ebonite plates 3 inches in diameter. The object of this was to prevent any tendency to sparking from the backs or edges of the brass discs. Three small ebonite distance pieces were placed between the brass plates, and the whole arrangement fastened together by means of three screws passing through the ebonite. Electrical connection to the brass plates was made by means of wires screwed into them, passing out axially through the ebonite.

The ebonite distance pieces were carefully measured before the apparatus was put together by means of a micrometer screw gauge. The lengths in millims. were :—

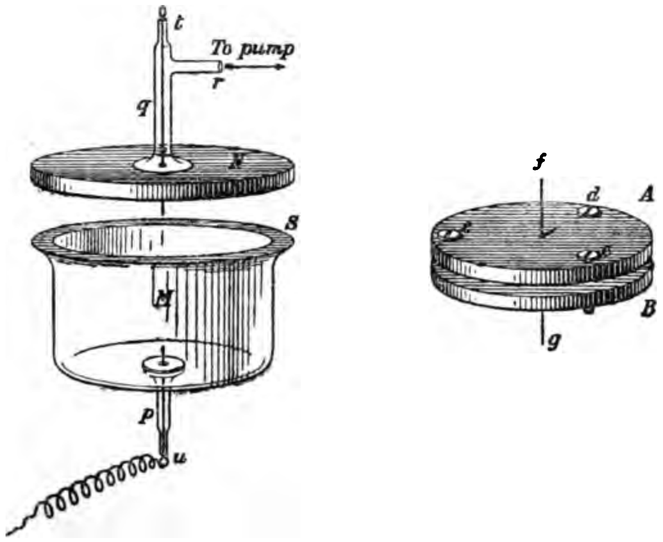
$$\text{No. 1}\begin{cases}0\cdot755\\0\cdot755\end{cases}\qquad\text{No. 2}\begin{cases}0\cdot755\\0\cdot752\end{cases}\qquad\text{No. 3}\begin{cases}0\cdot757\\0\cdot756\end{cases}$$

Mean 0·755 mm. = 0·0297 inch.

The brass plates being large compared with the distance between them, the electromotive intensity between them is sensibly uniform.

In order to be able to introduce various gases between the brass plates, and to vary the pressure, it was necessary to enclose them in an air-tight chamber, The arrangement is represented in fig. 1. A, B, are the ebonite discs, into the fronts of which the brass sparking plates are let. These latter are not visible in the figure. The ebonite plates are held together by the three screws *c*, *d*, *e*. The small distance between them is equal to the thickness of the distance pieces. The leading wires pass to the plates at *f* and *g*.

Fig. 1.

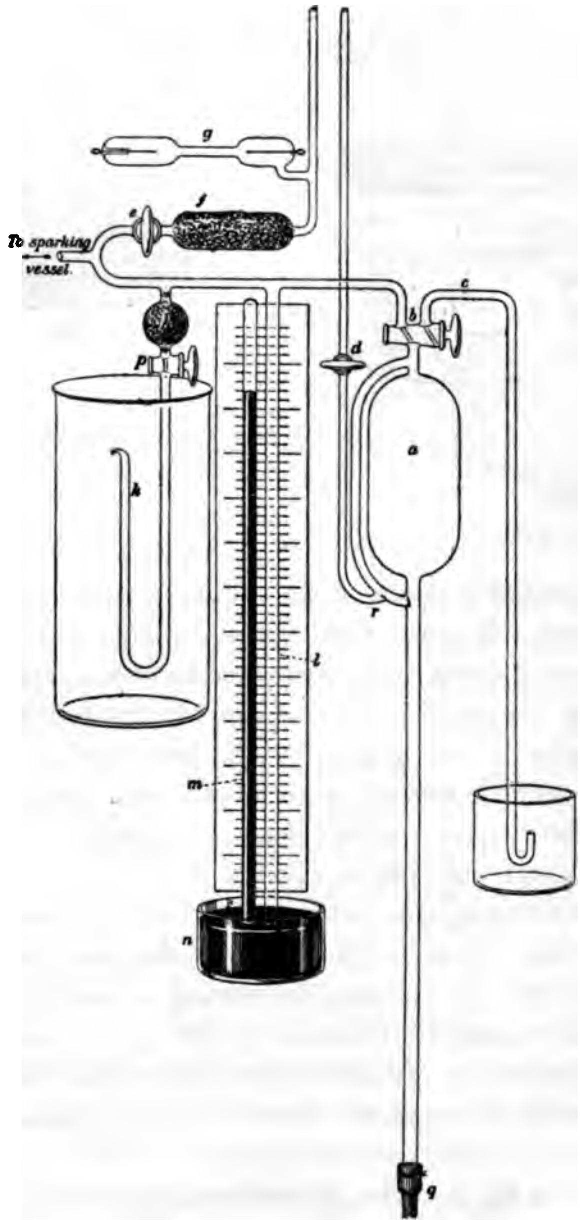


This whole arrangement is enclosed in the cylindrical glass pot M, shown separately for the sake of clearness. M has a broad ground flange S, which is covered by the glass lid N, the joint being made tight with sealing wax. The leads are introduced through holes drilled in the middle of the lid and the bottom of the pot. Over these holes are placed the glass tubes *p*, *q*, up which the wires pass. These tubes have flanges at the ends, ground flat so as to fit the flat surface to which they are cemented. A side tube *r* provides the means of admitting and withdrawing gas. The wires are fused through the tube at *t* and *u*.

In order conveniently to vary the pressure and adjust it to any desired value, it was necessary to provide a reservoir into which the gas could be drawn, or from which it could be expelled, by raising or lowering a mercury vessel. It was also necessary to provide a means of exhausting the apparatus. A special form of mercury pump was designed, by means of which both these objects could be attained. By thus dispensing with a separate reservoir, the apparatus was considerably simplified.

The pump is shown in fig. 2. The general arrangement of the various parts will, it is hoped, be sufficiently clear from the figure. *a* is the pump, which also serves for the adjustment of the pressure in the manner to be presently described. *g* is a vacuum tube, which serves for observing the spectrum of the gas as a test of its purity. *f* and *h* are phosphoric anhydride drying tubes. *d* and *e* and *p* are ordinary glass stopcocks. *b* is a three-way glass stopcock. *l* is the manometer tube. A barometer tube, *m*, is arranged by the side of it, and dips into the same mercury vessel *n*. The height of the mercury columns in these tubes can be read off to the $\frac{1}{10}$th of a millim. on the mirror glass scale behind them. The difference of these heights, of course, gives the pressure of gas in the apparatus.

Fig. 2.



k is a U-shaped tube immersed in a glass beaker, and serves for the introduction of gas into the apparatus. Before filling with gas, it is necessary to remove the air from the sparking vessel.

For this purpose stopcocks *d* and *e* are opened, and *b* is set so as to open communication between *a* and *c*. The mercury reservoir (not shown) attached to the india-rubber hose *q* is raised. Mercury rises in *a*, driving out the air through *c*, and on lowering the mercury, fresh air bubbles in at *r*, just as in the ordinary Toepler pump. When the apparatus has been exhausted, stopcocks *d* and *e* are shut. *b* is turned so as to open communication between *a* and *r*, and gas is admitted through *p* to any desired pressure. The pressure can then be varied at pleasure by raising or lowering the mercury in *a*. The mercury reservoir attached to the hose *q* can be fixed at any given height so as to make the mercury stand at a corresponding height in *a*. The object of shutting off the part between the stopcocks *d* and *e* is to make the volume in connection with *a* as small as possible, thus making it possible to obtain a greater range of pressure without admitting or removing gas permanently. Another advantage of this contrivance was that, when helium was to be used, less of it would be required. The volume required to fill the apparatus to atmospheric pressure was only about 40 cc., when *a* was entirely filled with mercury.

We now come to the arrangements for producing and measuring the striking potential. A large Wimshurst machine was used. It was driven at a constant speed by means of an electric motor. The potential difference between its terminals was measured by means of one of Lord KELVIN'S multicellular electrostatic voltmeters. This instrument has been checked by Mr. CAPSTICK. He compared it with a quadrant electrometer standardised by Clark cells, and found it practically correct. I have also verified it to some extent myself, by observing the *additional* deflection produced by 50 volts, this latter voltage being determined by a Weston voltmeter, believed to be trustworthy. No measurable discrepancy was detected.

Some previous experimenters have used a Wimshurst machine in direct connection with the spark terminal, and have turned it till the spark passed. This method succeeds fairly well when large spark lengths and, consequently, large spark potentials are to be measured. But when the spark potential is small it is scarcely possible to raise the potential slowly enough to prevent the needle of the instrument swinging violently; and if this occurs, it is of course impossible to make any accurate observations. PEACE (*loc. cit.*) failed to get consistent measurements with the Wimshurst machine. He was driven to the use of a battery of storage cells. The use of these, however, very much adds to the trouble of the investigation, and I was able to avoid it in the following manner. The machine was shunted by means of a fluid high resistance column of variable length. When running at a constant speed the difference of potential at the terminals of the resistance column was constant, and depended of course on the length of the resistance column in use. By gradually increasing the amount of this resistance the potential difference could be raised to any desired value and adjusted with the utmost nicety.

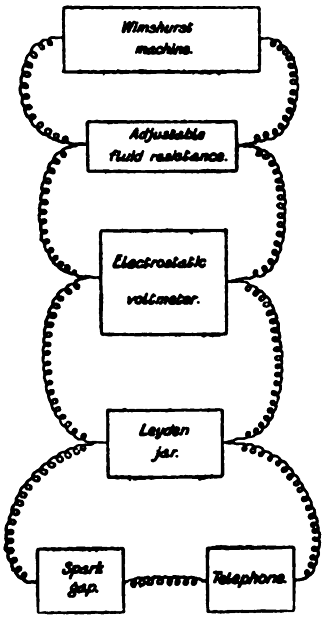
In order further to improve the electrical steadiness of the arrangement a large

Leyden jar was also connected across the terminals of the machine. The capacity thus added diminished the effect of any slight irregularities in its action.

It is not easy to observe the faint sparks which pass between the sparking plates unless the observer gives his undivided attention to watching for them, and if this be done it is impossible to read the potential difference at the exact moment when the spark takes place. A telephone was therefore inserted in series with the spark gap, and arranged in a clip so as to be against the observer's ear when his eye was in the proper position for reading the voltmeter. Whenever a spark passed, the telephone gave an easily audible click. By means of these arrangements the measurements could be taken with considerable rapidity and precision.

A diagram (fig. 3) of the electrical connections is given below.

Fig. 3.



It must be remembered that the first spark taken through a gas passes with far greater difficulty than those which succeed it. I have found a gas able to sustain for a short time a potential difference three times as great as that required to produce discharge through it when this initial resistance had been broken down. If the measurements of the spark potential are to be compared with the cathode fall measurements made while a continuous current was flowing through the gas, it is clear that they should be made with the gas in its electrically weakest condition, that is, immediately after it has been vigorously sparked through. This condition was complied with in my experiments.

In taking a measurement of the spark potential, the pressure of the gas was adjusted to the desired value and read off. The machine was started, and the resistance column lengthened until the potential rose sufficiently to cause the first spark to pass. This first spark was followed by a torrent of others, the telephone emitting a chattering noise. The resistance column was gradually shortened until these sparks ceased, and then cautiously lengthened until they just began again.

At the moment when this happened the voltmeter was read. The voltage was allowed to rise slowly enough to prevent any appreciable swing of the needle.

When an observation had been taken, the voltage was again reduced below the sparking value, and then again cautiously increased by lengthening the resistance column. When the telephone began to click the reading was again noted. Ten observations were usually taken at each pressure. They could be obtained in fairly rapid succession.

Experiments on Atmospheric Air.

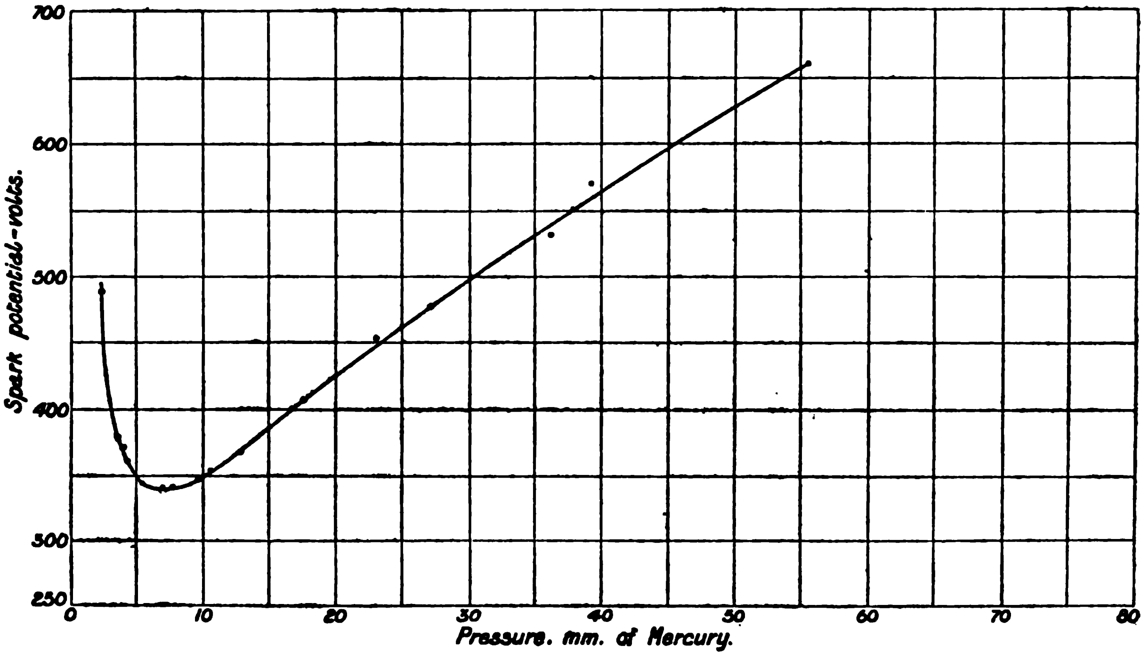
The first set of measurements was made on atmospheric air, not specially well dried. The readings will be given in full, as an example, to show the degree of concordance obtained.

Pressure (mm.) of mercury.	Spark potential (volts).	Mean value of spark potential.
72·4	800, 780, 760, 740, 740, 760, 780, 770, 770, 770 . .	767
55·6	640, 650, 665, 650, 660, 667, 662, 680, 670, 660 . .	660
39·1	570, 580, 555, 578, 580, 572, 563, 575, 560, 565 . .	570
36·1	540, 530, 520, 550, 510, 520, 520, 540, 525, 550 . .	531
27·2	470, 470, 466, 470, 490, 475, 485, 480, 484, 485, 468 .	477
23·1	461, 460, 458, 440, 462, 459, 451, 432, 457, 455 . .	454
17·5	410, 413, 421, 402, 405, 400, 412, 409, 407, 405 . .	408
13·1	370, 376, 360, 370, 359, 358, 380, 375, 365, 375 . .	369
10·7	350, 351, 358, 360, 353, 360, 350, 352, 353	354
9·6	355, 345, 348, 340, 349, 350, 351, 345, 350, 347 . .	348
7·9	341, 339, 341, 342, 342, 340, 341, 341, 346	341
7·0	342, 348, 330, 350, 340, 339, 339, 340, 343, 336 . .	341
5·4	345, 347, 343, 340, 348, 346, 347, 347, 345, 341 . .	345
4·4	360, 362, 362, 361, 365, 365, 360, 365, 355, 360 . .	362
4·0	370, 372, 380, 372, 372, 375, 365, 364, 375, 375 . .	372
3·8	372, 380, 380, 380, 381, 385, 380, 380, 380, 378 . .	380
2·4	500, 490, 483, 483, 502, 480, 470, 487, 500, 483 . .	488
1·8	755, 800, 780, 790, 790, 770, 790, 800, 780, 760 . .	782

The minimum spark potential is thus 341 volts. It will be noticed that the larger potential differences, *i.e.*, those furthest on either side from the minimum, could not be measured with the precision that was possible with those in the neighbourhood of the minimum. As, however, the value of the minimum is the point of chief interest, this is of small importance.

The results are plotted on Diagram No. 1. The smoothness of the curve is some guarantee of the accuracy of the observations.

Diagram No. 1.—Air.



Hydrogen.

We now come to the experiments on hydrogen. The hydrogen was prepared first by the electrolysis of caustic potash solution, and dried by means of phosphoric anhydride. The sample of gas thus prepared gave a minimum spark potential of 300 volts. The readings were as follows:—

Pressure (mm.).	Spark potential (volts.)
90·9	589
78·0	584
65·5	507
55·8	470
45·4	416
34·8	360
25·1	323
19·9	303
15·5	302
9·9	333
6·0	548
5·4	583

These results are plotted on Diagram No. 2.

It will be seen that the minimum is very nearly 300 volts.

found that the cathode fall in ordinary nitrogen was extremely inconstant. Thus in one case the value of this quantity, at first 315 volts, gradually rose to 410 volts during the passage of the current for many hours. He found that this variation in the value of the cathode fall was connected with the presence of traces of oxygen in the nitrogen—traces too small to be removed by the ordinary chemical absorbents, such as alkaline pyrogallol. I was anxious to try whether the same cause was operative in my case.

The method employed by WARBURG for getting rid of the last traces of oxygen from his nitrogen was to transport metallic sodium electrolytically through the heated glass of his discharge tube, so as to form a clean surface of that metal on the inside. At the high temperature used the sodium rapidly absorbed all traces of residual oxygen.

Such a method was obviously inapplicable to my apparatus, put together as it was with sealing wax joints. I used, therefore, a somewhat different method, which, though troublesome to carry out, gave good results.

This method was to bubble the sample of gas employed repeatedly through the liquid alloy of sodium and potassium.

The alloy must be manipulated in the absence of air to prevent its surface becoming fouled. The vessel in which it was contained is represented in the next figure.

a and *b* are glass bulbs blown on the side limbs of a Y-shaped tube *f*. On the bottom limb of this Y there is a stopcock *g*. The lower part of the limb is of capillary bore.

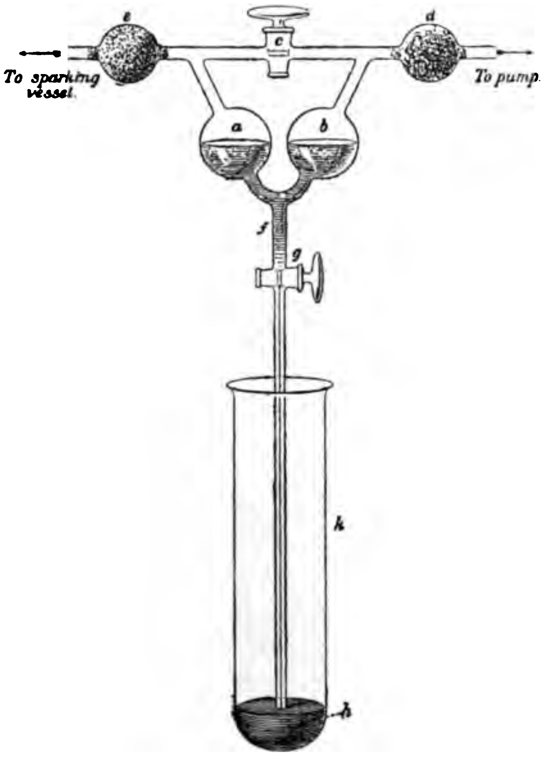
The bulbs *a* and *b* can be placed in communication by means of the stopcock *c*. *d* and *e* are bulbs containing phosphoric anhydride to constitute an additional safeguard against the access of moisture to the alloy. The exit tubes from these bulbs lead respectively to the pump-reservoir and to the sparking vessel.

The alloy was prepared in a test-tube *k*, by melting some sodium and adding potassium until the product remained liquid on cooling. A few drops of rock oil were from time to time placed on the surface of the alloy. The vapour from this sufficiently guarded against the access of air.

To introduce the alloy into the apparatus, *c* was opened and *g* closed. The air was then pumped out from the system of tubes.

The test-tube containing the alloy was then brought up as shown in the figure, the end *h* of the Y being *under the surface* of the alloy. *g* was then cautiously opened so that the alloy was sucked up into the bulbs *a* and *b*. When it had risen about half-way up them, as shown in the figure, *g* was closed, thus preventing the entry of any more of the alloy. In this way the alloy could be introduced without any contamination. It had all the appearance of clean mercury. This apparatus was used to absorb traces of oxygen from nitrogen in the following way :—*c* being open and in communication with the pump reservoir, the gas was admitted. *c* was then closed. On allowing the mercury to rise in the pump reservoir, the gas in the bulb *b* was compressed, and forced the surface of the alloy down to *f*. The gas then bubbled up through the bulb *a*, and was compressed into the sparking vessel. When the mercury reservoir was lowered again, the gas bubbled back again through the alloy, now in the bulb *b*.

Fig. 4.

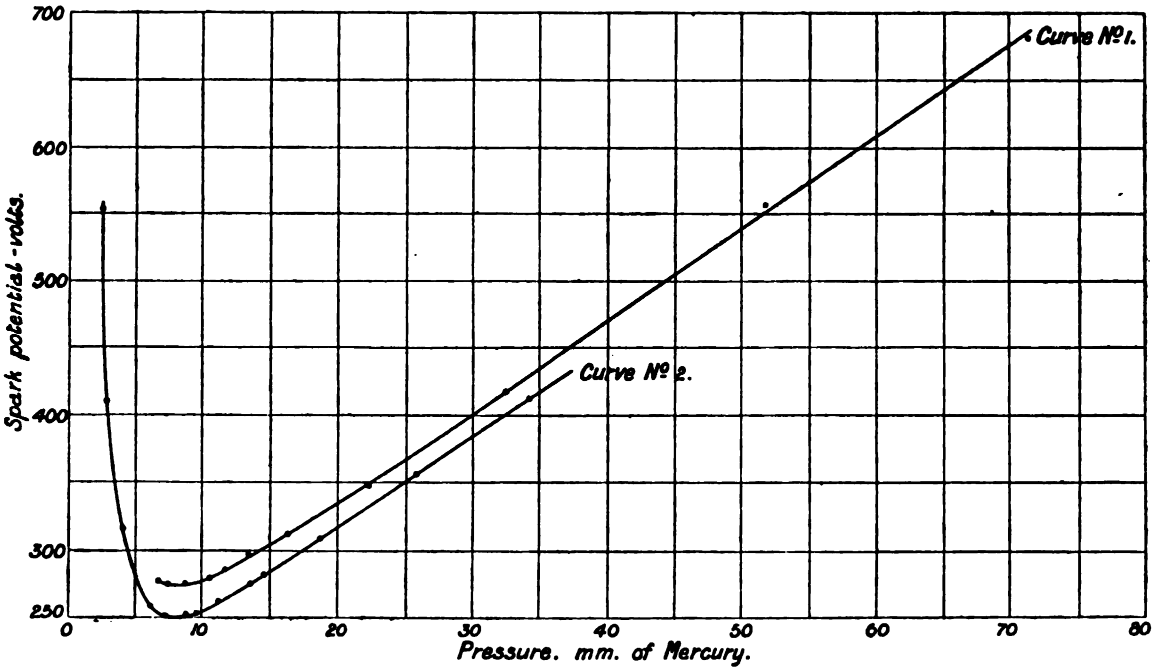


This process was repeated as often as required.

It was not possible to pass *all* the gas through the alloy at each operation; yet by giving time for the diffusion of the gas compressed into the sparking vessel, so as to allow it to become of uniform composition throughout, before being again passed through the absorbent, a very efficient purification could be effected by a few passages of the gas through the alloy. When as much had been drawn into the pump reservoir as circumstances allowed, the amount remaining behind in the sparking vessel was not more than one-fifth of the whole. Thus, supposing all the oxygen to be taken out of that part of the gas which had actually bubbled through the alloy, only $\frac{1}{5}$ would remain after the first operation, $\frac{1}{25}$ after the second, $\frac{1}{125}$ after the third, and so on.

When it was desired to measure the spark potential in the purified gas, the tap *c* was opened, so as to make the pressure the same in the manometer as in the sparking vessel. If this had not been done, the pressure due to the column of alloy would have destroyed the equality of these pressures. The nitrogen employed was prepared by the action of heat on ammonium nitrite. Ammonium chloride solution was contained in a flask, fitted with an india-rubber cork, through which passed a dropping funnel and an exit tube for the gas. Potassium nitrite solution was contained in the funnel, and could be dropped into the warmed ammonium chloride. The gas passed through strong sulphuric acid and caustic potash solutions, each contained in ordinary potash bulbs. From these it passed into a chamber containing phosphoric anhydride, closed at either end by taps. An approximate vacuum was first made in the entire arrangement by means of a water pump. This reduced the pressure to something less than a centimetre of mercury. Gas was then liberated so as to slowly fill the apparatus up to atmospheric pressure. A current was allowed to flow for some time through the phosphoric anhydride chamber, so as to wash out all traces of air. When this had been done the chamber was shut off, and the gas left in it all night, so that it might become thoroughly dry. It could then be admitted through a tap into the sparking vessel.

Diagram No. 3.—Nitrogen.



After the gas had been passed several times through the alloy, the following measurements were taken (Diagram No. 3, curve 1) :—

Pressure (mm.).	Voltage.
71·1	683
51·7	559
32·4	418
22·2	348
16·2	311
13·5	298
11·7	287
10·6	280
8·9	276
7·6	276
6·8	279

It will be seen that the minimum was about 276 volts—much lower than the values obtained with nitrogen not specially treated.

The treatment with the alloy was now continued for some hours.

The following set of readings was obtained after this :—

Pressure (mm.).	Voltage.
34·2	411
25·8	356
18·9	308
14·6	281
13·8	275
11·1	262
9·8	251
8·9	251
7·3	251
6·0	259
4·0	315
2·9	410
2·3	553

Further treatment with the alloy did not appreciably reduce the minimum voltage.

A repetition of this experiment with a fresh sample of nitrogen gave a result in close agreement with the above. The minimum may be therefore taken as 251 volts.

It will be seen that the perfect removal of oxygen does not much affect the slope of the curve in its straight part, where the pressure is above the critical value. The interpretation of this (on the view given above) is that the cathode fall of potential, not that near the anode, is affected by the presence of traces of oxygen.

Helium.

It has been shown by Professor RAMSAY and Dr. COLLIE ('Roy. Soc. Proc.,' vol. 59, p. 259) that helium at atmospheric pressure allows sparks to pass through it much more easily than does air. With a given E.M.F. ten times as long a spark was obtained in helium as in air. This is a very striking result. It seemed, therefore, well worth while to investigate the behaviour of helium as regards its spark potential in the same way as the other gases.

For the method of preparing this gas I must refer to a paper "On the Discharge of Electricity through Argon and Helium," which I hope shortly to publish, where it is described in detail. I merely state here that it was extracted from monazite by means of strong sulphuric acid, and that nitrogen and other impurities were removed from it by mixing with oxygen and exposing it to the action of electric sparks in presence of caustic alkali. The oxygen was then removed by suitable absorbents.

Although all the care I was able to give was spent on the purification of the gas, some cause, the nature of which remains a mystery, made the spark potential variable. It was found that if a specimen of helium remained in the apparatus, and measurements of its spark potential were taken at intervals, the value of this quantity went down. As an example of this, I will give a series of measurements made on a sample of helium from which the surplus oxygen had been removed by the copper-ammonia method of Hempel.*

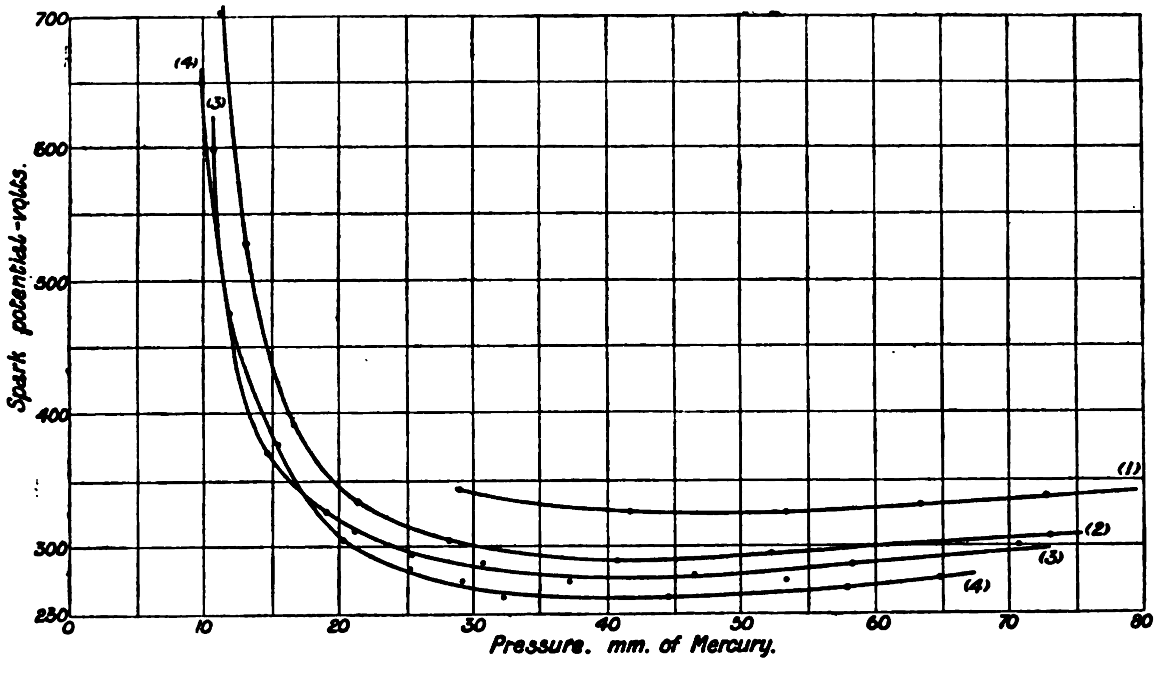


Diagram No. 4.—Helium.

Any gaseous ammonia which might remain in the helium was removed by means of dilute sulphuric acid. The helium was dried by phosphoric anhydride, and examined

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[*] This method consists in bringing the gas into contact with metallic copper at the ordinary temperature, the copper having just been washed free from oxide by a solution of ammonia to which some ammonium carbonate has been added. See Hempel, 'Methods of Gas Analysis,' p. 126.

spectroscopically. No trace of nitrogen bands could be seen. The red hydrogen line was visible, as it always is in a vacuum tube. This sample of gas gave the following readings when first put into the apparatus :—

1.

Pressure (mm.).	Voltage.
94·0	348
83·1	340
72·8	339
63·5	332
53·5	326
41·9	326
29·0	343

Minimum 326.

The gas was left in the apparatus untouched. Next day the readings obtained were :—

2.

Pressure (mm.).	Voltage.
91·7	341
73·0	309
52·3	295
40·9	288
28·1	303
21·5	332
16·5	391
13·2	528
11·3	703

Minimum 288.

The gas was again left untouched till next morning :—

3.

Pressure (mm.).	Voltage.
81·9	316
70·7	301
58·1	285
46·5	280
37·2	274
30·8	288
25·4	294
21·2	311
19·2	326
1·48	370
1·09	599

Minimum 274.

It will thus be seen that, after the gas has been sparked through, its sparking potential is lowered. In this case the minimum was lowered from 326 to 288, and then to 274.

To see what the effect of further sparking would be, a torrent of sparks was passed through the gas by means of the machine for an hour. The readings then taken were :—

4.

Pressure (mm.).	Voltage.
64·9	276
57·9	269
44·8	261
32·3	261
29·2	274
25·2	284
20·3	303
15·6	377
12·0	475
10·0	650

Minimum 261.

It will be seen that the minimum has gone down still further. Another three hours' sparking did not make any further difference. The minimum spark potential after this treatment was 262 volts, practically the same as before. Other specimens of helium from which the surplus oxygen had been removed by means of phosphorus (at the ordinary temperature) behaved very similarly.

It may be remarked that this behaviour of helium is very like that observed by WARBURG in the case of nitrogen which had not been put through any exceptional treatment for the removal of its surplus oxygen : with this difference, however, that he found the cathode fall go up (from 315 to 410 volts) with the sparking, whereas, in my measurements on helium, the spark potential went down.

It seems not unlikely that if the last traces of oxygen were removed from the helium by means of the sodium-potassium alloy, normal and constant results might be obtained. I hope to examine this point on some future occasion.

General Conclusions.

It remains to discuss the results obtained, and to inquire how far they bear out the conclusion that the minimum spark potential is equal to the cathode fall measured over the whole extent of the negative glow in a vacuum tube.

For this purpose it will be convenient to tabulate the values of these quantities for the various gases side by side.

Nature of gas.	Cathode fall given by Warburg.	Minimum spark potential found above volts.
Atmospheric air	340–350 volts ('Wied. Ann.,' vol. 31, p. 559) .	341
Hydrogen	About 300 volts ('Wied. Ann.,' vol. 31, p. 581) .	302, 308
Ordinary nitrogen, carefully dried	Varies 315–410 („ „ „ p. 557) .	347, 351 369, 388
Nitrogen specially freed from all traces of oxygen	230 volts ('Wied. Ann.,' vol. 40, p. 1)	251
Helium	. . . . [Values found by myself (see paper on "Discharge of Electricity through Argon and Helium"). 226 volts.]	Varies 326–261

In the case of air, the agreement is as good as could be expected, and, indeed, much better. It is also satisfactory in the case of hydrogen.

In ordinary nitrogen the results are in neither case constant. But my numbers all lie between the extremes found by WARBURG.

In the specially purified nitrogen a difference of about 10 per cent. is to be observed. For measurements of this kind the discrepancy is not enough to establish an essential want of equality. It is possible that my method of removing oxygen, which depended on the use of alkali metals at ordinary temperatures, was not quite so efficient as that of WARBURG, who used a high temperature.

In the case of helium alone is the value of the cathode fall seriously different from that of the minimum spark potential.

Although this discrepancy is large, I do not think that, in view of the satisfactory agreement in the other cases, it can be considered to weigh seriously against the equality. It must be remembered that the values found for helium differ much more among themselves than the lowest of them differs from the measured cathode fall. Measurements of the same quantity which do not agree with one another cannot be expected to agree with other independent measurements.

I think it may be considered to be established by these experiments that the minimum spark potential *is* equal to the cathode fall.

One point is very strikingly brought out by the curves. I mean the extremely small rate of increase of the spark potential with the pressure in the case of helium, and the relatively high pressure at which the minimum occurs. All the features shown by the curves for the common gases are seen in the curve for helium, but the difference in degree is very striking. The enormous length of spark at atmospheric pressure in helium as compared with air might almost suggest that the conduction is effected in some radically different way. But these curves show that the behaviour of helium when conveying the discharge, though very peculiar, is not radically different from that of the common gases.

In conclusion, I must record my best thanks to Professor J. J. THOMSON. I cannot adequately express how much I owe to his kind encouragement and advice.

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—— Part II...	0	14	0
1836. Part I...	1	10	0
—— Part II...	2	0	0
1837. Part I...	1	8	0
—— Part II...	1	8	0
1838. Part I...	0	13	0
—— Part II...	1	8	0
1839. Part I...	0	18	0
—— Part II...	1	1	6
1840. Part I...	0	18	0
—— Part II...	2	5	0
1841. Part I...	0	10	0
—— Part II...	1	10	0
1842. Part I...	0	16	0
—— Part II...	1	2	0
1843. Part I...	0	10	0
—— Part II...	1	10	0
1844. Part I...	0	10	0
—— Part II...	1	10	0
1845. Part I...	0	16	0
—— Part II...	1	0	0
1846. Part I...	0	7	6
—— Part II...	1	12	0
—— Part III...	1	12	0
—— Part IV...	1	12	0
1847. Part I...	0	14	0
—— Part II...	0	16	0
1848. Part I...	1	0	0
—— Part II...	0	14	0
1849. Part I...	1	0	0
—— Part II...	2	5	0
1850. Part I...	1	10	0
—— Part II...	3	5	0
1851. Part I...	2	10	0
—— Part II...	2	10	0
1852. Part I...	1	0	0
—— Part II...	2	5	0
1853. Part I...	0	18	0
—— Part II...	0	12	0
—— Part III...	1	2	0
1854. Part I...	0	12	0
—— Part II...	0	16	0
1855. Part I...	0	16	0
—— Part II...	1	6	0
1856. Part I...	2	0	0
—— Part II...	1	4	0
—— Part III...	1	4	0
1857. Part I...	1	8	0

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1857. Part II...	1	4	0
—— Part III...	1	2	0
1858. Part I...	1	8	0
—— Part II...	3	0	0
1859. Part I...	2	10	0
—— Part II...	2	5	0
1860. Part I...	0	16	0
—— Part II...	2	1	6
1861. Part I...	1	3	0
—— Part II...	1	5	0
—— Part III...	1	7	6
1862. Part I...	2	14	0
—— Part II...	3	0	0
1863. Part I...	1	14	0
—— Part II...	1	7	6
1864. Part I...	0	11	0
—— Part II...	1	7	6
—— Part III...	1	10	0
1865. Part I...	2	2	0
—— Part II...	1	5	0
1866. Part I...	1	14	0
—— Part II...	2	7	6
1867. Part I...	1	3	0
—— Part II...	1	15	0
1868. Part I...	2	5	0
—— Part II...	2	0	0
1869. Part I...	2	10	0
—— Part II...	3	3	0
1870. Part I...	1	10	0
—— Part II...	1	18	0
1871. Part I...	1	10	0
—— Part II...	2	5	0
1872. Part I...	1	12	0
—— Part II...	2		
1873. Part I..			
—— Part II.			
1874. Part I.			
—— Part II.			
1875. Part I			
—— Part II			
1876. Part I			
—— Part II			
1877. Part I			
—— Part II			
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1878. Part I			
—— Part II			
1879. Part I			
—— Part II			
1880. Part I			
—— Part II			
—— Part III			

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1881. Part I...	2	10	0
—— Part II...	1	10	0
—— Part III...	2	2	0
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—— Part II...	2	0	0
—— Part III...	2	10	0
—— Part IV...	1	0	0
1883. Part I...	1	10	0
—— Part II...	2	10	0
—— Part III...	1	12	0
1884. Part I...	1	8	0
—— Part II...	1	16	0
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—— Part II...	2	5	0
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—— Part II...	1	15	0
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—— (B)	1	16	0
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—— (B.)	2	17	6
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—— (B.)	1	14	0
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—— (B.)	1	5	0
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—— (B.)	3	3	0
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—— (B.)	2	2	0
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—— (B.)	2	13	0
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